Can. J. Math., Vol. XXXVI, No. 1, 1984, pp. 22-37

SERIAL RIGHT NOETHERIAN RINGS

SURJEET SINGH

A module M is called a serial module if the family of its submodules is linearly ordered under inclusion. A ring R is said to be serial if R_R as well as ${}_RR$ are finite direct sums of serial modules. Nakayama [**8**] started the study of artinian serial rings, and he called them generalized uniserial rings. Murase [**5**, **6**, **7**] proved a number of structure theorems on generalized uniserial rings, and he described most of them in terms of quasi-matrix rings over division rings. Warfield [**12**] studied serial both sided noetherian rings, and showed that any such indecomposable ring is either artinian or prime. He further showed that a both sided noetherian prime serial ring is an $(R:J)$-block upper triangular matrix ring, where R is a discrete valuation ring with Jacobson radical J. In this paper we determine the structure of serial right noetherian rings (Theorem 2.11). We also study right noetherian rings whose proper homomorphic images are serial; Theorem 3.3 shows that any such semiprime ring is either serial or prime. Thereby we improve [**11**, Theorem 6] and its generalization given by Levy and Smith [**4**]. Finally in Theorem (4.1) we establish another characterisation of artinian serial rings.

1. Preliminaries. All rings considered here are with identity $1 \neq 0$ and modules are unital right modules, unless otherwise specified. A ring R is said to be noetherian (artinian) if it is right as well as left noetherian (artinian). For definition and basic properties of semiprime Goldie rings we refer to [**2**]. Let R be a prime right Goldie ring. The following properties and concepts about R are well known. R is said to be right bounded if every essential right ideal of R contains a non-zero ideal. In this paper any module M over a semi-prime Goldie ring is said to be torsion (torsion-free) if it is torsion (torsion-free) in the Goldie torsion theory. If R is right bounded no non-zero torsion injective R-module is finitely generated. If R is both sided Goldie any two finitely generated uniform, torsion-free R-modules are embeddable in each other.

For definition and basic properties of hereditary noetherian prime ((hnp)) rings, we refer to [**1**]. By [**12**, Theorem 5.11] any prime, serial,

Received April 22, 1982 and in revised form November 26, 1982.

noetherian ring is hereditary. It is clearly both side bounded. As defined by Warfield [**12**] a simple module T is called a successor of a simple module S if Ext $(S, T) \neq 0$; under the same conditions, S is called a predecessor of T. The results collected in the following theorem are all due to Warfield [**12**, (5.1), (5.3), (5.6), (5.11)].

THEOREM 1.1. *Let R be a serial, right noetherian ring. Then*:

(a) *Any simple R-module S has at most one successor and one predecessor up to isomorphism. Further S has a successor unless S is projective.*

(b) *If there exists an indecomposable projective R-module P such that $\cap_n PJ^n \neq 0$, where J is the Jacobson radical of R, then R has a simple module S with no predecessor.*

(c) *Any uniform R-module is serial. In particular an indecomposable injective R-module is serial.*

(d) *If R is also left noetherian, then $\cap_n J^n = 0$ and R is the product of an artinian serial ring and finitely many prime serial rings.*

For any ring R, $J(R)$ (or simply J) and $N(R)$ (or simply N) will denote its Jacobson radical and its nil radical respectively. For any module M_R, $E_R(M)$ (or simply $E(M)$) will denote its injective hull. For any ordinal α, J^α is defined inductively as follows: $J^0 = R$. If α is a limit ordinal $J^\alpha = \cap_{\beta<\alpha} J^\beta$ and if $\alpha = \beta + 1$, $J^\alpha = J^\beta J$. For any module M_R, $\text{ann}_R\,(M)$ (or simply ann (M)) will denote the annihilator of M in R. The symbol $N \subset' M$ will mean that N is an essential submodule of M.

2. Serial right noetherian rings. We start with the following:

LEMMA 2.1. *Let R be a serial ring with Jacobson radical J. If $\cap J^n = 0$, then R is noetherian. If for some n, $J^n = 0$, R is artinian.*

Proof. Now $R = e_1R \oplus e_2R \oplus \ldots \oplus e_kR$ for some orthogonal indecomposable idempotents e_i. Consider $x \neq 0$ in e_iR. As $\cap e_iJ^n = 0$, for some n,

$$x \notin e_iJ^n \setminus e_iJ^{n+1}.$$

Then $xR = e_iJ^n$. This immediately gives that e_iR is right noetherian. Hence R_R is noetherian. Similarly ${}_RR$ is noetherian. The second part is obvious.

