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A Case When the Fiber of the Double Suspension is the Double Loops on Anick's Space

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Abstract. The fiber W_n of the double suspension $S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ is known to have a classifying space BW_n. An important conjecture linking the *EHP* sequence to the homotopy theory of Moore spaces is that $BW_n \simeq \Omega T^{2np+1}(p)$, where $T^{2np+1}(p)$ is Anick's space. This is known if $n = 1$. We prove the $n = p$ case and establish some related properties.

1 Introduction

Let p be an odd prime and localize all spaces and maps at p. Let W_n be the homotopy fiber of the double suspension $E^2\colon S^{2n-1}\Omega^2 \longrightarrow S^{2n+1}$. A long-standing conjecture in homotopy theory is that W_n has a double classifying space. Gray [G] showed that W_n does have at least a single classifying space BW_n. He proved that there are homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n,$$
$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1},$$

where BW_n is an H-space, ν and j are H-maps (the $p = 3$ case of this being proved in [T]), and the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{j} \Omega^2 S^{2np+1}$$

is homotopic to the loops on the p-th James–Hopf invariant. The map ϕ is similar to the np-case of a map $\varphi\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ constructed by Cohen, Moore, and Neisendorfer [CMN1,CMN2] in their investigation of the mod-p Moore space. They proved that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p-th power map and used this fact to inductively show that the homotopy exponent of S^{2n+1} is p^n. It remains open whether ϕ and the np-case of φ are homotopic.

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The map φ fits in a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \longrightarrow T^{2n+1}(p) \longrightarrow \Omega S^{2n+1}.$$

This was proved by Anick [A] for $p \geq 5$ and by different means in [GT2] for $p \geq 3$. The space $T^{2np+1}(p)$ is a candidate for the double classifying space of W_n: potentially, $W_n \simeq \Omega^2 T^{2np+1}(p)$. In view of Gray's work, we could ask for the stronger property that $BW_n \simeq \Omega T^{2np+1}(p)$. Going further, one could ask that this be a homotopy equivalence of H-spaces. The existence of any homotopy equivalence would establish a deep connection between the homotopy theory of Moore spaces and the *EHP* sequence and would be useful in calculating the homotopy groups of spheres. To date, the only known case was when $n = 1$. It follows from [S3], although not explicitly stated there in this form, that there is a multiplicative homotopy equivalence $BW_1 \simeq \Omega T^{2p+1}(p)$.

In this note, we prove the $n = p$ case of the conjecture and do so in its strongest form.

Theorem 1.1 *For $p \geq 3$ there is a homotopy equivalence of H-spaces*

$$BW_p \simeq \Omega T^{2p^2+1}(p).$$

The homotopy equivalence in Theorem 1.1 depends on the existence of a certain splitting that occurs in only one case when $n > 1$, so our method of proof has no hope of extending to other cases, let alone to the general case. To describe the splitting, let $S^{2n+1}\{p\}$ be the homotopy fiber of the degree p map on S^{2n+1}. In [S2], Selick showed that $\Omega S^{2n+1}\{p\}$ is indecomposable if $n \notin \{1, p\}$, and in [S1] he showed that when $n = p$, there is a homotopy decomposition $\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times \Omega^2 X$, where $S^3\langle 3\rangle$ is the three-connected cover of S^3 and X is a space with the property that $\Omega^2 X \simeq W_p$. In [S3], this was improved upon by using a homotopy equivalence $\Omega S^3\langle 3\rangle \simeq T^{2p+1}(p)$ to show that there is a homotopy decomposition $\Omega S^{2p+1}\{p\} \simeq T^{2p+1}(p) \times \Omega X$. Notice that X is a double delooping of W_p, although it is a space that is not satisfactorily described in the sense that it is not identified as some other known space. We address this by refining Selick's decomposition to show that the space X can be chosen to be $T^{2p^2+1}(p)$, and we go further by showing that the equivalence can be chosen to be multiplicative.

Theorem 1.2 *For $p \geq 3$ there is a homotopy decomposition of H-spaces*

$$\Omega S^{2p+1}\{p\} \simeq T^{2p+1}(p) \times \Omega T^{2p^2+1}(p).$$

2 The Proof of Theorem 1.2

We make use of the following Lemma originally proved in [AG] for $p \geq 5$, and extended to the $p = 3$ case in [GT2]. For a space X, $m \geq 3$, and $r \geq 1$, let $\pi_m(X; \mathbb{Z}/p^r\mathbb{Z})$ be the set of homotopy classes of maps $[P^m(p^r), X]$.

