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SUBGROUP SEPARABILITY OF GENERALIZED FREE PRODUCTS OF FREE-BY-FINITE GROUPS

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ABSTRACT. We prove that generalized free products of finitely generated free-by-finite groups amalgamating a cyclic subgroup are subgroup separable. From this it follows that if $G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \, ; \, a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}(uv)^t \rangle$ where $t \geq 1$ and u, v are words on $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ respectively then G is subgroup separable thus generalizing a result in [9] that such groups have solvable word problems.

1. **Introduction.** Subgroup separability is of interest in both group theory and topology. The former because of its relation to the generalized word problem and the latter because of its relation to embedding theory. Unfortunately the classes of groups known to be subgroup separable are relatively few. In the last few years a number of new classes of such groups have been discovered, see [2], [4], [6], [7], [10]. In this note we prove that generalized free products of finitely generated (f. g.) free-by-finite groups amalgamating a cyclic subgroup are subgroup separable. As a contrast, it is interesting to note that the generalized free products of two f. g. free abelian-by-finite groups amalgamating a cyclic subgroup need not be subgroup separable. An example of two copies of a finite extension of a free abelian group of rank 2, amalgamating a cyclic subgroup that is not subgroup separable was constructed by Long and Niblo [5].

Since a large number of Fuchsian groups are free-by-finite, this result overlaps somewhat Niblo's result [6] that generalized free products of Fuchsian groups amalgamating a cyclic subgroup are subgroup separable. Moreover, our proof is completely group theoretical instead of using topological arguments.

We also show that

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \, ; \, a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}, (uv)^t \rangle$$

where $t \geq 1$ and u, v are words on $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ respectively are subgroup separable. This improves Rosenberger's result [8] which says G is RF. Our main tool is to make use of a criterion proved by Brunner, Burns and Solitar [2].

THE BBS-CRITERION. Let $g = A *_{\langle x \rangle} B$. Then G is subgroup separable if A, B and $\langle x \rangle$ satisfy the following condition: For any $m + 1$ nontrivial elements g_i of A, (B), and

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any n f. g. subgroups $H_1, \ldots, H_n$ of A, (B), m, n finite, there exists a positive integer k such that to each positive integer l there is $N \lhd_f A$, (B), with the following properties:

1. $N \cap \langle x \rangle = \langle x^{kl} \rangle$;
2. for each H_j such that $H_j \cap \langle x \rangle = 1$, $NH_j \cap \langle x \rangle = \langle x^{kl} \rangle$;
3. for each H_j such that $H_j \cap \langle x \rangle \neq 1$, $NH_j \cap \langle x \rangle = H_j \cap \langle x \rangle$;
4. for each $g_i \notin \langle x \rangle$, $g_i \notin N\langle x \rangle$;
5. for each pair (i, j) such that $H_j \cap \langle x \rangle g_i = \emptyset$, we have $NH_j \cap \langle x \rangle g_i = \emptyset$.

2. **Results.** In order to prove our main result we need to prove some results parallel to Propositions 1 and 2 of [2] for free-by-finite groups.

LEMMA 1. *Let G be a finite extension of a* f. g. *free group F. Let $x, g \in G$, where x is of infinite order. If H is a* f. g. *subgroup of G such that $H \cap \langle x \rangle g = \emptyset$, then there exists $N \lhd_f G$ such that $N \subseteq F$ and $NH \cap \langle x \rangle g = \emptyset$.*

PROOF. Let $K = H \cap F$. Clearly K is of f. i. in H. Thus $H = Kh_1 \cup \cdots \cup Kh_r$, where $h_i \in H$ with $h_1 = 1$ and r is finite. Let k be the smallest positive integer such that $x^k \in F$. Consider the element $x^l g h_i^{-1}$, where $0 \leq l < k$ and $1 \leq i \leq r$. Since $H \cap \langle x \rangle g = \emptyset$, we have $K \cap \langle x^k \rangle x^l g h_i^{-1} = \emptyset$. If $x^l g h_i^{-1} \in F$, then by Proposition 2 of [2], there exists $M_{l,i} \lhd_f F$ such that $M_{l,i} K \cap \langle x^k \rangle x^l g h_i^{-1} = \emptyset$. If $x^l g_i h_i^{-1} \notin F$, let $M_{l,i} = F$. In this case it is easy to see $FK \cap \langle x^k \rangle x^l g h_i^{-1} = \emptyset$. Let $M = \bigcap_{l,i} M_{l,i}$. Clearly $M \lhd_f F$. Since F is f. g., there exists a characteristic subgroup N in M of finite index (f. i.) in F. It follows that $N \lhd_f G$. Moreover $N \subset M$ implies $NH \cap \langle x \rangle g = \emptyset$ as required.