The following is immediate from (1.1) (c):

LEMMA 2.2. *Any uniform module over a serial right noetherian ring is either injective or finitely generated.*

THEOREM 2.3. *Any semiprime serial right noetherian ring R is left noetherian, and is a direct sum of prime serial rings.*

Proof. Let R be a semiprime serial right noetherian ring. Let R be not left noetherian. By (2.1) $\cap_n J^n \neq 0$. So for some indecomposable idempotent $e \in R$ $\cap_n eJ^n \neq 0$. Consequently by (1.1) (b), R admits a simple module S having no predecessor. It can be seen that [**12**, Lemma (5.5)] is valid for serial, right noetherian rings. Consequently there exists a projective R-module which is artinian. This yields socle $(R) \neq 0$. As R is a semiprime right noetherian ring, $R = \text{socle}(R) \oplus T$, where T is an ideal of R with soc $(T) = 0$. Since T is a semiprime serial right noetherian ring with zero socle, we get T is left noetherian. Consequently R is also left noetherian, as socle (R) is left artinian. Hence the result follows.

LEMMA (2.4). *Let R be a serial right noetherian ring.*

(I) *Any two non-comparable prime ideals of R are comaximal.*

(II) *For any indecomposable idempotent e, eR/eN is either simple, or for a unique non-maximal prime ideal P, $eN = eP$ and eR/eN is a projective R/P-module with socle $(eR/eN) = 0$.*

(III) *For any non-maximal prime ideal P, $P^2 = P$.*

Proof. Since in any (hnp)-ring every non-zero prime ideal is maximal, the same holds in a prime serial noetherian ring. Using this and (1.1) (d) it follows that in any serial (both sided) noetherian ring, any two non-comparable prime ideals are comaximal. Now R/N is semiprime. So by (2.4), it is also left noetherian. Thus given any two non-comparable prime ideals P and Q of R, P/N and Q/N are non-comparable prime ideals of the serial noetherian ring R/N. So that P/N and Q/N are comaximal. Hence P and Q are comaximal. This proves (I).

By (1.1) (d), R/N is a finite direct sum of prime serial rings, each of which is either simple artinian, or non-artinian. Now eR/eN is isomorphic to an indecomposable summand of $(R/N)_R$. So for some unique prime ideal P, eR/eN is isomorphic to a summand of R/P. Consequently eR/eN is simple if R/P is artinian; notice that in this situation there is no prime ideal P' properly contained in P. If R/P is not artinian, then socle $(R/P) = 0$ gives socle $(eR/eN) = 0$. That $eN = eP$ for some unique prime ideal P is now immediate. This proves (II).

Let P be a non-maximal prime ideal of R. Let S be a simple R/P-module. As $\bar{R} = R/P$ is bounded, $E_{\bar{R}}(S)$ is not finitely generated. By [**10**, Theorem 2.8] it has an infinite properly ascending chain of submodules

$$0 = S_0 < S_1(=S) < S_2 < S_3 < \ldots\ldots$$

such that $E_{\bar{R}}(S) = \cup_i S_i$, each S_i/S_{i-1} is simple, and there exists n such that $S_i/S_{i-1} \approx S_j/S_{j-1}$ if and only if $i \equiv j \pmod n$. This immediately yields that S has a successor as well as a predecessor. Since simple modules over R/P^2 are the same as those over R/P, we get that every simple R/P^2-module admits a predecessor. Consequently by (1.1) (b)

$$\cap J^k(R/P^2) = 0.$$

So by (2.1) R/P^2 is noetherian. Since it is indecomposable and non-artinian it must be prime. Hence $P^2 = P$.

LEMMA 2.5. *Let U be a uniform module over a serial right noetherian ring R and P be a non-maximal prime ideal of R such that $UP = 0$. Then $E_R(U) = E_{\bar{R}}(U)$, where $\bar{R} = R/P$. Further $E_R(U)P = 0$.*

Proof. Since $\bar{R}$ is a bounded (hnp)-ring, and is not artinian, $E_{\bar{R}}(U)$ is not finitely generated. So by (2.2) it is an injective R-module. Hence $E_R(U) = E_{\bar{R}}(U)$. The last part is obvious.

LEMMA 2.6. *Let R be a serial right noetherian ring, and $N = N(R)$. Let e and f be any two indecomposable idempotents in R. Then*:

(i) *If eR/eN is not simple,*

$$\mathrm{Hom}_R\,(eR, fN) = 0 = fNe.$$

(ii) *If eR/eN is not simple and fR/fN is simple,*

$$\mathrm{Hom}\,(eR, fR) = 0 = fRe.$$

Proof. (i) Let $0 \neq \lambda: eR \rightarrow fN$ be an R-homomorphism. Let $A = \mathrm{Im}\,\lambda$. Then $AN \neq A$, and we get an epimorphism

$$\bar{\lambda}: eR/eN \rightarrow A/AN.$$

Consider $E = E_R(eR/eN)$. Then λ extends to an R-homomorphism

$$\mu: E \rightarrow E_R(A/AN).$$

If $P = \mathrm{ann}_R\,(eR/eN)$ (2.4) gives that P is a non-maximal prime ideal. By (2.5)

$$E_R(eR/eN) = E_{\bar{R}}(eR/eN),$$

where $\bar{R} = R/P$, and it is not finitely generated. So $EP = 0$, and by (2.2) we get that every homomorphic image of E is injective. Consequently μ is onto. As $(\mathrm{Im}\,\mu)P = 0$ and $fR/AN \subset E_R(A/AN)$, we get

