IDEALS IN A RING OF EXPONENTIAL POLYNOMIALS

Dedicated to the memory of Hanna Neumann

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An exponential polynomial is a finite linear combination of terms $u_n e_a : t \to t^n e^{at}$ where n is any non-negative integer and a is any complex number. The set X of exponential polynomials is clearly a vector space over the field of complex numbers $\mathbb{C}$ and this set is identical with the set of solutions to all homogeneous linear ordinary differential equations with constant coefficients.

With the truncated convolution product of two functions x and y defined by

$$x * y : t \to \int_0^t x(t-r)y(r)\,dr,$$

$X(+,*)$ is a commutative ring and an algebra over $\mathbb{C}$. An ideal I of X is a subspace of X that contains $x * y$ whenever y belongs to I and x is an exponential polynomial.

As usual $D^k x$ denotes the kth derivative of x so that $D^k x$ is an exponential polynomial when x is one. We define the *degree* of an exponential polynomial x to be zero if $x(0) \neq 0$ and n if $D^k x(0) = 0$ for $k = 0, 1, 2, \cdots, n-1$ but $D^n x(0) \neq 0$. Also we define the set Y_n as

$$Y_n = \{y \in X : D^k y(0) = 0 \qquad \text{for } k = 0, 1, \cdots, n\}.$$

It is clear that for a fixed non-negative integer n, Y_n is a subspace of X and Y_n consists of all exponential polynomials of degree greater than n. Moreover, each Y_n is an ideal and

$$\ldots \subset Y_n \subset Y_{n-1} \subset \ldots \subset Y_1 \subset Y_0 \subset X.$$

Our main result (Theorem 6) is that any non-trivial proper ideal I is equal to Y_n for some non-negative integer n. Thus, the set of all ideals of X forms a single descending chain.

To show this, we will employ the space $C(R)$ of all continuous complex-valued functions defined on the real line R and taken with the topology of convergence uniform on all compact subsets of R. Also, we will be concerned with the two kinds of integral equations of convolution type.

As well, it is possible to construct from the ring X, a field of convolution quotients in a manner innovated by Mikusinski (see, for example, Erdélyi, [1]).

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Under the operations of addition and truncated convolution, $C(R)$ is a commutative ring and an algebra over $\mathbb{C}$ (see Erdélyi, [1], page 15 for details of $C[0, \infty)$ that apply to $C(R)$). Other elementary properties that Erdélyi shows (loc. cit., pages 43 and 45) for $C[0, \infty)$ that also hold for $C(R)$ are:

a) If $\{f_n\} \in C(R)$ and if $f_n \to f$ uniformly on all compact subsets of R as $n \to \infty$ (hereafter referred to as locally uniform convergence), then $f \in C(R)$. Moreover, if $g \in C(R)$, then $f_n * g \to f * g$ locally uniformly as $n \to \infty$.

b) If $f \in C(R)$ and if f^{*n} is defined by $f^{*1} = f$ and $f^{*n} = f^{*(n-1)} * f$ for $n = 2, 3, \cdots$, then $f^{*n} \to 0$ locally uniformly as $n \to \infty$.

It follows from a) and b) that if $f, g \in C(R)$ and if $f * g = f$, then $\mathrm{f} = 0$, since $f = f * g^{*n}$ for any positive integer n. Thus $C(R)$ has no idempotents or identity.

LEMMA 1. *Let $f, g \in C(R)$. Then the integral equation*

$$x - x * f = g$$

has a unique continuous solution.

PROOF. A continuous solution to this equation is the limit in $C(R)$ of the Cauchy sequence $(g + f * g + \cdots + f^{*n} * g)$. If y is the difference between any two contions, then $y = y * f$ and so $y = 0$.

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The remaining propositions will be stated in terms of exponential polynomials. By elementary calculus, it may be shown that

$$u_m e_a * u_n e_a = A_{m,n} u_{m+n+1} e_a$$

and

$$u_m e_a * u_n e_b = P_m e_a + Q_n e_b \qquad (a \neq b)$$

where $A_{m,n}$ is a complex number and P_m and Q_n are polynomials of degree m and n. Hence the truncated convolution product of two exponential polynomials is an exponential polynomial, and so, X is a commutative ring and an algebra over $\mathbb{C}$.

