THE DESCENDING CHAIN CONDITION IN MODULAR LATTICES

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In a recent paper Kovács [1] studied join-continuous modular lattices which satisfy the following conditions:

(i) *every element is a join of finitely many join-irredicibles, and,*

(ii) *the set of join-irreducibles satisfies the descending chain condition.* He was able to prove that such a lattice must itself satisfy the descending chain condition. Interest was expressed in whether or not one could obtain the same result without the assumption of modularity and/or of join-continuity. In this paper we give an elementary proof of this result without the assumption of join-continuity (which of course must then follow as a consequence of the descending chain condition). In addition we give a suitable example to show that modularity may not be omitted in general. We first state the main result:

THEOREM. *If L is a modular lattice in which (i) and (ii) hold, then L satisfies the descending chain condition.*

The proof will be given after establishing a preliminary result.

LEMMA. *Let L be a modular lattice and let K be the set of all $x \in L$ such that the principal ideal (x) generated by x satisfies the descending chain condition. Then K is an ideal of L (possibly void). Moreover, K satisfies the descending chain condition.*

PROOF. It is enough to prove that K is join closed since everything else is obvious and holds in general. Let $a, b \in K$ and suppose we are given a chain $a \vee b \geqq x_1 \geqq x_2 \geqq \cdots$. Observe that the descending chain condition holds in the interval $[a, a \vee b]$—by the isomorphism theorem in modular lattices—since it is transposed to $[a \wedge b, b]$. Now consider the chains $\{a \wedge x_n\}$ in (a) and $\{a \vee x_n\}$ in $[a, a \vee b]$. By the descending chain condition there is an n such that for $m \geqq n$ we have $a \vee x_n = a \vee x_m$ and $a \wedge x_n = a \wedge x_m$. But $x_m \leqq x_n$ so that by modularity $x_m = x_n$ for $m \geqq n$. This shows that $a \vee b \in K$ and completes the proof.

PROOF OF THE THEOREM. Let J denote the set of join irreducibles of L and let K be as in the Lemma. We intend to show that $J \subseteq K$. To show an $x \in J$ is an element of K we may assume that each $y \in J$, $y < x$, is a member of K. This is because (ii) holds. Now by (*i*) and the Lemma this implies that any $y < x$ is a member of K. It is now obvious that $x \in K$ since any chain $x \geqq x_1 \geqq x_2 \geqq \cdots$ is either constant or eventually in K. Hence $J \subseteq K$ and by (*i*) we have $K = L$. Therefore L satisfies the descending chain condition.

Close examination of the proof shows that we can state:

COROLLARY. *If L is any lattice in which* (*i*) *and* (*ii*) *hold and in which K* (*as defined above*) *is an ideal, then L satisfies the descending chain condition.*

We now give an example of a join-continuous lattice which satisfies (*i*) and (*ii*) but not the descending chain condition. Let L consist of the following elements in the real plane: (a) the origin (0,0); (b) the line segment from (0,1) to (1,0) and; (c) the line segment from (0,1) to (1,2). It is easily checked that L is a join-continuous lattice with the order on L induced by the pointwise order on the plane (a sketch is most helpful). The points $(t, 1-t)$, $0 \leqq t \leqq 1$, in the line segment described in (b) are atoms and thereby join-irreducible. An element $(t, 1+t)$, $0 \leqq t \leqq 1$, is given irredundantly as the join of (0,1) and $(t, 1-t)$. It is now clear that (*i*) and (*ii*) hold in L, yet the line segment described in (c) has infinite descending chains so that the descending chain conditions does not hold.

Reference

[1] L. G. Kovács, 'The descending chain condition in join continuous modular lattices' *J. Aust. Math. Soc.* 10 (1969), 1–4.

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