$$(fR/AN)P = 0.$$

This in turn gives $fN = AN$. As $A \subset fN$, we get $fN = fN^2$ and hence $fN = 0$. This is a contradiction. Hence

$$\text{Hom}(eR, fN) = 0 = fNe.$$

(ii) Let $\lambda: eR \to fR$ be a non-zero R-homomorphism. Let $\text{Im}\,\lambda \not\subset fN$. Then $\text{Im}\,\lambda = fR$, as fR/fN is simple. Thus $eR \approx fR$. This is a contradiction. Consequently $\text{Im}\,\lambda \subset fN$. This contradicts part (i). Hence

$$\text{Hom}(eR, fR) = 0 = fRe.$$

THEOREM 2.7. *Let R be a serial right noetherian ring. Then*:

(i) *$N = N(R)$ has finite length as a right R-module, and $N^{k+1} = N^kJ$.*

(ii) *Let P be a non-maximal prime ideal of R, and e, f be two indecomposable idempotents of R such that eR/eN and fR/fN are projective R/P-modules. Then $eN \approx fN$, and eN is the largest finite length submodule of $E(eR)$. If $eN \neq 0$, eN/eN^2 has no predecessor.*

Proof. Let g be any indecomposable idempotent of R such that $gN^k \neq 0$, for some $k \geqq 1$. As gN^k is serial, there exists an indecomposable idempotent h together with an R-epimorphism

$$\lambda: hR \to gN^k.$$

By (2.6) hR/hN is simple. Thus λ induces an isomorphism

$$hR/hN \approx gN^k/gN^{k+1}.$$

Consequently gN^k/gN^{k+1} is simple. Since N is nilpotent, we get

$$gN > gN^2 > gN^3 > \ldots > gN^l = 0,$$

for some l, is a composition series of gN. Consequently N_R also has finite length, and $N^{k+1} = N^k.J$.

(ii) By hypothesis eR/eN and fR/fN are torsion-free R/P-modules. Thus socle $(eR/eN) = 0$. As $E(eR)$ is serial we get there exists no finite length submodule of $E(eR)$ containing eN properly, since otherwise socle $(eR/eN) \neq 0$. Hence eN is the largest, finite length submodule of $E(eR)$. Since any two finitely generated uniform, torsion free modules over a prime noetherian ring are embeddable in each other, eR/eN is embeddable in fR/fN. Consequently there exists an R-homomorphism

$$\lambda: eR \to fR/fN$$

with $\text{Ker}\,\lambda = eN$. The projectivity of eR gives an R-homomorphism

$\mu:eR \rightarrow fR$ such that $\lambda = \pi\mu$, where $\pi:fR \rightarrow fR/fN$ is the natural homomorphism. Then

$$\mu(eN) \subset fN < \mu(eR).$$

Since $\mu(eR)/fN$ is a torsion-free R/P-module, which is a homomorphic image of the uniform R/P-module $\mu(eR)/\mu(eN)$, we get $\mu(eR)/\mu(eN)$ is a torsion-free R/P-module. So its socle is zero. Consequently using the fact that fN has finite length we get $\mu(eN) = fN$. Consequently composition length $d(fN) \leqq d(eN)$. Similarly $d(eN) \leqq d(fN)$. Hence $d(eN) = d(fN)$. So it follows from $\mu(eN) = fN$ that $eN \approx fN$ under μ. Since socle (eR/eN) is zero, the last part is obvious.

LEMMA 2.8. *Let P and P' be two distinct non-maximal prime ideals in a serial right noetherian ring R. Let e and f be two indecomposable idempotents of R such that eR/eN and fR/fN are projective as R/P-module and R/P'-module respectively. Then $eRf = 0 = fRe$.*

Proof. Let $eRf \neq 0$. This gives a non-zero homomorphism $\lambda:fR \rightarrow eR$. By (2.6), $\operatorname{Im} \lambda \not\subset eN$. Further (2.7) gives $\lambda(fN) \subset eN$. Thus we get an R-epimorphism

$$\bar{\lambda}:fR/fN \rightarrow \operatorname{Im} \lambda/eN.$$

Consequently

$$[(\operatorname{Im} \lambda)/eN]P' = 0.$$

This yields $P' \subset P$, as $\operatorname{Im} \lambda/eN$ is a torsion free R/P-module. Hence $P' = P$, as P is not maximal. This is a contradiction. Hence $eRf = fRe = 0$.