LEMMA 2. *X has no non-zero divisors of zero.*

PROOF. Suppose that $x, y \in X$ and $x * y = 0$. Suppose also that $y \neq 0$. With

$$0 = D(x * y) = x(0)y + (Dx) * y$$

and $y \neq 0$, it follows that $x(0) = 0$. Inductively, $D^n x(0)$ is zero for $n = 0, 1, 2, \cdots$ and since any exponential polynomial is an entire function, $x = 0$. Hence X has no non-zero divisors of zero.

PROPOSITION 3. *Let f, g be exponential polynomials. Then the integral equation*

$$x - x * f = g$$

has a unique exponential polynomial solution.

PROOF. By Lemma 1, this equation has an unique continuous solution. With f being continuously differentiable, $x * f$ along with g is continuously differentiable. Thus x is continuously differentiable. By induction, and as f and g are indefinitely differentiable, it follows that x is indefinitely differentiable.

Since f is an exponential polynomial, there is a linear differential operator, $L(D)$, with constant coefficients, such that $L(D)f = 0$. Using

$$D^k(f * x) = f(0)D^{k-1}x + \cdots D^{k-1}f(0)x + (D^k f) * x,$$

we see that

$$L(D)(f * x) = M(D)x$$

and so

$$(L(D) - M(D))x = L(D)g$$

where $M(D)$ is a linear differential operator with order less than that of $L(D)$. Since $L(D) \neq M(D)$ and $L(D)g \in X$, x is an exponential polynomial.

PROPOSITION 4. *Let f, g be exponential polynomials where f is not identically zero. Then the integral equation $x * f = g$ has an unique exponential polynomial as a solution if, and only if, the degree of g exceeds the degree of f.*

PROOF. Let the degree of f be n. Since f is a non-identically zero entire function, n is finite. If $n = 0$ and the degree of g is positive, then $f(0) \neq 0$ and $g(0) = 0$. By Proposition 3, there exists an $x \in X$ such that

$$f(0)x + x * Df = Dg .$$

Using $e * Df = f - f(0)e$ where $e : t \to 1$, we obtain

$$f(0)x * e + x * (e * Df) = e * Dg \quad \text{or} \quad x * f = g .$$

Now if $n > 0$ and the degree of g exceeds n, $D^k f(0) = 0$ for $k = 0, 1, \cdots, n-1$, $a = D^n f(0) \neq 0$ and $D^n g(0) = 0$. Again, by Proposition 3, there exists an $x \in X$ such that

$$ax + x * D^{n+1}f = D^{n+1}g,$$

and so,

$$ax * e + x * (D^n f - ae) = D^n g \quad \text{or} \quad x * D^n f = D^n g.$$

On repeated integration, with

$$D^k f(0) = 0 = D^k g(0) \quad \text{for} \quad k = 0, 1, \cdots, n-1,$$

we obtain $x * f = g$.

Conversely, suppose that m is the degree of g and that m does not exceed the degree of f. Suppose also that there is exponential polynomial x for which $x * f = g$.

If m is zero so that $g(0)$ is non-zero, there is a contradiction of $x * f(0) = 0$. If m is positive, the relation $x * f = g$ leads, after differentiating m times, to a similar contradiction. Hence it is necessary that the degree of g exceeds the degree of f for $x * f = g$ to have a solution in X.

The uniqueness of any exponential polynomial solution follows from Lemma 2.

REMARKS. Since the set $X(+, *)$ is a commutative ring with no non-zero divisors of zero, it is possible to construct a field F_X of convolution quotients of exponential polynomials in the same manner that Mikusinski constructed the field of convolution quotients F from the ring $C[0, \infty)$. One difference between the two fields is that F_X is not complete whereas F is complete.

The equation $x * f = g$ where $f, g \in X$ and $f \neq 0$ always admits a solution in F_X. Moreover, if

$$p = \text{degree} f - \text{degree}\, g + 1$$

is a positive integer, by the above Proposition, there is an exponential polynomial y satisfying $y * f = g * e^{*p}$ where $e : t \to 1$. Thus if s is the inverse of e in F_X, then $x = s^{*p} * y$ satisfies $x * f = g$ and so this equation has a pth extended derivative (Erdélyi, [1], page 29) of an exponential polynomial as a solution.

PROPOSITION 5. *Let f be an exponential polynomial of degree n and*

$$I_f = \{h = f * g : g \in X\}.$$

Then $I_f = Y_n$.

PROOF. To show that $I_f \subset Y_n$, one may make use of the formula

$$D^k(f * g)(0) = \sum_{l=0}^{k} D^{k-l}f(0) \cdot D^{l-1}g(0).$$

The reverse inclusion follows from Proposition 4.

