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NEARLY OPEN, ACCORDING TO A SUBSET, LINEAR MAPS WHICH ARE OPEN

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ABSTRACT. For a linear map we introduce two notions of being open according to a subset. With these two definitions, we characterize the linear, nearly open mappings which are open. Thus we generalize a famous theorem of V. Pták.

Introduction. As a generalization of the classical notion of open linear mappings, the author studied in [3] and [4] the notion of open mappings according to a subset A of the value space which was introduced by P. Lévine. This notion was shown in [3] to be related to the closedness of the image of a closed convex set by a linear mapping. The applications to mathematical programming were investigated in [4].

In this paper we prove a generalization of Pták's theorem [5, Th. 4.1, p. 54] (also [6, Th. IV.8.3, p. 163]) with respect to this notion of open mappings. In section 2, we introduce a new notion of open linear mappings according to a subset of the definition space and give some characterizations of it in the dual space. Besides the relation between the former and the latter definitions, we obtain another generalization of Pták's theorem.

1. **Open mappings according to a subset of the value space.** We consider two real, Hausdorff, locally convex topological vector spaces (L.C.T.V.S.) E and F. Let u be a linear mapping from E into F. We denote by E' and F' the topological duals of E and F respectively, endowed with their weak topologies $\sigma(E', E)$ and $\sigma(F', F)$. For a given subset $A \subset E$, we define the polar set of A as the set $A^0 = \{y \in E' / \forall x \in A \ \langle x, y\rangle \geq -1\}$; A^{00} is the bipolar in the dual system $\langle E, E'\rangle$. The adjoint mapping of u is denoted, when it exists, by u'.

1.1 DEFINITION ([3, p. 413] and [4, p. 86]). Let A be a *subset of* F. We say that u is *open* (resp. nearly open) *according to* A, if for every closed, convex 0-neighborhood W which contains $u^{-1}(A^{00})$, there exists a closed, convex 0-neighborhood V which contains A such that:

$$u(E) \cap V \subset u(W) \qquad (\text{resp. } u(E) \cap V \subset \overline{u(W)}).$$

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It is easy to show that applying this definition to $A=\{0\}$ we obtain the usual notion of open (or nearly open) mapping (see [4, Prop. 2.5, p. 87]). We get two duality theorems.

1.2 THEOREM [4, p. 88]. *Let* A *be a* **subset of F,** *and* u *be a continuous linear mapping from* E *into* F. *Then the following assertions are equivalent*:

(i) u *is nearly open according to* A;

(ii) *For every equicontinuous subset* B *contained in* $u'(F')$, *there exists a closed, convex, equicontinuous subset* B' *in* F' *such that* $B'\subset A^0$ *and* $\overline{u'(A^0)}\cap B\subset u'(B')$;

(iii) *For every closed, convex, equicontinuous subset* B *in* E', *there exists a closed, convex, equicontinuous subset* B', *contained in* A^0, *such that* $u'(A^0)\cap B\subset u'(B')$ *and* $u'(A^0)$ *is closed in* $u'(F')$ (*for the induced topology*).

1.3 THEOREM [3, p. 413] and [4, p. 91]. *Let* A *be a* **subset of F,** *and* u *be a continuous linear mapping from* E *into* F. *Then the following assertions are equivalent*:

(i) u *is open according to* A;

(ii) *For every closed, convex, equicontinuous subset* B *in* E', *there exists a closed, convex, equicontinuous subset* B', *contained in* A^0, *such that* $\overline{u'(A^0)}\cap B\subset u'(B')$;

(iii) *For every closed, convex, equicontinuous subset* B *in* E', *there exists a closed, convex, equicontinuous subset* B', *contained in* A^0, *such that* $u'(A^0)\cap B\subset u'(B')$ *and* $u'(A^0)$ *is closed.*

1.4 REMARK. Assuming that $A=\{0\}$ the above theorems reduce to known results, see [4, Ch. XIII]. Theorem 1.3 in particular is a classical one, see Schaefer [6, Th. IV. 7.5, p. 158]. One can find Theorem 1.2 and some related results in De Wilde [1, Ch. VI]. More generally, for linear relations, many close results are given by Mennicken and Sagraloff [2].