Consider any non-maximal prime ideal P in a serial right noetherian ring R. Consider any indecomposable idempotent $e \in R$ with eR/eN a projective R/P-module. Let $eN \neq 0$. Consider a composition series

$$eN > eN^2 > \ldots > eN^{t+1} = 0$$

of eN. Consider the simple modules $S_i = eN^i/eN^{i+1}$ for $i \leqq t$. Because of (2.7) (ii), the finite sequence of simple modules $(S_1, S_2, \ldots, S_t)$ is uniquely determined by P. We call it the successor sequence of P. We extend it further to a sequence of simple modules

$$S_1, S_2, \ldots, S_t, S_{t+1}, \ldots$$

where each one is followed by its successor. Since by (2.7), S_1 has no predecessor, all the members of this sequence are distinct. However R admits only finitely many simple modules. So the above sequence is finite.

https://doi.org/10.4153/CJM-1984-003-5 Published online by Cambridge University Press

Thus we get a finite sequence

$$(S_1, S_2, \dots, S_t, \dots, S_u)$$

of simple modules, which extends the successor sequence of P, in which each S_i is followed by its successor and S_u has no successor. This sequence is called the *extended successor sequence of* P, and is uniquely determined by P. We understand that if $eN = 0$, the above sequence is an empty sequence.

LEMMA 2.9. *Let P and P' be two distinct non-maximal prime ideals in a serial right noetherian ring R. Then the extended successor sequences of P and P' are disjoint.*

Proof. Let $(S_1, S_2, \dots, S_m)$, $(T_1, T_2, \dots, T_n)$ be the extended successor sequences of P and P' respectively. Suppose for some i, j that $S_i = T_j$. Let i be smallest. If $i > 1$, then S_{i-1} is the predecessor of S_i. So T_j has S_{i-1} as its predecessor. As T_1 has no predecessor, we get $j > 1$ and $T_{j-1} = S_{i-1}$. This is a contradiction to the choice of i. So $i = 1$. Thus as S_1 has no predecessor we get $j = 1$. Consequently the two sequences are the same. In the notation of (2.8), we have $eN/eN^2 \approx fN/fN^2$. This gives an R-isomorphism

$$\lambda: E(eR/eN^2) \rightarrow E(fR/fN^2).$$

Using (2.7) (ii) and (2.5) we get $E(eR/eN^2)/eN/eN^2$ is a torsion-free R/P-module and is isomorphic to $E(fR/fN^2)/fN/fN^2$. The latter is a torsion-free R/P'-module. This gives $P = P'$ and we get a contradiction. Hence the result follows.

Since the structure of a prime serial right noetherian (hence noetherian) ring is known, we are interested to study non-prime, non-artinian serial rings which are right noetherian, but not left noetherian. It is enough to study such indecomposable rings.

THEOREM 2.10. *Let R be an indecomposable serial right noetherian ring, which is not left noetherian. Then*:

(i) *R has only one non-maximal prime ideal P.*

(ii) *The successor sequence of P is non-empty.*

(iii) *There exists a unique simple projective R-module.*

Proof. Since R is not artinian it has a non-maximal prime ideal P. If $P = 0$, by (2.3) R is left noetherian; this is a contradiction. Hence $P \neq 0$. Write

$$R = e_1R \oplus e_2R \oplus \dots \oplus e_nR$$

for some orthogonal indecomposable idempotents e_i. Let A be the sum of those e_iR for which either e_iR/e_iN is a projective R/P-module or e_iR/e_iN is a simple module occurring in the extended successor sequence of P. Let B be the sum of other e_jR's. Clearly $A \neq 0$. Let $B \neq 0$. Notice that if for a finite length serial module K_R, some composition factor of K_R is in the extended successor sequence of P, then all the composition factors of K are in the sequence. Consider any $e_i \in A$, $e_j \in B$. We want to show that $e_iRe_j = 0 = e_jRe_i$. Let $e_iRe_j \neq 0$. If e_iR/e_iN is simple, then every composition factor of e_iR is in the extended successor sequence of P and in particular e_jR/e_jN is in the extended successor sequence of P; this is a contradiction. So e_iR/e_iN is not simple and

$$\text{ann}\,(e_iR/e_iN) = P.$$

Then (2.8) gives that either $e_iRe_j \subset e_iN$ or, e_iR/e_iN and e_jR/e_jN both are projective R/P-modules; this again leads to a contradiction. Hence $e_iRe_j = 0$. Let $e_jRe_i \neq 0$. If e_iR/e_iN is not simple, (2.6) and (2.8) give that e_jR/e_jN is not simple and that

$$\text{ann}\,(e_jR/e_jN) = P.$$

This is a contradiction. Thus e_iR/e_iN is simple. Now using (2.9) it follows that $e_jRe_i = 0$. Hence A and B are ideals of R and $R = A \oplus B$. This is a contradiction. Hence $B = 0$. The construction of A and the fact that $B = 0$, shows that there is no non-maximal prime ideal in R other than P. Let the successor sequence of P be empty. Then each e_iR/e_iN is a projective R/P-module and $e_iN = e_iP = 0$. This gives $P = 0$; which is a contradiction. Hence the successor sequence of P is non-empty. So let $(S_1, S_2, \dots, S_u)$ be the extended successor sequence of P; which is non-empty. For some i,

$$S_u \approx e_iR/e_iN.$$

As S_u has no successor, [**12**, Lemma (5.3)] yields $e_iN = 0$ and S_u becomes projective. This S_u is unique to within isomorphism. This proves the theorem.