Lemma 2.1 *Let $p \geq 3$. Let X be an H-space such that $p^k \cdot \pi_{2np^k-1}(X; \mathbb{Z}/p^{k+1}\mathbb{Z}) = 0$ for $k \geq 1$. Then any map $P^{2n}(p) \to X$ extends to a map $T^{2n+1}(p) \to X$.*

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Proof of Theorem 1.2 By [GT2], for $p \geq 3$, there is a homotopy fibration

$$T^{2n+1}(p) \xrightarrow{\varpi} \Omega S^{2n+1}\{p^r\} \xrightarrow{h} BW_n, \tag{2.1}$$

where ϖ is an H-map. We first construct a specific right homotopy inverse of h when $n = p$ and then go on to establish the multiplicative decomposition of $\Omega S^{2p+1}\{p\}$.

In general for any n, it is well known that there is an isomorphism

$$H_*(\Omega S^{2n+1}\{p^r\}) \cong \big(\bigotimes_{j=0}^{\infty} \Lambda(a_{2np^j-1})\big) \otimes \big(\bigotimes_{j=0}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^j-2}]\big).$$

In (2.1), we have $H_*(T^{2n+1}(p)) \cong \Lambda(a_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[b_{2n-2}]$ and ϖ_* is the inclusion. As well, h_* is the projection onto

$$H_*(BW_n) \cong \big(\bigotimes_{j=1}^{\infty} \Lambda(a_{2np^j-1})\big) \otimes \big(\bigotimes_{j=1}^{\infty} \mathbb{Z}/p\mathbb{Z}[b_{2np^j-2}]\big).$$

When $n = p$, [S2] shows that ϖ has a left homotopy inverse, and so there is a homotopy decomposition $\Omega S^{2p+1}\{p\} \simeq T^{2p+1}(p) \times BW_p$. Let f be the composite

$$f\colon S^{2p^2-2} \longrightarrow BW_p \longrightarrow \Omega S^{2p+1}\{p\},$$

where the left map is the inclusion of the bottom cell and the right map is a right homotopy inverse for h. Let $\iota \in H_{2p^2-2}(S^{2p^2-2})$ be a generator. Then $f_*(\iota) = b_{2p^2-2}$.

By [N], $p \cdot \pi_*(S^{2p+1}\{p\}) = 0$, so f extends to a map $g\colon P^{2np-1}(p) \longrightarrow \Omega S^{2p+1}\{p\}$. Taking the adjoint gives a map $g'\colon P^{2np}(p) \longrightarrow S^{2p+1}\{p\}$. By [N], $S^{2p+1}\{p\}$ is an H-space and $p \cdot \pi_*(S^{2p+1}\{p\}; \mathbb{Z}/p^r\mathbb{Z}) = 0$ for any $r \geq 1$. Thus Lemma 2.1 can be applied to g', and doing so we obtain a map

$$s\colon T^{2p^2+1}(p) \longrightarrow S^{2p+1}\{p\}.$$

Looping, the restriction of Ωs to the bottom cell is f, and so $(\Omega s)_*(\iota) = b_{2p^2-2}$.

By the definition of f, the composite

$$S^{2p^2-2} \xrightarrow{f} \Omega S^{2p+1}\{p\} \xrightarrow{h} BW_p$$

is the inclusion of the bottom cell, so as Ωs extends f the composite

$$\Omega T^{2p^2+1}(p) \xrightarrow{\Omega s} \Omega S^{2p+1}\{p\} \xrightarrow{h} BW_p$$

is degree one in $H_{2p^2-2}(\ )$. In general, as shown in [GT1], any map $s\colon \Omega T^{2np+1}(p) \longrightarrow BW_n$ that is degree one in $H_{2np-2}(\ )$ is a homotopy equivalence. Thus, in our case, $h \circ \Omega s$ is a homotopy equivalence.

Next, let μ be the loop multiplication on $\Omega S^{2n+1}\{p^r\}$. Since Ωs is a right homotopy inverse of h, the homotopy fibration

$$T^{2p+1}(p) \xrightarrow{\varpi} \Omega S^{2p+1}\{p\} \xrightarrow{h} BW_n$$

splits, and so the composite

$$e\colon T^{2p+1}(p) \times \Omega T^{2p^2+1}(p) \xrightarrow{\varpi \times \Omega s} \Omega S^{2p+1}\{p\} \times \Omega S^{2p+1}\{p\} \xrightarrow{\mu} \Omega S^{2p+1}\{p\}$$

is a homotopy equivalence. Observe that both ϖ and Ωs are H-maps, and so is μ because it is homotopic to the loop of the multiplication on $S^{2p+1}\{p\}$. Thus e is an H-map, and so it is a multiplicative homotopy equivalence. ■

The proof of Theorem 1.2 also showed that there is a homotopy equivalence $BW_p \simeq \Omega T^{2p^2+1}(p)$. However, this need not be multiplicative. In the next section, we produce a potentially different homotopy equivalence that is multiplicative.

