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A LOWER BOUND FOR THE VOLUME OF HYPERBOLIC 3-MANIFOLDS

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1. Introduction. The motivation for this paper was the work of Thurston and Jørgensen on volumes of hyperbolic 3-manifolds. They prove, among other things, that the set of all volumes of complete hyperbolic 3-manifolds is well-ordered. In particular, there is a hyperbolic 3-manifold which has minimum volume among all complete hyperbolic 3-manifolds. Further, there is a minimum volume member in the collection of complete hyperbolic 3-manifolds with one cusp; and similarly for n cusps. Computer studies to date show that the manifold obtained by performing (5,1) Dehn surgery on the figure-eight knot in the 3-sphere is the leading candidate for the minimum volume hyperbolic 3-manifold. Its volume is about 0.98. The leading one-cusp minimum volume candidate is the figure-eight knot complement in the 3-sphere. Its volume is about 2.03.

This paper gives an explicit construction for a solid tube around a short geodesic in a complete hyperbolic 3-manifold. Using this construction and a similar construction for cusp neighborhoods, the following theorems are proved.

THEOREM 1. 0.00064 *is a lower bound for the volume of a complete hyperbolic* 3-*manifold.*

THEOREM 2. *In the "thick and thin" decomposition of a complete hyperbolic* 3-*manifold,* $M = M_{thick} \cup M_{thin} = M_{(0,\epsilon]} \cup M_{(\epsilon,\infty)}$, ϵ *can be taken to be at least* 0.104, *i.e., the injectivity radius of the thick part is at least* 0.052.

The construction of solid tubes around short geodesics in (complete) hyperbolic 3-manifolds will be given in Sections 3 and 4. The elements of $PSL(2, \mathbf{C})$ act as isometries on the upper-half-space model of hyperbolic 3-space; and complete hyperbolic 3-manifolds correspond to discrete torsion-free subgroups of $PSL(2, \mathbf{C})$. Further a geodesic in a hyperbolic 3-manifold corresponds to an element in this subgroup, and length information can be read from the trace of the matrix. In fact, the trace of the matrix associated to a geodesic tells us not only the length of the geodesic but also its "torsion" (or "holonomy") the amount of rotation a

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transverse disc undergoes when it is parallel translated once around the geodesic; that is, the trace tells us the complex length of the geodesic.

In Section 3, Jørgensen's trace inequality is exploited to give constraints on the complex lengths of geodesics, thereby giving us a means of constructing solid tubes around geodesics of small real length and small holonomy. But we really want constraints in terms of the real length of the geodesic, thereby giving us solid tubes around short geodesics. This interplay between real and complex lengths is the essential difference between the 2-dimensional case and the 3-dimensional case. The 2-dimensional case is covered in [**11**].

The pigeonhole principle can be used to overcome the real-complex problem (and this was how I originally did it); however, a lemma pointed out to me by Don Zagier improved on the pigeonhole principle results. Thus, we present Zagier's Lemma and the rest of the solid tube construction in Section 4. The pigeonhole principle approach will be put in an appendix to Section 4; it turns out to be useful in [**9**]. It is important to note that the shorter the geodesic, the larger (the volume of) the constructed solid tube.

In Section 5, a standard construction of cusp neighborhoods is given; and, as Troels Jørgensen pointed out to me, this leads to the following

THEOREM. $\left(\frac{\sqrt{3}}{4}\right)n$ *is a lower bound for the volume of a complete hyperbolic 3-manifold with n cusps.*

The existence of these neighborhoods and the solid tube construction will be used in Section 6 to prove Theorem 1. The fact that all of the relevant neighborhoods are disjoint is proved in Section 7 and 8, and is used in Section 9 to prove Theorem 2.

Independent work on related topics has been done by Brooks and Matelski [**2**], Buser [**1**], Gallo [**3**], and Waterman [**13**].

I would like to thank Bill Dunbar, Michael Handel, Troels Jørgensen, Steve Kerckhoff, and Bill Thurston for helpful conversations.

2. Preliminaries. In this paper we will be dealing exclusively with complete orientable hyperbolic 3-manifolds (Chapters 1 through 6 of [**12**] is a good reference). A *hyperbolic manifold* is a differentiable manifold with a Riemannian metric of constant sectional curvature -1. *Complete* means complete as a metric space.

An important model for hyperbolic 3-space is the upper-half-space model, H.

$$H = \{ (x, y, t) : x, y, t \text{ are real numbers and } t \text{ is positive}\},$$

the Riemannian (hyperbolic) metric is

$$ds^2 = (dx^2 + dy^2 + dt^2)/t^2.$$

The boundary of H is the Riemann sphere $\{z = x + iy\} \cup \{\infty\}$, and the orientation-preserving isometries of H correspond to elements of

$$PSL(2, \mathbf{C}) = \{2 \times 2 \text{ complex matrices of determinant } 1\}/\pm\text{Id}.$$

These matrices act on the bounding Riemann sphere by

$$z \rightarrow (az + b)/(cz + d),$$

i.e., they are Moebius transformations (which take circles to circles), and this action can be extended to H (taking hemispheres to hemispheres) by using a hemisphere associated to a circle in the bounding complex plane. A complete orientable hyperbolic 3-manifold M can be described as $M = H/\Gamma$ where Γ is a discrete, torsion-free subgroup of $PSL(2, \mathbf{C})$.