Henceforth let R be an indecomposable serial right noetherian ring, which is not left noetherian, and let P be its unique non-maximal prime ideal. As seen in the proof of the above theorem, we can write

$$R = (e_1R \oplus e_2R \oplus \dots \oplus e_sR) \oplus (f_1R \oplus f_2R \oplus \dots \oplus f_tR)$$

for some orthogonal indecomposable idempotents e_i, f_j such that e_iR/e_iN is a projective right R/P-module, and each f_jR/f_jN is a simple module occurring in the extended successor sequence of P. Let $e = \sum e_i, f = \sum f_j$.

(2.6) gives $fRe = 0$. Further (2.4) yields $eN = eP$, and by (2.6) $eRf = eNf = ePf = eN$. Notice that $fP = fR$. As each f_jR is of finite length, fRf is a serial artinian ring. Also $R/P \approx eRe/ePe$, being a right noetherian prime serial ring, is also left noetherian. However by (2.6) $ePe = eNe = 0$. Hence eRe is a prime serial noetherian ring. Thus we can write

$$R = \begin{bmatrix} eRe & eRf \\ 0 & fRf \end{bmatrix}$$

where eRe is a prime, serial noetherian ring, fRf is a serial, artinian ring. We are now ready to state and prove the main structure theorem.

THEOREM 2.11. *Let R be an indecomposable, non-prime, non-artinian ring. Then R is a serial right noetherian ring if and only if* $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ *such that*

(a) *S is a prime, serial noetherian ring, which is not artinian, and T is an indecomposable artinian serial ring admitting a simple projective module.*

(b) *M is an (S, T)-bimodule such that ${}_SM$ is a divisible torsion free module with rank* $({}_SM) =$ rank (T/B), $B =$ ann_T (M) *and ${}_TB$ is a summand of ${}_TT$.*

(c) *M_T is a direct sum of finitely many isomorphic serial modules and* rank $(M_T) =$ rank (S_S).

Proof. Let R be a serial right noetherian ring. We have shown just before this theorem that

$$R = \begin{bmatrix} eRe & eRf \\ 0 & fRf \end{bmatrix}.$$

Here eRe is a prime serial noetherian ring, fRf is an artinian serial ring. Write $S = eRe$, $M = eRf$, $T = fRf$. We can write

$$f = f_1 + f_2 + \ldots + f_t$$

for some orthogonal indecomposable idempotents such that $f_iR/f_iN (1 \leqq i \leqq u)$ constitute the set of all members of the successor sequence of P, and for some v, $u \leqq v \leqq t$, $f_iR/f_iN (1 \leqq i \leqq v)$ constitute the set of all members of the extended successor sequence of P, and $f_{i+1}R/f_{i+1}N$ is the successor of f_iR/f_iN for $i < v$. As $fRe = 0$, each $S_i = f_iR/f_iN$ is a simple fRf-module, and we have

$$\mathrm{Ext}_T (S_i, S_{i+1}) \neq 0 \quad \text{for } i < v.$$

By construction, $eRf_i \neq 0$ for $1 \leqq i \leqq u$, $eRf_j = 0$ for $u < j \leqq v$ and f_1R, $f_2R, \ldots, f_vR$ is a maximal set of non-isomorphic summands of R_R among $f_iR (1 \leqq i \leqq t)$. Thus

$$\text{Ext}_T(S_i, S_{i+1}) \neq 0 \quad \text{for } i < v,$$

proves that $T = fRf$ is indecomposable. Since S_v has no successor, $f_vN = 0$ and hence S_v is also a projective T-module. Since Rf is a finite direct sum of serial left R-modules, eRf is a finite direct sum of serial left eRe-modules. Since any uniform left S-module (here $S = eRe$) is either injective or finitely generated we get

$$eRf = M = A \oplus L$$

where L is a divisible left S-module, and ${}_SA$ is a finitely generated left S-module. Clearly L is an ideal of R. Since R/M is left noetherian and $M/L \approx A$ is left noetherian, we get R/L is left noetherian. Let $A \neq 0$, then R/L is representable as $\begin{bmatrix} S & M/L \\ 0 & T \end{bmatrix}$ with S, T indecomposable rings and M/L a non-zero (S, T)-bimodule. This gives R/L is an indecomposable noetherian serial ring, which is neither prime nor artinian. This is a contradiction. Hence $A = 0$ and $M = L$. We now show that ${}_SM$ is torsion free. Take any indecomposable torsion injective module E over a bounded (hnp)-ring R'. We know that E is serial and its proper submodules are of finite lengths (see [**10**, Theorem 2.8]). But E itself is not of finite length. So it gives that each proper R'-submodule of E is an $\text{End}_{R'}(E)$-submodule and E as an $\text{End}_{R'}(E)$-module is not of finite length. Consequently any injective R'-module F with its torsion submodule non-zero, is not of finite length as $\text{End}_{R'}(F)$-module. Since N_R is of finite length by (2.8), $M = eNf$ is of finite length as a T-module, since $T = fRf$. Let $B = \text{ann}_T(M)$. As T/B is embeddable in $\text{End}_S(M)$, we get that M is of finite length over $\text{End}_S(M)$. So by what we have shown above, ${}_SM$ is torsion free. Now