3 The Proof of Theorem 1.1

To prepare for the proof, we first establish a general property of the homotopy fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ of H-spaces and H-maps. In [G] it was shown that there is a homotopy equivalence $\Sigma^2\Omega^2 S^{2n+1} \simeq \Sigma^2(S^{2n-1} \times BW_n)$. Thus $\Sigma^2\nu$ has a right homotopy inverse $t\colon \Sigma^2 BW_n \to \Sigma^2\Omega^2 S^{2n+1}$. The map t can be used to modify the decomposition of $\Sigma^2\Omega^2 S^{2n+1}$. The canonical Hopf construction on $\Omega^2 S^{2n+1}$ gives a map $\Sigma\Omega^2 S^{2n+1} \wedge \Omega^2 S^{2n+1} \to \Sigma\Omega^2 S^{2n+1}$. Suspending and freely moving the suspension coordinates gives a composite

$$a\colon S^{2n-1} \wedge \Sigma^2 BW_n \xrightarrow{E^2 \wedge t} \Omega^2 S^{2n+1} \wedge \Sigma^2\Omega^2 S^{2n+1} \longrightarrow \Sigma^2\Omega^2 S^{2n+1}.$$

It follows that the wedge sum of $\Sigma^2 E^2$, t, and a gives a homotopy equivalence

(3.1) $\Sigma^2(S^{2n-1} \times BW_n) \xrightarrow{\simeq} \Sigma^2 S^{2n-1} \vee \Sigma^2 BW_n \vee (\Sigma^2 S^{2n-1} \wedge BW_n) \longrightarrow \Sigma^2\Omega^2 S^{2n+1}.$

Lemma 3.1 *Suppose there is an H-map $f\colon \Omega^2 S^{2n+1} \to \Omega Z$, where Z is an H-space, and suppose $f \circ E^2$ is null homotopic. Then there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, "f"] & BW_n \arrow[dl, "g"] \\
\Omega Z &
\end{tikzcd}$$

where g is an H-map.

Proof We first prove the special case when $Z = \Omega X$. The Hopf construction is natural for H-maps, so the assumptions that f is an H-map and $f \circ E^2$ is null homotopic imply that $\Sigma^2 f \circ a$ is also null homotopic. Thus the homotopy equivalence in (3.1) implies that there is a factorization

$$\begin{tikzcd}
\Sigma^2\Omega^2 S^{2n+1} \arrow[r, "\Sigma^2\nu"] \arrow[d, "\Sigma^2 f"] & \Sigma^2 BW_n \arrow[dl, "\lambda"] \\
\Sigma^2\Omega^2 X &
\end{tikzcd}$$

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where $\lambda = \Sigma^2 f \circ t$. Composing with the evaluation map $ev\colon \Sigma^2\Omega^2 X \longrightarrow X$ and taking the double adjoint gives a homotopy commutative diagram

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, "f"] & BW_n \arrow[dl, "g"] \\
\Omega^2 X &
\end{tikzcd}$$

where $g = \Omega^2(ev \circ \Sigma^2\lambda) \circ E^2$. Thus f factors through ν, as asserted.

It remains to show that g is an H-map. Consider the diagram

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \times \Omega^2 S^{2n+1} \arrow[r, "\nu\times\nu"] \arrow[d, "\mu"] & BW_n \times BW_n \arrow[r, "g\times g"] \arrow[d, "\mu"] & \Omega^2 X \times \Omega^2 X \arrow[d, "\mu"] \\
\Omega^2 S^{2n+1} \arrow[r, "\nu"] & BW_n \arrow[r, "g"] & \Omega^2 X
\end{tikzcd}$$

where the maps labelled μ are the H-space multiplications. Since ν is an H-map, the left square homotopy commutes, and, by hypothesis, f (that is, $g \circ \nu$) is an H-map, so the outer rectangle also homotopy commutes. We wish to show that the right square homotopy commutes, proving that g is an H-map. Observe that if $\nu \times \nu$ had a right homotopy inverse, then the commutativity of the outer rectangle would imply the commutativity of the right square. Now $\nu \times \nu$ does not have a right homotopy inverse, but $\Sigma^2(\nu \times \nu)$ does because $\Sigma^2\nu$ has one. Since $\Omega^2 X$ is a double loop space, we can double adjoint. The right homotopy inverse for $\Sigma^2(\nu \times \nu)$ implies that the double adjoint of the right square homotopy commutes. Hence the right square itself homotopy commutes.