The short geodesic which we will put a solid tube around will be denoted g. Lifts of g to H will be studied, and after a suitable conjugation one such lift will be the t-axis. The isometry which fixes this axis and yields the short geodesic will be denoted X. So,

$$X = \begin{bmatrix} p & 0 \\ 0 & p^{-1} \end{bmatrix} \quad \text{where } p \in C \text{ and } |p| \neq 0 \text{ or } 1.$$

The *complex length* of X, which we denote by $lh(X)$, is defined to be $ln(p^2) = 2ln(p)$ which is a complex number, say $x = x_1 + ix_2$. The complex length describes the action of X on a totally geodesic disc perpendicular to the axis of X; the disc is moved a distance x_1 along the axis of X and spun through an angle of x_2 radians. (Warning: If a capital letter, say X or C, is used to represent a matrix, then the associated lower-case letter, say x or c, will automatically represent the complex length associated with that matrix.)

In our study of lengths of geodesics by traces of matrices the following formulas will be useful.

(0) $\text{tr}(X) = p + p^{-1} = \text{trace}(X)$

(1) $\text{tr}(X)^2 - 4 = (p - p^{-1})^2$

(2) $2(\cosh(x) - 1) = e^x + e^{-x} - 2 = p^2 + p^{-2} - 2$
$= (p - p^{-1})^2 = \text{tr}(X)^2 - 4.$

Since trace (X) is a conjugacy invariant, complex length makes sense for the geodesic g and we define $lh(g)$ to equal

$$lh(X) = x_1 + ix_2.$$

Further, x_1 is the length of g determined by the Riemannian metric; we will denote this length by $l(g)$ or $l(X)$. Here, we are assuming that X is a primitive generator of g.

In constructing solid tubes around geodesics we will need to study the distance between axes corresponding to loxodromic transformations X

and Y acting on H. (In some references, "loxodromic" transformations are called "hyperbolic" transformations.) In particular, we will study the hyperbolic isometry C taking the axis of X to the axis of Y for which the axis of C is their unique common perpendicular. Such a C exists if X and Y have no common fixed point on the bounding Riemann sphere (at the end of this section we will point out why this is the relevant case for our purposes).

Up to normalization to have determinant 1, C^2 is given by

$$C^2 = (X - X^{-1})(Y - Y^{-1}).$$

This equation can be seen geometrically as follows:

$$X - X^{-1} = \begin{bmatrix} (p - p^{-1}) & 0 \\ 0 & (p^{-1} - p) \end{bmatrix}$$

is a 180° rotation about the axis of X (this can be seen by checking fixed points). Similarly, $Y - Y^{-1}$ is a 180° rotation about the axis of Y. Now, it can be seen geometrically that composing a 180° rotation about the axis of X with a 180° rotation about the axis of Y yields a loxodromic transformation whose axis is the unique common perpendicular from X to Y, and that this transformation is, in fact, C^2.

It can be computed that

$$(3) \quad \operatorname{tr}(C^2)^2 = \frac{\pm 4(a - d)^2(p - p^{-1})^2}{(p - p^{-1})^2((a + d)^2 - 4)}$$

where $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; and

$$(4) \quad \operatorname{tr}(C^2)^2 - 4 = \frac{\pm 16(\operatorname{tr}(XYX^{-1}Y^{-1}) - 2)}{(\operatorname{tr}(X)^2 - 4)(\operatorname{tr}(Y)^2 - 4)}.$$

Setting $k(X) = \frac{1}{2}|\operatorname{tr}(X)^2 - 4|$ will be convenient.

Some hyperbolic trigonometric formulae will be used:

$$(5) \quad |\cosh(a + ib) - 1| = \cosh(a) - \cos(b)$$

therefore,

$$k(X) = \cosh(x_1) - \cos(x_2).$$

For notational convenience we will generally write $k(X)$ as k.

$$(6) \quad \cosh(a) = \left(\frac{1}{2}(\cosh(2a) + 1)\right)^{1/2}$$

$$(7) \quad \sinh^2(a/2) = \frac{1}{2}(\cosh(a) - 1).$$

Again, because trace is a conjugacy invariant all relevant formulas are independent of the choice of X as having axis the t-axis.

Finally, Jørgensen's trace inequality (see [**5**]) will be crucial.

THEOREM [**5**]. *If X and Y generate a non-elementary discrete subgroup of $SL(2, \mathbf{C})$, then*

(8) $|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2| \geqq 1 - |\mathrm{tr}(X)^2 - 4|.$

Actually, Jørgensen proves more, but for our purposes this is sufficient. Also, since we will be dealing with torsion-free groups "non-elementary" can be replaced by "non-abelian." Further, although our X and Y are elements of $PSL(2, \mathbf{C})$ there are no problems, according to Jørgensen's proof, as long as our lift of $XYX^{-1}Y^{-1}$ is the one forced by the lifts of X and Y (from $PSL(2, \mathbf{C})$ to $SL(2, \mathbf{C})$).

As mentioned before, the perpendicular bisector construction only works if X and Y have disjoint fixed points. But this is the case if X and Y generate a non-abelian discrete group (see [**8**] p. 15) and in what follows we restrict our attention to this case.

3. The solid tube construction – part I; the radius of the solid tube in terms of complex length.

Definition. A *solid tube* of radius r around a geodesic g in a hyperbolic 3-manifold M is the set of all points in M which are a distance less than or equal to r from the (core) geodesic g. We will focus our attention on embedded (i.e., non-self-intersecting) solid tubes. The *length of the tube* is defined to be the (real) length of the geodesic g. If the radius is zero then the solid tube is actually the core geodesic. Sometimes it is helpful to think of the sold tube dynamically: pump air into the geodesic (zero radius tube) in a uniform fashion until a tube of the desired radius is obtained.