$$B = \oplus \sum_{i=1}^{t} Bf_i.$$

Let for some f_i, $Bf_i \neq 0$ and also $Mf_i \neq 0$. Choose $u \neq 0$ in Bf_i. Then

$$R\begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Tu \end{bmatrix} \text{ and } \begin{bmatrix} 0 & Mf_i \\ 0 & 0 \end{bmatrix}$$

are non-comparable left ideals, contained in Rf_i; this is a contradiction. Hence $Bf_i \neq 0$ implies $Bf_i = Tf_i$. Consequently ${}_TB$ is a summand of ${}_TT$. Thus $Mf_i \neq 0$ if and only if $B \cap Tf_i = 0$. Consequently $M = \oplus \sum Mf_i$ gives that the number of non-zero Mf_i is the same as the rank of T/B. Hence

$$\text{rank}({}_SM) = \text{rank}(T/B).$$

This proves (b). Since $e_iN \approx e_jN$ for all i, j by (2.7), (c) also follows.

We now outline the proof of the converse. Let $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ satisfy the given conditions. Let

$$S = e_1S \oplus \ldots \oplus e_nS$$

for some orthogonal indecomposable idempotents. Then

$$M_T = \oplus \sum_{I=1}^{n} e_iM.$$

As rank $(M_T) = n$, each e_iM is a finite length serial T-module. As S is bounded, for any $x \neq 0$ in e_iS there exists a non-zero ideal A of S such that $e_iA \subset xS$. The divisibility of ${}_SM$ yields $AM = M$. Consequently

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} R = \begin{bmatrix} xS & e_iM \\ 0 & 0 \end{bmatrix}.$$

This in turn yields that $\begin{bmatrix} e_i & 0 \\ 0 & 0 \end{bmatrix} R$ is a serial module. Further also T is a serial ring. We get R is right serial. Now $B = \text{ann}_T(M)$ and by hypothesis $T = B \oplus A$ for some left ideal A of T. So we can write

$$T = Tg_1 \oplus \ldots \oplus Tg_l \oplus Tg_{l+1} \oplus \ldots \oplus Tg_t$$

for some orthogonal indecomposable idempotents g_i's such that

$$B = \oplus \sum_{i=1}^{l} Tg_i.$$

By hypothesis

$$\text{rank } ({}_SM) = \text{rank } (T/B) = t - l.$$

Thus

$${}_SM = \oplus \sum_{i=l+1}^{t} Mg_i,$$

with each Mg_i a serial injective torsion free left S-module. Consequently for any $i > l$, and any $xg_i \neq 0$ in Tg_i, $Mxg_i = Mg_i$; using this we get $R \begin{bmatrix} 0 & 0 \\ 0 & g_i \end{bmatrix}$ is serial. For $i \leqq l$, as $Mg_i = 0$,

$$R\begin{bmatrix} 0 & 0 \\ 0 & g_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Tg_i \end{bmatrix};$$

which is a serial left R-module, as Tg_i is a serial left T-module. Further each

$$R\begin{bmatrix} e_i & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Se_i & 0 \\ 0 & 0 \end{bmatrix}$$

is a serial left R-module. This all shows that R is left serial. Hence R is serial. This proves the theorem.

3. Proper homomorphic images serial. Singh [**11**] proved the following:

THEOREM 3.1. *Let R be a prime right bounded, right Goldie ring such that for each non-zero ideal A of R, R/A is an artinian serial ring. Then R is right hereditary.*

Recently Levy and Smith [**4**] have proved the following generalization of the above theorem.

THEOREM 3.2. *Let R be a right noetherian, essentially right bounded semi-prime ring, all of whose homomorphic images are serial rings, then R is right hereditary.*

In this section we improve on the above theorem, and give an alternative proof. First of all we prove the following:

THEOREM 3.3. *Let R be a right noetherian semiprime ring (not necessarily essentially right bounded), all of whose proper homomorphic images are serial rings. Then either R is a serial noetherian ring or a prime ring.*

Proof. Let R be not a prime ring. Now

$$0 = \bigcap_{i=1}^{t} P_i \quad \text{for some primes } P_i.$$

Clearly $t \geqq 2$. Take t to be minimal. It is clear from (1.1) (a) that in a serial, noetherian ring any two non-comparable prime ideals are comaximal. So if $t \geqq 3$, $P_i \cap P_j \neq 0$ gives that $R/P_i \cap P_j$ is a serial noetherian ring. Thus $P_i + P_j = R$ for $i \neq j$. Consequently $R \approx \oplus \sum R/P_i$, a finite direct sum of prime rings. As each R/P_i is serial, we get R is serial, and hence noetherian by (2.3).