THEOREM 6. *Let I be any non-trivial proper ideal of X. Then $I = Y_n$, where*

$$Y_n = \{y \in X : D^k y(0) = 0 \quad \text{for} \quad k = 0, 1, 2, \cdots, n\}$$

for some non-negative integer n.

PROOF. It is trivial that each Y_n is an ideal of X. If J is any ideal of X that contains an element x for which $x(0) \neq 0$ and $z(0) = bx(0)$ where z is any element of X, then, by Proposition 3, the equation $Dz - bDx = (Dx) * y + x(0)y$ has a solution y in X. On integration, this equation yields $z = bx + x * y$ and so $z \in J$. Hence $J = X$.

Let I be any non-trivial proper ideal of X and x be any element of I so that $x(0) = 0$. Thus $x = e * Dx$ and so $x \in I_{Dx}$.

Now let $h \in I_{Dx}$ with $h = g * Dx$ where $g \in X$. With $x(0) = 0$,

$$h = g * Dx = D(g * x) = g(0)x + (Dg) * x$$

and so $h \in I$. Thus $x \in I_{Dx} \subset I$ for all $x \in I$.

By Proposition 5, $I_{Dx} = Y_{n(x)}$ for some non-negative integer $n(x)$. Hence $I = \bigcup_{x \in I} Y_{n(x)}$ and as

$$\cdots \subset Y_m \subset Y_{m-1} \cdots \subset Y_1 \subset Y_0,$$

we see that $I = Y_n$ for some non-negative integer n.

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Other aspects of the ring structure of X may be of interest. With the observations that $Y_n = I_{u_n}$ where $u_n : t \to t^n$ and that the trivial ideal is none other than I_0, we see that any proper ideal is of the form $I_f = \{g * f : g \in X\}$ for some $f \in X$.

It is clear that no non-trivial proper ideal of X is prime since for any f, I_f contains $f * f$ but not f.

However, if I is a non-trivial proper ideal of X so that $I = Y_n$ for some non-negative integer n, it can be shown that $x^{*(n+2)}$ belongs to Y_n for all $x \in X$. Adopting the definition of the radical of an ideal given by Jacobson ([2], page 173) as the set

$$\{x \in X : x^{*m} \in I \quad \text{for some positive integer } m\}$$

we see that the radical of any non-trivial ideal is X, so all ideals of X are primary.

The ring X may be embedded in a ring U consisting of ordered pairs (a, x) where $a \in \mathbb{C}$ and $x \in X$ and where addition and multiplication are defined in the usual manner (Jacobson, [2], page 85) for embedding a ring without identity into a ring with identity. U may easily be shown to be an integral domain and to have all ideals forming a single descending chain $\cdots \subset (0, Y_{n+1}) \subset \cdots \subset (0, Y_0) \subset (0, X) \subset U$.

Moreover, U may be shown to be an Euclidean Domain (loc. cit., page 122) by defining $\delta(a,x)$ to be 1, if $a \neq 0$, $2^{(1+\deg x)}$ when $a = 0$ and $x \neq 0$ and zero otherwise. These claims may be verified by use of the results in this note and the observation that for $x, y \in X$, degree $(x * y) =$ degree $x +$ degree $y + 1$.

Note added in proof. These results may be v'ewed n two other ways that have been kindly suggested by Professor H. K. Farahat of the University of Calgary and Professor J.-P. Kahane of the University of South Paris respectively.

The first is that if $d(p)$ denotes the ordinary degree of a polynomial p (that differs from our definition of the degree of an exponential polynomial), then

$$T = \{p/q : p, q \text{ are polynom als}, \ q \neq 0, \ d(p) \leqq d(q)\}$$

with the operations of addition and multiplication is a ring. It may be shown that

$$S = \{p/q \in T : \ d(p) < d(q)\}$$

is a unique maximal ideal of T. T is also a Euclidean Domain and the ideals of T and S form single descending chains.

The Laplace transform is an isomorphism between our ring $X(+,*)$ and S whose elements are also rational functions. Also U is isomorphic to T.

The second is that if f is an exponential polynomial, it is entire. For $f(t) = \sum_{k=0}^{\infty} a_k t^k/k!$, set $Jf(\xi) = \sum_{k=0}^{\infty} a_k \xi^{k+1}$. Then J is an isomorphism between $X(+,*)$ and a subring of the familiar ring of formal power series in one indeterminate.

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[2] N. Jacobson, *Lectures in Abstract Algebra*, (Van Nostrand, 1951).

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