THEOREM. *If g is a geodesic in a complete hyperbolic 3-manifold M, with complex length $lh(g) = x = x_1 + ix_2$ satisfying*

$$\cosh(x_1) - \cos(x_2) < \sqrt{2} - 1$$

then there exists a solid tube aroung g with radius r satisfying

$$\sinh^2(r) = \frac{1}{2}\left(\frac{\sqrt{1 - 2k}}{k} - 1\right) \quad \textit{where } k = \cosh(x_1) - \cos(x_2).$$

Proof. We will be working in H which is the universal cover of M and we will be focusing on a particular lift of g to H. As in Section 2, this lift will be the axis of some hyperbolic transformation X. Dynamically constructing a solid tube around g in M by gradually increasing the radius corresponds to the following picture in H. Infinitely long tubes of the same radius will be growing around Axis (X) and all of its images under the action of $\Gamma(M = H/\Gamma)$. The tube in M will fail to be embedded as soon as the tube around Axis (X) in H intersects the tube around some image Axis (Y) of Axis (X) in H(X and Y are conjugate under Γ). The radius of the solid tube when it first bumps into itself is $c_1/2$ where c_1 is the (real) length of the unique common perpendicular to Axis (X) and Axis (Y) (see Section 2 where the matrix C associated with this perpendicular is described; the complex length of C is $c_1 + ic_2$). See Figure 1.

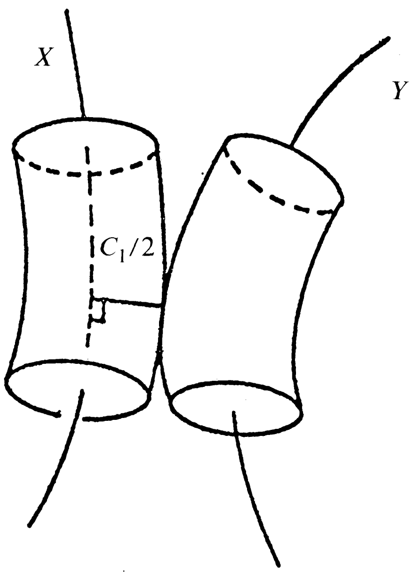

Figure 1

So, the construction of a solid tube around g reduces to estimating the (real) length of C.

Since Y is conjugate to X, $\mathrm{tr}(Y) = \mathrm{tr}(X) = p + p^{-1}$ and the derivation of a constraint on $\sinh^2(c_1/2)$ is as follows:

$$\begin{aligned} 2|\cosh(2c) - 1| &= |\mathrm{tr}(C^2)^2 - 4| \\ &= 16|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2|/(\,|\mathrm{tr}(X)^2 \\ &\qquad - 4||\mathrm{tr}(Y)^2 - 4|\,). \end{aligned}$$

But by (8),

$$|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2| \geqq 1 - |\mathrm{tr}(X)^2 - 4| = 1 - 2k$$

so that

$$2|\cosh(2c) - 1| \geqq 4(1 - 2k)/(k^2)$$

$$\cosh(2c_1) - \cos(2c_2) \geqq 2(1 - 2k)/(k^2)$$

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$$\cosh(2c_1) + 1 \geqq 2((1 - 2k)/k^2)$$

$$\cosh(c_1) = \left(\frac{1}{2}(\cosh(2c_1) + 1)\right)^{1/2} \geqq ((1 - 2k)/k^2)^{1/2}$$

$$= (1 - 2k)^{1/2}/k$$

$$\sinh^2(c_1/2) = \frac{1}{2}(\cosh(c_1) - 1) \geqq \frac{1}{2}(((1 - 2k)^{1/2}/k) - 1).$$

This completes the proof of the theorem.

So, given that

$$0 < k(X) = \cosh(x_1) - \cos(x_2) < \sqrt{2} - 1,$$

which is a constraint on the complex length of g, we have obtained some control over $c_1/2$, the radius of the solid tube. However, we need this control in terms of x_1, not k. In particular, a short g, i.e., a small x_1, could have a large

$$k = \cosh(x_1) - \cos(x_2)$$

if x_2 were close to $\pi/2$, say. This problem will be dealt with in the next section, but first we prove the following.

COROLLARY. *The theorem holds if* $k = \cosh(nx_1) - \cos(nx_2)$, *where n is a positive integer.*

Proof. In computing the distance between Axis (X) and Axis (Y) in H we could have used the matrix X^n in place of X because the axes of X^n and its conjugates are the same as the axes of X and its conjugates. And in applying Jørgensen's trace inequality we would use

$$\operatorname{tr}(X^n)^2 - 4 = 2(\cosh(nx) - 1)$$

because

$$lh(X^n) = nlh(X) = nx_1 + inx_2 = nx.$$

Finally, it should be noted that the cases where Jørgensen's trace inequality do not apply can be handled by elementary means.

4. The solid tube construction – part II; radius of the solid tube in terms of length. The solid tube construction of Section 3 only works if

$$k = k(X) = \cosh(x_1) - \cos(x_2) < \sqrt{2} - 1;$$

but x_1 small does not ensure that k is small because x_2 may be "large". To overcome this problem we need to take iterates X^n of X until the angle of the iterate X^n is small (mod 2π), while at the same time not taking too many iterates in which case the real length of X^n would be too long. That is, we want to find an integer n so that

$$k(X^n) = \cosh(nx_1) - \cos(nx_2) < \sqrt{2} - 1$$

and then use the corollary in Section 3.

Our original approach to finding n was to use a pigeonhole principle technique (outlined in the appendix to this section). Later, Don Zagier showed us the following lemma which improves on the pigeonhole principle results.