Let $t = 2$. Then $P_1 \cap P_2 = 0$ gives that P_1 and P_2 are the minimal prime ideals of R. Further each is the annihilator of the other. Write $P = P_1$, $Q = P_2$. Then $P \oplus Q \subset' R_R$ and P_R is a complement of Q_R etc. We claim $P + Q = R$. On the contrary let $P + Q \neq R$. Now

$$(P + Q)/Q \subset' (R/Q)_R.$$

https://doi.org/10.4153/CJM-1984-003-5 Published online by Cambridge University Press

Take any $E \subset' R_R$, then $E \cap P \subset' P_R$ yields

$$[(E \cap P) + Q]/Q \subset' R/Q.$$

As R/Q is a bounded ring, there exists a non-zero ideal A/Q of R/Q contained in $[(E \cap P) + Q]/Q$. Then $A = B \oplus Q$, where $B = E \cap A = P \cap A$ is a non-zero ideal of R. Similarly there exists a non-zero ideal C of R contained in $E \cap Q$. Then $B + C$ is an ideal of R contained in E and this ideal is an essential right ideal of R. Now $P + Q \neq R$ gives that $P + Q \subset M$, some maximal right ideal of R. Then $S = R/M$ is a simple module such that $SP = 0 = SQ$. So $E_{R/P}(S)$ is not finitely generated and is contained in $E_R(S)$. Consider any $x\ (\neq 0) \in E(S)$. There exists $K \subset' R_R$ such that $xK \subset S$. As proved above we can find an ideal $A \subset K$ such that $A \subset' R_R$. Thus $AP \neq 0$, and xR is an R/AP-module. As R/AP is a serial right noetherian ring, we get xR is serial by (1.1) (c). Since P/AP is a non-maximal prime ideal, by (2.5) $E_{\bar{R}}(xR)P = 0$, where $\bar{R} = R/AP$. This gives $E(S)P = 0$. Similarly $E(S)Q = 0$. However $P + Q$ contains a regular element of R. So

$$E(S)(P + Q) = E(S).$$

This leads to a contradiction. Hence $R = P \oplus Q$. Thus again R is a direct sum of prime rings and is serial.

THEOREM 3.4. *Let R be a prime, right bounded, right noetherian ring such that for each ideal $A \neq 0$, R/A is serial. Then for each ideal $A \neq 0$, R/A is artinian.*

Proof. Let P be a non-zero prime ideal of R which is not maximal. Let E_P be an indecomposable summand of $E(R/P)$. Then E_P is a torsion uniform right R-module. Consider any finitely generated submodule U of E. As R is right bounded there exists a non-zero ideal A of R such that $UA = 0$. Clearly $A \subset P$ and U is an R/A-module. Since R/A is serial, by (2.5) $E_{R/A}(U) = E_{R/P}(U)$. Hence $UP = 0$. This gives $E_P.\ P = 0$. This is a contradiction, since E_P is a faithful R-module. Hence R has no non-zero, non-maximal prime ideal. So given any ideal $B \neq 0$ of R, every prime ideal of R/B is maximal. As R is a right FBN-ring, we get R/B is artinian. This proves the theorem.

Combining (3.1), (3.3) and (3.4) we get the following:

THEOREM 3.5. *Let R be a semiprime right noetherian ring such that for each ideal $A \neq 0$, R/A is serial. Then R is a finite direct sum of prime rings. If R is not prime, then R is serial. If R is prime and right bounded then R is right hereditary.*

4. Artinian serial rings. Consider the following two conditions on a Module M_R:

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of finite length serial modules.

(II) Given two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U any homomorphism $f:W \to V$ can be extended to a homomorphism $g:U \to V$ provided the composition length $d(U/W) \leqq d(V/f(W))$.

The study of modules satisfying (I) and (II) was initiated by Singh [**11**]. Any module over an artinian serial satisfies these conditions. Here we prove the following:

THEOREM 4.1. *If a ring R is such that R_R satisfies* (I) *and* (II), *then R is an artinian serial ring.*

We prove this result through various lemmas. Throughout all the lemmas R_R satisfies (I) and (II). Without any loss of generality we take R to be indecomposable. The following is immediate from the given conditions:

LEMMA 4.2. (i) *R is a right artinian right serial ring.*
(ii) *any uniform cyclic* (*right*) *R-module is serial and quasi-injective.*
(iii) *Any simple R-module admits at most one successor.*

LEMMA 4.3. (i) *Any uniform injective R-module is serial.*
(ii) *Any simple R-module admits at most one predecessor.*

Proof. Consider a simple module S_R and $E = E_R(S)$. Since R is right artinian $E = \mathrm{soc}^n(E)$ for some n. By induction we show that $\mathrm{soc}^k(E)$ is serial. Clearly $\mathrm{soc}^1(E) = S$ is serial. To apply induction let $k > 1$ such that $\mathrm{soc}^{k-1}(E)$ is serial and $E \neq \mathrm{soc}^{k-1}(E)$. Let A and B be two submodules of E of length k each. Then

$$\mathrm{soc}^{k-1}(E) \subset A \cap B.$$

There exist indecomposable idempotents e and f in R such that $A \approx eR/eN^k$, $B \approx fR/fN^k$. If $eR \approx fR$, $A \approx B$. Let $eR \not\approx fR$. Then e and f can be chosen to be orthogonal. Then $A \times B$ is embeddable in

$$eR/eN^k \oplus fR/fN^k \subset R/N^k.$$

So by condition (II) the identity map of $\mathrm{soc}^{k-1}(E)$ can be extended to an isomorphism of A onto B. Thus in any case there exists an isomorphism σ of A onto B. As $A + B \subset E(S)$ and by (4.2) A is quasi-injective, $\sigma(A) \subset$