Next, consider the more general case when Z is an H-space. Then there is a map $r\colon \Omega\Sigma Z \to Z$ that is a left homotopy inverse of the suspension $E\colon Z \to \Omega\Sigma Z$. If we let $X = \Omega\Sigma Z$ and let $\overline{f} = \Omega E \circ f$, then the first case above applies to the pair $(X, \overline{f})$ and we obtain an H-map $\overline{g}\colon BW_n \to \Omega^2\Sigma Z$ such that $\overline{g} \circ \nu \simeq \overline{f}$. Let $g = \Omega r \circ \overline{g}$. Then g is an H-map and $g \circ \nu \simeq \Omega r \circ \overline{g} \circ \nu \simeq \Omega r \circ \overline{f} \simeq \Omega r \circ \Omega E \circ f \simeq f$. ∎

Proof of Theorem 1.1 We will construct a map $BW_p \to \Omega T^{2p^2+1}(p)$, which is both an H-map and a homotopy equivalence. Let γ be the composite

$$\gamma\colon \Omega S^{2p+1}\{p\} \xrightarrow{\simeq} T^{2p+1}(p) \times \Omega T^{2p^2+1}(p) \longrightarrow \Omega T^{2p^2+1}(p),$$

where the left map is the inverse homotopy equivalence from Theorem 1.2 and the right map is the projection. In general, the homotopy inverse of an H-equivalence is itself an H-equivalence. So in our case, both maps in the composite defining γ are H-maps and so γ is an H-map. Let $\delta\colon \Omega S^{2p+1} \to S^{2p+1}\{p\}$ be the connecting map for the homotopy fibration

$$S^{2p+1}\{p\} \longrightarrow S^{2p+1} \xrightarrow{p} S^{2p+1}.$$

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Let f be the composite

$$f\colon \Omega^2 S^{2p+1} \xrightarrow{\Omega\delta} \Omega S^{2p+1}\{p\} \xrightarrow{\gamma} \Omega T^{2p^2+1}(p).$$

Observe that f is an H-map as it is the composite of H-maps. By connectivity, the composite $f \circ E^2$ is null homotopic. By [GT2], $T^{2p^2+1}(p)$ is an H-space for all $p \geq 3$. Thus Lemma 3.1 implies that there is a homotopy commutative diagram

$$\begin{tikzcd} \Omega^2 S^{2p+1} \arrow[r, "\nu"] \arrow[d, "f"] & BW_p \arrow[dl, "e"] \\ \Omega T^{2p^2+1}(p) & \end{tikzcd}$$

where e is an H-map.

We wish to show that e is also a homotopy equivalence. This could be done by analyzing ν_* and f_* to show that e_* is degree one in $H_{2np^2-2}(\,\cdot\,)$ and applying the same theorem from [GT1] used in the proof of Theorem 1.2. Alternatively, the definition of f as $\gamma \circ \Omega\delta$ gives a homotopy pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \Omega^2 S^{2p+1} \arrow[r, "f"] \arrow[d, "\Omega\delta"] & \Omega T^{2p^2+1}(p) \arrow[d, equal] \\ T^{2p+1}(p) \arrow[r] \arrow[d] & \Omega S^{2p+1}\{p\} \arrow[r, "\gamma"] \arrow[d] & \Omega T^{2p^2+1}(p) \\ \Omega S^{2p+1} \arrow[r, equal] & \Omega S^{2p+1} & \end{tikzcd}$$

that defines the space X. Consider the homotopy fibration in the left column. The composite $T^{2p+1}(p) \to \Omega S^{2p+1}\{p\} \to \Omega S^{2p+1}$ is onto in homology and a Serre spectral sequence calculation shows that $H_*(X) \cong H_*(S^{2p-1})$. Hence $X \simeq S^{2p-1}$. The homotopy fibration along the top row shows that the map $S^{2p-1} \to \Omega^2 S^{2p+1}$ is degree one in $H_{2p-1}(\ )$ by connectivity, and so is homotopic to E^2. Having identified the homotopy fiber of f, the factorization of f as $e \circ \nu$ implies that there is a homotopy fibration diagram

$$\begin{tikzcd} S^{2p-1} \arrow[r, "E^2"] \arrow[d, "t"] & \Omega^2 S^{2p+1} \arrow[r, "\nu"] \arrow[d, equal] & BW_p \arrow[d, "e"] \\ S^{2p-1} \arrow[r, "E^2"] & \Omega^2 S^{2p+1} \arrow[r, "f"] & \Omega T^{2p^2+1}(p) \end{tikzcd}$$

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where t is an induced map of fibers. The homotopy commutativity of the left square shows that t is degree one and so is homotopic to the identity map. The Five-Lemma therefore implies that e is a homotopy equivalence. ■

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