LEMMA. *For x_1 and x_2 with $0 < x_1 < \pi\sqrt{3}$, there exists an integer n greater than or equal to one for which*

$$\cosh(nx_1) - \cos(nx_2) \leqq \cosh\left(\sqrt{\frac{4\pi x_1}{\sqrt{3}}}\right) - 1.$$

Proof. Let $u = x_2/2\pi$ and $v = x_1/2\pi$, then there exist integers m and n such that

$$(nu - m)^2 + n^2v^2 \leqq 2v/\sqrt{3}.$$

(This is true because the modular group has a fundamental domain with all points having imaginary part at least $\sqrt{3}/2$; thus, there exists an element A in $SL(2, \mathbf{Z})$ with bottom row $(n, -m)$ such that the imaginary part of $A(u + iv)$ is at least $\sqrt{3}/2$. But this imaginary part is

$$v/[(nu - m)^2 + n^2v^2],$$

and the result follows.) Here, $x_1 < \pi\sqrt{3}$ implies $v < \sqrt{3}/2$ which forces n to be non-zero. Now,

$$\begin{aligned}\cosh(nx_1) - \cos(nx_2) &= \cosh(nx_1) - \cos(nx_2 - 2m\pi)\\ &\leqq \sum_{r=1}^{\infty} \frac{1}{(2r)!}[(nx_1)^{2r} - (-1)^r(nx_2 - 2m\pi)^{2r}]\\ &\leqq \sum_{r=1}^{\infty} \frac{1}{(2r)!}[(nx_1)^2 + (nx_1 - 2m\pi)^2]^r\\ &\leqq \sum_{r=1}^{\infty} \frac{(4\pi x_1/\sqrt{3})^r}{(2r)!} = \cosh\left(\sqrt{\frac{4\pi x_1}{\sqrt{3}}}\right) - 1.\end{aligned}$$

COROLLARY. *The longest short geodesic obtainable by using the lemma has length*

$$\frac{\sqrt{3}}{4\pi}[\log(\sqrt{2} + 1)]^2,$$

which is a bit more than 0.107.

Proof. To construct a non-trivial solid tube we need $k < \sqrt{2} - 1$. But

$$x_1 = \frac{\sqrt{3}}{4\pi}[\log(\sqrt{2} + 1)]^2$$

yields

$$k = \cosh(nx_1) - \cos(nx_2) \leqq \cosh\left(\sqrt{\frac{4\pi x_1}{\sqrt{3}}}\right) - 1$$
$$= \cosh(\log(\sqrt{2} + 1)) - 1 = \sqrt{2} - 1.$$

Using the lemma and the theorem in Section 3, we have

THEOREM. *Let g be a geodesic in a complete hyperbolic 3-manifold. If the (real) length $l(g)$ of g is less than*

$$\frac{\sqrt{3}}{4\pi}[\log(\sqrt{2} + 1)]^2 \approx 0.107,$$

then there exists an embedded solid tube around g whose radius r satisfies

$$\sinh^2 r = \frac{1}{2}\left(\frac{\sqrt{1 - 2k}}{k} - 1\right) \quad \textit{where } k = \cosh\left(\sqrt{\frac{4\pi l(g)}{\sqrt{3}}}\right) - 1.$$

COROLLARY 1. *For r and k as in the theorem, r is a decreasing function of $l(g)$. That is, as the geodesic shrinks the solid tube grows.*

Proof.

$$k = \cosh\left(\sqrt{\frac{4\pi l(g)}{\sqrt{3}}}\right) - 1$$

is an increasing function of $l(g)$; and r as in the theorem is a decreasing function of k, at least for $0 < k < \sqrt{2} - 1$.

COROLLARY 2. *As the length of the geodesic approaches zero the radius of the solid tube approaches infinity.*

COROLLARY 3. *For geodesics g with*

$$l(g) < \frac{\sqrt{3}}{4\pi}[\log(\sqrt{2} + 1)]^2,$$

the volume of the solid tube constructed in the theorem increases when $l(g)$ decreases.

Proof. The volume of a solid tube is $\pi l(g)\sinh^2(r)$ which, in our case, is

$$\pi l(g)\left(\frac{\sqrt{1-2k}}{k} - 1\right)/2.$$

As in Section 2, we will use x_1 in place of $l(g)$. So, we must show that

$$\frac{d}{dx_1}\left(x_1\left(\frac{\sqrt{1-2k}}{k} - 1\right)\right) < 0.$$

That is,

$$\frac{\sqrt{1-2k}}{k} - 1 + x_1\left(\frac{k-1}{k^2\sqrt{1-2k}}\right)\frac{dk}{dx_1} < 0?$$

$$1 - 2k - k\sqrt{1-2k} < \left(x_1\frac{dk}{dx_1}\right)\left(\frac{1-k}{k}\right)?$$

This holds if

$$x_1\frac{dk}{dx_1} > k.$$

But,

$$k = \cosh\left(\sqrt{\frac{4\pi x_1}{\sqrt{3}}}\right) - 1$$

so

$$x_1\frac{dk}{dx_1} = x_1\left(\sqrt{\frac{\pi}{\sqrt{3}x_1}}\right)\sinh\left(\sqrt{\frac{4\pi x_1}{\sqrt{3}}}\right)$$

and expanding in power series shows that

$$x_1\frac{dk}{dx_1} > k.$$

Example. If we have a geodesic of length $l(g) = 0.10695$ then the volume of an embedded solid tube around g can be computed. In particular,

$$k = \cosh\left(\sqrt{\frac{4\pi l(g)}{\sqrt{3}}}\right) - 1 = .4137166\ldots$$

and the volume of the solid tube is

$$\pi l(g)(\sqrt{1-2k}/k - 1)/2 = 0.0006882\ldots$$

Appendix (*To Section* 4): *The Pigeonhole Principle*. Originally, the results of Section 4 were proved by using the pigeonhole principle (some of the results were weaker). The advantage of this approach is that it is geometric and "hands-on." In particular, it is well-suited to the needs of [**9**]. The disadvantage is that the results obtained are not quite as good as those obtained via Zagier's lemma. For example, the longest short geodesic from the pigeonhole principle has length roughly equal to $\pi/50$; whereas the Zagier long short geodesic has length roughly equal to 0.107.