A. Hence $A = B$. This proves that $\mathrm{soc}^k (E)$ is serial. Hence E is serial. Now (ii) is obvious.

Let $S_1, S_2, \dots, S_t$ be a maximal length sequence of non-isomorphic simple R-modules such that each S_{i+1} is the successor of S_i. We can find orthogonal indecomposable idempotents $e_1, e_2, \dots, e_t$ in R such that

$$S_i \approx e_iR/e_iN.$$

Then

$$S_{i+1} \approx e_iN/e_iN^2.$$

Let $S \approx eR/eN$, for some indecomposable idempotent e be such that $S \neq S_i$ for any i. It is clear from (4.2) and (4.3) that if a composition factor of a serial R-module K is among $S_i's$ then every composition factor of K is among $S_i's$. Thus every composition factor of e_iR is among $S_i's$. As $S \neq S_i$, it gives $eRe_i = 0 = e_iRe$. This in turn shows that R is decomposable. This is a contradiction. Hence $e_1R, e_2R, \dots, e_tR$ constitute a maximal set of non-isomorphic serial summands of R_R.

LEMMA 4.4. *If* $e_tN^2 \neq e_tN$, *then* R *is serial.*

Proof. $e_tN^2 \neq e_tN$ implies that R/N^2 is a direct sum of serial right modules each of length 2. In view of (4.3), each of these serial R/N^2-modules is injective. Consequently R/N^2 is quasi-Frobenius. Thus as R/N^2 is right serial, the duality between the right ideals and left ideals of a quasi-Frobenius ring gives R/N^2 is also left serial. Hence by [**5**, Theorem 10] R is serial.

Proof of (4.1). In view of (4.4) we take $e_tN = 0$. So that s_t is a simple projective R-module. Further in view of [**5**, Theorem 10] we take $N^2 = 0$. Let T be the basic ring of R. Then T also satisfies (I) and (II). Further R is serial if and only if T is serial. Thus without loss of generality we can take $R = T$. In that case

$$R = e_1R \oplus e_2R \oplus \dots \oplus e_tR.$$

Each e_iR $(i < t)$ being of length 2 is injective, and e_tR is simple. Every e_iRe_i is a division ring and e_iRe_{i+1} is a one-dimension right $e_{i+1}Re_{i+1}$-vector space. Using the fact that for $i < t$, e_iR is injective and that $e_iRe_i = \mathrm{End}_R (e_iR)$ we get e_iRe_{i+1} is a one-dimensional left e_iRe_i-vector space; hence $Ne_{i+1} = e_iRe_{i+1}$ is a simple left R-module. So each of $Re_2, \dots, Re_t$ is serial. As $e_jRe_1 = 0$ for $j \neq 1$, gives $Ne_1 = 0$. Consequently Re_1 is simple, and R is left serial. This proves the theorem.

Acknowledgment. A part of this work was done while the author was visiting McMaster University, in August 1981, on the invitation of Professor B. J. Muller. I express my sincerest thanks to Professor Muller for his many suggestions, and also to Kuwait University for awarding me research grant No. SM15. I thank the referee for pointing out some misprints and other suggestions.

References

1. D. Eisenbud and J. C. Robson, *Hereditary noetherian prime rings*, J. Algebra *16* (1970), 86-104.

2. A. V. Jategaonka, *Left principal ideal rings*, Springer-Verlag *123* (1970).

3. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. Lon. Math. Soc. *36* (1961), 260-268.

4. L. S. Levy and P. Smith, *Semi-prime rings whose homomorphic images are serial*, Can. J. Math. *34* (1982), 691-695.

5. I. Murase, *On the structure of generalized uniserial rings I*, Sci. Papers, College Gen. Ed., Univ. Tokyo *13* (1963), 1-22.

6. ——— *On the structure of generalized uniserial rings II*, Sci. Papers, College Gen. Ed., Univ. Tokyo *13* (1963), 131-158.

7. ——— *On the structure of generalized uniserial rings III*, Sci. Papers College Gen. Ed., Univ. Tokyo *14* (1964), 11-25.

8. T. Nakayama, *On Frobeniusean algebras II*, Ann. Math. *42* (1941), 1-21.

9. S. Singh, *Quasi-injective and quasi-projective modules over hereditary noetherian prime rings*, Can. J. Math. *26* (1974), 1173-1185.

10. ——— *Modules over hereditary noetherian prime rings II*, Can. J. Math. *28* (1976), 73-82.

11. ——— *Some decomposition theorems in abelian groups and their generalizations*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker *25* (1976), 183-189.

12. R. B. Warfield Jr., *Serial rings and finitely presented modules*, J. Algebra *37* (1975), 187-222.

Kuwait University,
Kuwait