The following lemma is extracted from the proof of Proposition 1 of [2].

LEMMA 2. *Let F be a free group and $x \in F$. Let K be a* f. g. *subgroup of F such that $K \cap \langle x \rangle = 1$. Then there exists $Q \lhd_f F$ such that there is a positive integer r such that to each positive integer s there exists a verbal subgroup Q_s of* f. i. *in Q with the property that*

$$KQ_s \cap \langle x \rangle = Q_s \cap \langle x \rangle = \langle x^{rs} \rangle.$$

The verbal subgroup Q_s of Q is defined by the laws $[y, z] = 1$ and $y^{2s} = 1$.

In fact, using exactly the same argument, Lemma 2 can be extended to the following:

LEMMA 3. *Let F be a free group and $x \in F$. Let K be a* f. g. *subgroup of F such that $K \cap \langle x \rangle = 1$. Then there exists $Q \lhd_f F$ such that if $L \lhd_f F$ and $L \subseteq Q$ then there is a positive integer r such that for each positive integer s there exists a verbal subgroup L_s of* f. i. *in L with the property that*

$$KL_s \cap \langle x \rangle = L_s \cap \langle x \rangle = \langle x^{rs} \rangle.$$

Here L_s is again the verbal subgroup of L defined by $[y, z] = 1$ and $y^{2s} = 1$.

LEMMA 4. *Let G be a finite extension of a* f. g. *free group F. Let $x \in G$ be of infinite order and H be a* f. g. *subgroup of G such that $H \cap \langle x \rangle = 1$. Then there exists a positive*

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integer ρ such that to each positive integer s there is $Y_s \lhd_f G$ with the property that $Y_s \subseteq F$ and

$$HY_s \cap \langle x \rangle = Y_s \cap \langle x \rangle = \langle x^{\rho s} \rangle.$$

PROOF. Let k be the smallest positive integer such that $x^k \in F$. Let $K = H \cap F$. Since $[H : K]$ is finite, we have $H = Kh_1 \cup \cdots \cup Kh_t$ where $h_i \in H$ and $h_1 = 1$. Clearly $K \cap \langle x^k \rangle = 1$ and $K \cap \langle x^k \rangle x^l h_i^{-1} = \emptyset$ for all $0 \leq l < k$ and $1 \leq i \leq t$ with the exception of $x^0 h_1^{-1}$. Now K is a f. g. subgroup of F. Therefore, by Lemma 2, there exists $Q \lhd_f F$ such that there is a positive integer r such that to each positive integer s there is $Q_s \lhd_f F$ with the property that $KQ_s \cap \langle x^k \rangle = Q_s \cap \langle x^k \rangle = \langle x^{krs} \rangle$. In fact, since $[F : Q]$ is finite, we can by Lemma 3 assume that Q is characteristic of f. i. in F. This implies $Q \lhd_f G$. By Lemma 1, to each element $x^l h_i^{-1}$, where $0 \leq l < k$ and $1 \leq i \leq t$, except $x^0 h_1^{-1}$, there exists $M_{l,i} \lhd_f G$ such that $KM_{l,i} \cap \langle x^k \rangle x^l h_i^{-1} = \emptyset$. Let Y be the intersection of all the $M_{l,i}$ and Q. Clearly $Y \lhd_f G$ and $Y \subseteq Q$. Thus, by Lemma 3, there exists a positive integer r' such that to each positive integer s there is a verbal subgroup Y_s of f. i. in Y such that $KY_s \cap \langle x^k \rangle = \langle x^{kr's} \rangle$. We shall show that $HY_s \cap \langle x \rangle = Y_s \cap \langle x \rangle = \langle x^{kr's} \rangle$. Let $u \in HY_s \cap \langle x \rangle$. Then $u = y_s h = y_s v h_i$, where $y_s \in Y_s$, $h \in H$ and $v \in K$. Since $u \in \langle x \rangle$, $u = x^{nk} \cdot x^l$ where n is an integer and $0 \leq l < k$. This implies $y_s v \in KY_s \cap \langle x^k \rangle x^l h_i^{-1} = \emptyset$. This is impossible unless $l = 0$ and $i = 1$. Thus $u = y_s v \in KY_s \cap \langle x^k \rangle = \langle x^{kr's} \rangle$. Hence $HY_s \cap \langle x \rangle = \langle x^{kr's} \rangle$. Let $\rho = kr'$. Then Y_s is the required normal subgroup of f. i. in G for each positive integer s.