We will now sketch the original pigeonhole principle proof of the theorem; but we will not prove the corollaries; the arguments are essentially the same.

THEOREM. *Let g be a geodesic in a complete hyperbolic 3-manifold. If the length of g is less than or equal to $\pi/50$ then there exists an embedded solid tube around g whose radius r satisfies*

$$\sinh^2 r = \frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right) \quad \textit{where } k = \cosh\left(\frac{2\pi}{M}\right) - \cos\left(\frac{2\pi}{M}\right)$$

and M is the positive integer such that

$$\frac{2\pi}{(M+1)^2} < x_1 \leqq \frac{2\pi}{M^2}.$$

Proof. We will exploit the corollary in the previous section. Assume the length of g which, as usual, we denote x_1 is $2\pi/M^2$. Break the unit disc into M wedges each of $2\pi/M$ radians. Then by the pigeonhole principle there exists an integer $N \leqq M$ such that

$$\cos(Nx_2) \geqq \cos\left(\frac{2\pi}{M}\right). \text{ Thus}$$

$$k(X^N) = \cosh(Nx_1) - \cos(Nx_2) \leqq \cosh(Mx_1) - \cos\left(\frac{2\pi}{M}\right)$$

$$= \cosh\left(\frac{2\pi}{M}\right) - \cos\left(\frac{2\pi}{M}\right).$$

By the corollary of the previous section, taking

$$k = \cosh\left(\frac{2\pi}{M}\right) - \cos\left(\frac{2\pi}{M}\right)$$

yields a solid tube of radius r satisfying

$$\sinh^2(r) = \frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right)$$

and $M \geqq 10$ makes $k < \sqrt{2} - 1$. Now assume

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$$\frac{2\pi}{(M+1)^2} < x_1 < \frac{2\pi}{M^2}$$

and use the M pigeonhole principle again. Since

$$Mx_1 < \frac{2\pi}{M}$$

and $\sqrt{1 - 2k/k}$ is a decreasing function of k, we get the desired r.

5. Cusp neighborhoods.

THEOREM. *$(\sqrt{3}/4)n$ is a lower bound for the volume of a complete hyperbolic 3-manifold with n cusps.*

Proof. As usual, we will be working in H.

We want to put embedded horoball neighborhoods around the cusps. The problem is to decide how big one horoball neighborhood should be as compared to the size of another. The solution is to demand that each horosphere boundary of the horoball neighborhood of a cusp have minimum translation length equal to one. This translation length is measured in the induced metric on the horosphere boundary. These horoball neighborhoods do not intersect: Conjugate the group so that one cusp is at infinity with parabolics

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } X' = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \text{ with } |w| \geqq 1;$$

and another cusp is at the origin with matrix

$$Y = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Here the boundary of the cusp neighborhood at infinity is at height one. Jørgensen's trace inequality applied to X and Y forces c to have absolute value at least one. Conjugating by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ interchanges the two cusps by rotating 180 degrees about the axis through $+1$ and -1. In particular, Y is conjugated to $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ and we see that a horoball neighborhood of the cusp with minimum translation length one must be at height $|c|$ which is greater than or equal to one. Thus, no two cusp neighborhoods intersect.

The torus determined by the horosphere boundary is characterized by the complex numbers 1 and w. The fact that the minimum translation distance is 1 forces w to have imaginary part at least $\sqrt{3}/2$. That is, in Figure 2 (which is a view from the cusp at ∞) we see that if the imaginary part of w were less than $\sqrt{3}/2$ then we could translate back into the unit disc. This is a contradiction. Hence, the volume contribution of the cusp at infinity is at least $(1/2)(\sqrt{3}/2) = \sqrt{3}/4$.

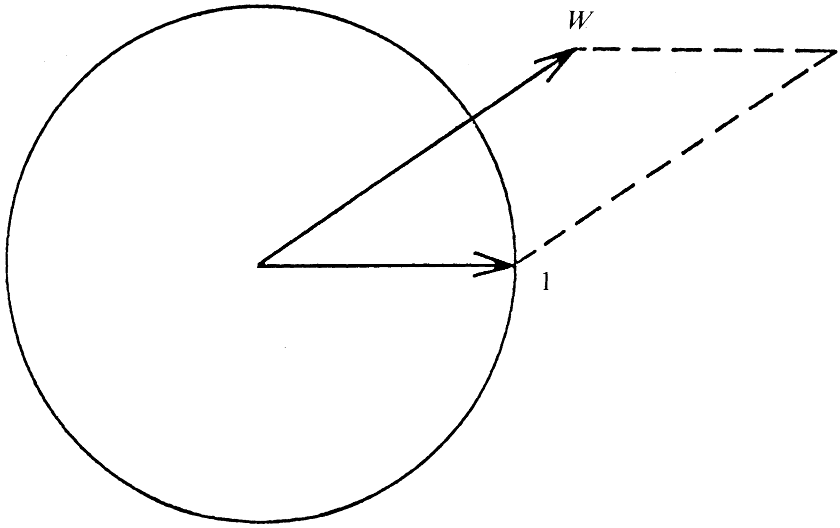

Figure 2

This analysis also holds for the other cusps and the volume of the entire manifold is at least $(\sqrt{3}/4)n$.

6. Lower bound for volume.

THEOREM 1. 0.00064 *is a lower bound for the volume of a complete hyperbolic* 3-*manifold.*

Proof. If the manifold has a cusp then we are done by the result of Section 5. If not then the manifold has the property that all non-trivial closed loops correspond to geodesics (see Proposition 5.3.1 of [**12**]) and the length of the geodesic is less than the length of the loop. In light of this, there are two possibilities

(1) The manifold has a geodesic of length less than or equal to 0.10695 in which case the example in Section 4 shows that the geodesic has an embedded solid tube of volume greater than 0.00064.

(2) The manifold has an embedded sphere of radius 0.10695/2. Since the volume of a hyperbolic sphere is $\pi(\sinh(2r) - 2r)$, plugging in $r = 0.10695/2$ yields a volume just barely greater than 0.00064.

Note. If we had chosen a number less than 0.10695 then the volume of the solid tube would have been bigger, but the volume of the embedded ball would have been smaller. In fact, 0.10695 is the optimal "trade-off" length.

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7. Multiple short geodesics.

THEOREM. *The solid tubes constructed previously about different short geodesics do not intersect.*

Proof. We want to prove Figure 3 is correct. That is, is $l_1/2 + l_2/2 \leqq l_{12}$ where l_{12} is the distance between X and Y?

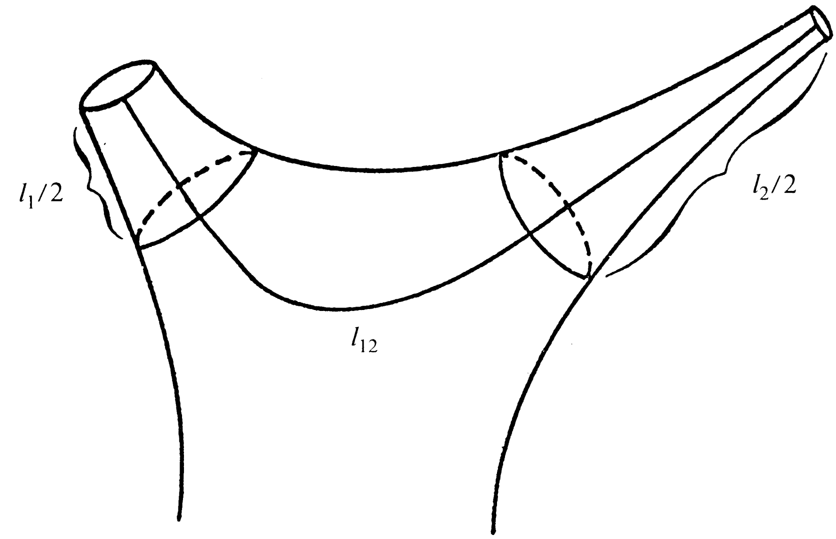

Figure 3

For notational convenience let $k = k(X^N)$ and $k' = k(Y^M)$ where M and N are any positive integers. We will thus be proving the theorem independent of the choices of N and M, that is, we are working in the context of Section 3, not Section 4.

By construction,

$$\cosh(l_1 + l_2) = \cosh(l_1)\cosh(l_2) + \sinh(l_1)\sinh(l_2)$$
$$= \frac{\sqrt{1-2k}}{k}\frac{\sqrt{1-2k'}}{k'}$$
$$+ \frac{\sqrt{1-2k-k^2}}{k}\frac{\sqrt{1-2k'-(k')^2}}{k'}$$

and

$$\cosh(2l_{12}) \geqq \frac{8|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2|}{|\mathrm{tr}(X)^2 - 4||\mathrm{tr}(Y)^2 - 4|} - 1 \geqq \frac{2(1-2k)}{kk'} - 1.$$

(Here we are using the fact that Jørgensen's trace inequality works for Y and X as well as for X and Y (since $\mathrm{tr}(XYX^{-1}Y^{-1}) = \mathrm{tr}(YXY^{-1}X^{-1})$) so we can switch k and k' in the numerator of the $\cosh(2l_{12})$ formula.) So,

$$\cosh(2l_{12}) \geqq \frac{1 - 2k}{kk'} + \left(\frac{1 - 2k}{kk'} - 1\right).$$

Assume $k \leqq k'$ hence

$$1 - 2k \geqq 1 - 2k' \quad \text{and} \quad \frac{\sqrt{1 - 2k}}{k}\frac{\sqrt{1 - 2k'}}{k'} \leqq \frac{1 - 2k}{kk'}.$$

So to show that $\cosh(l_1 + l_2) \leqq \cosh(2l_{12})$ it remains to show that

$$\frac{\sqrt{1 - 2k - k^2}}{k}\frac{\sqrt{1 - 2k' - (k')^2}}{k'} \leqq \frac{1 - 2k - kk'}{kk'}$$

or, better, to show that

$$(1 - 2k - k^2)(1 - 2k' - k'^2) \leqq (1 - 2k - kk')(1 - 2k' - kk').$$

Multiplying out we have to show

$$1 - 2k' - k'^2 - 2k + 4kk' + 2kk'^2 - k^2 + 2k^2k' + k^2k'^2$$
$$\leqq 1 - 2k' - kk' - 2k + 4kk' + 2k^2k' - kk' + 2kk' + k^2k'^2.$$

That is, is $-k^2 - k'^2 \leqq -2kk'$? Yes, since $0 \leqq (k - k')^2$.