Lemmas 1 and 4 correspond to Propositions 2 and 1 of [2]. Our main result follows immediately.

THEOREM 5. *Let $G = A *_C B$, where C is cyclic and A, B are* f. g. *free-by-finite groups. Then G is subgroup separable.*

PROOF. We note that finite extensions of subgroup separable groups are subgroup separable. Since free groups are subgroup separable, both A and B are subgroup separable. If C is finite then, by Lemma 3 [1], G is subgroup separable. Therefore we can assume $C = \langle x \rangle$ is infinite. By Lemmas 1 and 4, and applying the argument of the lemma of [2], A, B and $\langle x \rangle$ satisfy the BBS criterion. It follows that G is subgroup separable.

Since *HNN* extensions of finite groups are free-by-finite, we have:

COROLLARY 5.1. *Generalized free products of HNN extensions of finite groups amalgamating a cyclic subgroup are subgroup separable.*

Noting that free products of finite numbers of cyclic groups are f. g. free-by-finite, it follows immediately that generalized free products of free products of cyclic groups amalgamating a cyclic subgroup are subgroup separable. In fact we prove a further generalization of the above result.

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THEOREM 6. *Let* $G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n\ ;\ a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}, (uv)^t \rangle$, *where* $t \geq 1$, *and* u, v *are words on* $\{a_1, \ldots, a_m\}$ *and* $\{b_1, \ldots, b_n\}$ *respectively. Then* G *is subgroup separable.*

PROOF: CASE 1. At least one of u and v, say v is of infinite order. Let $X = \langle x, a_1, \ldots, a_m\ ;\ a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, (ux)^t \rangle$, $t \geq 1$. Applying Tietze transformation,

$$\begin{aligned} X &= \langle x, a_1, \ldots, a_m, w\ ;\ a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, (ux)^t, w = ux \rangle \\ &= \langle a_1, \ldots, a_m, w\ ;\ a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, w^t \rangle. \end{aligned}$$

It follows that X is a free product of cyclic groups, whence free-by-finite. Let $B = \langle b_1, \ldots, b_n\ ;\ b_1^{\beta_1}, \ldots, b_n^{\beta_n} \rangle$. Then B is free-by-finite. Thus, by Theorem 5, $G = X *_{x=v} B$ is subgroup separable.

CASE 2. Both u and v are of finite orders. Then u is a conjugate of, say, a_1^k and v is a conjugate of, say, b_1^l. Thus by a change of generators we can assume $u = a_1^k$ and $v = b_1^l$. This implies $G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n\ ;\ a_1^{\alpha_1}, \ldots, a_m^{\alpha_m}, b_1^{\beta_1}, \ldots, b_n^{\beta_n}, (a_1^k b_1^l)^t \rangle$. Let $c = (\alpha_1, k)$ and $d = (\beta_1, l)$. Let $W = \langle x, y\ ;\ x^\gamma, y^\delta, (x^r y^s)^t \rangle$, where $\gamma = \frac{\alpha_1}{c}$, $\delta = \frac{\beta_1}{d}$, $r = \frac{k}{c}$ and $s = \frac{l}{d}$. Since $(\gamma, r) = (\delta, s) = 1$, $W \approx W_1 = \langle x, y\ ;\ x^\gamma, y^\delta, (xy)^t \rangle$. Since W_1 is either finite or is a finite extension of a surface group which is subgroup separable by [2] or [7], W_1 is therefore subgroup separable, whence W is subgroup separable. Let $L = \langle a_1, a_1^{\alpha_1} \rangle *_{a_1^c = x} W$. Since $|a_1^c| = |x|$ is finite, L is subgroup separable. Now, let $M = L *_{y = b_1^d} \langle b_1\ ;\ b_1^{\beta_1} \rangle$. Again M is subgroup separable. Let $P = \langle a_2, \ldots, a_m, b_1, \ldots, b_n\ ;\ a_2^{\alpha_2}, \ldots, a_m^{\alpha_m}, b_2^{\beta_2}, \ldots, b_n^{\beta_n} \rangle$. Then it is not difficult to see $G = P * M$. Hence G is subgroup separable. This completes the proof.

Theorem 6 generalizes a result of Rosenberg [8] where it was proved that groups of the form G are RF. Since subgroup separable groups have solvable generalized word problem, Theorem 6 also generalizes Theorem 4.8 [9].

REMARK. In Theorem 6, if the elements u, v of G are restricted to be of infinite order then G is called a *group of F-type* by Fine and Rosenberger [3]. Thus Theorem 6 answers a question put to the second author by Rosenberger whether groups of F-type are subgroup separable.

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