So, $l_1 + l_2 \leqq 2l_{12}$ and the individual solid tubes work as embedded simultaneous collars.

8. Non-intersection of cusp neighborhoods and solid tubes.

THEOREM. *The solid tubes of the corollary of Section* 3, *and the cusp neighborhoods of Section* 5 *do not intersect.*

Proof. This is an application of Jørgensen's trace formula. After suitable conjugacy we can think of the cusp as defined by the matrices

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \quad \text{with } |w| \geqq 1;$$

and the short geodesic as defined by

$$Y = \begin{pmatrix} a & bd \\ d/b & a \end{pmatrix}$$

where

$$a = \mathrm{tr}(Y)/2, \quad d = \frac{\sqrt{\mathrm{tr}(Y)^2 - 4}}{2},$$

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and $\pm b$ are the fixed points of Y in the bounding Riemann sphere (see Figure 4, which is a "side-view" of H with the dotted semi-circle representing the axis of Y).

Thus

$$|\mathrm{tr}(XYX^{-1}Y^{-1}) - 2| = \left| \left(\frac{d}{b}\right)^2 \right| \quad \text{and} \quad |\mathrm{tr}(X)^2 - 4| = 0,$$

and by the trace inequality

$$\left| \left(\frac{d}{b}\right)^2 \right| \geqq 1,$$

i.e. $|b|^2 \leqq |d|^2$. So,

$$|b|^2 \leqq |d|^2 = \left| \frac{\mathrm{tr}(Y)^2 - 4}{4} \right| = k(Y)/2 = k/2.$$

We are now concerned with whether the solid tube for Y reaches above height 1. The "highest point" of the tube of radius r round the axis of Y is $|b|e^r$, which we need to control (see figure 4).

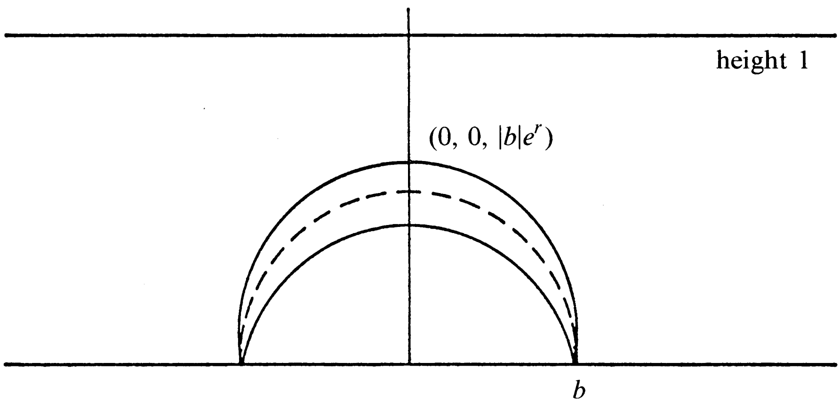

Figure 4

But,

$$\begin{aligned} e^{2r} &= \cosh(2r) + \sinh(2r) \\ &= \frac{\sqrt{1 - 2k}}{k} + \frac{\sqrt{1 - 2k - k^2}}{k} \leqq \frac{2}{k}\sqrt{1 - 2k} \quad \text{and} \end{aligned}$$

$$|b|^2 e^{2r} \leqq \frac{k}{2}\left(\frac{2}{k}\sqrt{1 - 2k}\right) = \sqrt{1 - 2k} < 1.$$

Thus, in this set-up the solid tube never reaches above height 1. The solid tubes and the cusp neighborhoods do not intersect.

Also, by the corollary of Section 3, this works for $k = k(Y^N)$ where N is a positive integer.

9. The thick and thin decomposition. The Margulis lemma (see [**12**], 5.10.1) is of great importance in understanding the structure of complete hyperbolic 3-manifolds. The most concrete form of this lemma is given in Theorem 5.10.2 of [**12**] which states that there exists an ϵ such that for all complete hyperbolic 3-manifolds the thin part of the manifold is made up of non-intersecting solid tubes and cusp neighborhoods. In this section we prove that $\epsilon = 0.104$ works. That is, we will essentially reprove the Margulis lemma for complete hyperbolic 3-manifolds, but for a specific ϵ.

$$M = M_{(0,\epsilon]} \cup M_{(\epsilon,\infty)} = M_{thin} \cup M_{thick}$$

where $M_{(0,\epsilon]}$ is the set of all points p such that there exists a closed (non-trivial) loop of length less than or equal to ϵ through p. So p is in $M_{(0,\epsilon]}$ if there exists an element X in $\Gamma = \pi_1(M)$ such that the distance from p to $X(p)$ is less than ϵ. It follows (see the proof of 5.10.2 of [**12**]) that $M_{(0,\epsilon]}$ is a collection of solid tubes and cusp neighborhoods. We have only to prove that these solid tubes and cusp neighborhoods do not intersect for ϵ equal to 0.104.

The solid tubes constructed in Sections 3 and 4, and the cusp neighborhoods constructed in Section 5 of this paper do not intersect (see Sections 5, 7 and 8). If these solid tubes and cusp neighborhoods contain the $M_{(0,\epsilon]}$ solid tubes and cusp neighborhoods then we will have the desired proof of the non-intersection of the $M_{(0,\epsilon]}$ solid pieces. It is easy to see that our cusp neighborhoods work, so we now restrict our attention to X not parabolic. So, we need to prove

LEMMA 1. *If there exists $X \in \Gamma = \pi_1(M)$ such that the distance from p to $X(p)$ is less than $\epsilon = 0.104$ (i.e., $p \in M_{(0,\epsilon]}$) then p is contained in a solid tube around the axis of X constructed in Sections 3 and 4 of this paper.*

Proof. First we will need the following lemma.

LEMMA 2. *Let $C = d(p, X(p))$ and s = distance from the axis of X to p, and $x = x_1 + ix_2 = lh(X)$ = complex length of X; then*

$$\cosh(C) = \cosh(x_1) + \sinh^2(s) \cdot (\cosh(x_1) - \cos(x_2)).$$

Thus,

$$\sinh^2(s) = \frac{-\cosh(x_1) + \cosh(C)}{\cosh(x_1) - \cos(x_2)}.$$

Proof. (Lemma 2). Use hyperbolic trigonometry (see [**12**] Section 2.6) on the triangle formed by the points p, $X(p)$ and the point at infinity in the upper-half-space model of hyperbolic 3-space. For simplicity, the axis of X

can be taken to be the t-axis, and thus p and $X(p)$ are the same distance from the t-axis.

Proof (Lemma 1). We must show that s is less than the radius r of the constructed solid tube. So, is $\sinh^2(s) \leqq \sinh^2(r)$?

$$\frac{-\cosh(x_1) + \cosh(C)}{\cosh(x_1) - \cos(x_2)} \leqq \frac{1}{2}\left(\frac{\sqrt{1 - 2k(X^n)}}{k(X^n)} - 1\right)?$$

The corollary of Section 3 implies that we are free to make use of any n. Once we have an n we will set k equal to $k(X^n)$ and $m = \sqrt{1 - 2k}/k$.Then what we have to show is that

$$2 \cosh(C) \leqq (m - 1) \cosh(x_1) - (m - 1) \cos(x_2) + 2 \cosh(x_1)$$

or

$$2 \cosh(C) \leqq (m + 1) \cosh(x_1) - (m - 1) \cos(x_2)$$

where

$$C = 0.104 \quad \text{and} \quad 2 \cosh(C) \approx 2.01.$$

We will break this down into two cases:

Case 1. $\cos(x_2) \geqq 1 - x_1$. Then

$$k(X) = \cosh(x_1) - \cos(x_2)$$

and the $k(X^n)$ which will be used is this one ($n = 1$). So,

$$k = k(X) = \cosh(x_1) - \cos(x_2)$$

and the desired inequality

$$2\frac{\cosh(C) - \cosh(x_1)}{\cosh(x_1) - \cos(x_2)} \leqq \left(\frac{\sqrt{1 - 2k}}{k} - 1\right)$$

reduces to

$$2(\cosh(C) - \cosh(x_1)) \leqq \sqrt{1 - 2k} - k,$$

which is easily seen to be true for $x_1 \leqq .104$. The key to the proof of this case is that we had the flexibility to take

$$k = k(X) = \cosh(x_1) - \cos(x_2)$$

not some other, inappropriate, $k(X^n)$.

Case 2. $\cos(x_2) \leqq 1 - x_1$. In this case we will use the n found by Zagier's lemma in Section 4. Notationally, m is as before and we must show that

$$2 \cosh(C) \leqq (m + 1) \cosh(x_1) - (m - 1)(1 - x_1)$$

$$\leqq (m + 1)\cosh(x_1) - (m - 1)\cos(x_2).$$

Since we are dealing with x_1 small this essentially reduces to

$$2.01 \leqq (m + 1)(1 + x_1^2/2) - (m - 1)(1 - x_1)$$
$$= 2 + (m + 1)(x_1^2/2) + (m - 1)x_1.$$

But $(m - 1)x_1 = (2/\pi)$ times the volume of the solid tube and when $x_1 \leqq 0.104$ we can compute that the volume of the associated solid tube is at least 0.017. So the inequality is easily satisfied.

REFERENCES

1. P. Buser, *On Cheeger's inequality* $\lambda_1 \geqq h^2/4$, Proc. of Symposia in Pure Math. *36* (1980), 29-77.
2. R. Brooks and J. Matelski, *Collars in Kleinian groups*, Duke Math. J. *43* (1982), 163-182.
3. D. Gallo, *A 3-dimensional hyperbolic collar lemma*, in *Kleinian groups and related topics*, Lecture Notes in Mathematics *971* (Springer-Verlag, New York, 1983).
4. N. Halpern, *A proof of the collar lemma*, Bull. Lond. Math. Soc. *13* (1981), 141-144.
5. T. Jorgensen, *On discrete groups of Moebius transformations*, Amer. J. Math. *98* (1976), 739-749.
6. L. Keen, *Collars on Riemann surfaces*, Ann. of Math. Studies *79* (1974), 239-246.
7. D. Kazhdan and G. Margulis, *A proof of Selberg's hypothesis*, (Russian) Mat. Sb. (N.S.) *75*(117) (1968), 163-168.
8. J. Lehner, *A short course in automorphic functions* (Holt, Reinhart and Winston, New York, 1966).
9. R. Meyerhoff, *A lower bound for the volume of hyperbolic 3-orbifolds*, preprint.
10. J. Matelski, *A compactness theorem for Fuchsian groups of the second kind*, Duke Math J. *43* (1976), 829-840.
11. B. Randol, *Cylinders in Riemann surfaces*, Comm. Math. Helvetici *54* (1979), 1-5.
12. W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University preprint (1978).
13. P. Waterman, *An inscribed ball for Kleinian groups*, Bull. London Math. Soc. *16* (1984), 525-530.

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