Observing that $(u(A))^0=u'^{-1}(A^0)$ and $u'(u'^{-1}(A^0))=A^0\cap u'(F')$, it is not difficult to deduce from Theorems 1.2 and 1.3 the following proposition:

1.5 PROPOSITION. *Let* A *be a* **subset of E** *and* u *be a continuous, linear mapping from* E *into* F. *Then the assertions* (i) *and* (ii) *are equivalent as well as the assertions* (iii) *and* (iv).

(i) u *is nearly open according to* $u(A)$;

(ii) *For every closed, convex, equicontinuous subset* B *in* E'; *there exists a closed, convex, equicontinuous subset* B', *contained in* $u'^{-1}(A^0)$, *such that* $A^0\cap u'(F')\cap B\subset u'(B')$;

(iii) u *is open according to* $u(A)$;

(iv) *For every closed, convex, equicontinuous subset* B *in* E', *there exists a closed, convex, equicontinuous subset* B', *contained in* $u'^{-1}(A^0)$, *such that* $A^0\cap u'(F')\cap B\subset u'(B')$ *and* $A^0\cap u'(F')$ *is closed.*

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We are now ready to obtain our result which contains, for $A=\{0\}$, Theorem 4.1 of Pták [5, p. 54], see also [6, Th. IV.8.3, p. 163].

1.6 THEOREM. *Let E be a* L.C.T.V.S. *and A a nonempty* **subset of E**. *Then the following assertions are equivalent*:

(i) *Every continuous, nearly open according to* $u(A)$, *linear mapping u from E into some* L.C.T.V.S. *F is open according to* $u(A)$;

(ii) *For any vector subspace M of E′ such that* $A^0\cap M\cap V^0$ *is closed for every* 0-*neighborhood V in E, one has* $A^0\cap M=\overline{A^0\cap M}$.

Proof. We begin by (ii) ⇒ (i). Let u be a continuous, nearly open according to $u(A)$, linear mapping. Then we have for adequate equicontinuous subsets $A^0\cap u'(F')\cap B\subset u'(B')$ (Prop. 1.5). Since $B'\subset u'^{-1}(A^0)$ it follows that $A^0\cap u'(F')\cap B=u'(B')\cap B$. This last subset is closed because B' is weak *-compact and u' is weakly continuous. Setting $u'(F')=M$ we get from (ii) that $A^0\cap u'(F')$ is closed, whence u is open according to $u(A)$ (Prop. 1.5).

We attack now (i) ⇒ (ii). Let M be a subspace of E' such that $A^0\cap M\cap V^0$ is closed for every 0-neighborhood V in E. Let us consider the subspace H generated by $A^0\cap M$. One has $H\subset M$ and $A^0\cap M=A^0\cap H$. We denote by u the quotient map from E onto E/H^0. The family of subsets of the type $u((A^0\cap M\cap V^0)^0)$ where V is closed, convex and belongs to a 0-neighborhood basis in E, defines on E/H^0 a locally convex topology which we denote by t. We claim that E/H^0 endowed with t is a Hausdorff space. To see that, take $u(x)\in E/H^0$ and $u(x)\neq 0$. Thus there exists $y\in A^0\cap M$ such that $\langle x,y\rangle\neq 0$. Then there exists a convex, circled 0-neighborhood V such that $y\in A^0\cap M\cap V^0$. Taking x' equal either to x or to $-x$, one can find an integer n such that $n\langle x',y\rangle<-1$. It follows that $nx'\notin(A^0\cap M\cap V^0)^0$ whence, $nx'\notin(A^0\cap M\cap V^0)^0+H^0$ because this last subset is contained in $(A^0\cap M\cap V^0)^0$. Finally $u(x)\notin(n^{-1}u((A^0\cap M\cap V^0)^0))\cap(n^{-1}u((-A^0\cap M\cap V^0)^0))$ which is a 0-neighborhood for t.

From the assumption $A^0\cap M\cap V^0$ is closed and hence weak *-compact, its polar set (in the dual system $\langle E/H^0,H\rangle$) is $u((A^0\cap M\cap V^0)^0)$ which consequently is a 0-neighborhood for the MacKey topology $\mathcal{T}(E/H^0,H)$. Thus the topological dual H' of E/H^0 (endowed with t) is contained in H.

Let us show that the quotient map u from E onto E/H^0 (endowed with t) is continuous and nearly open according to $u(A)$. It is continuous because $V\subset(A^0\cap M\cap V^0)^0\subset u^{-1}(u((A^0\cap M\cap V^0)^0))$.

According to Definition 1.1, let W be a closed convex 0-neighborhood containing $u^{-1}((u(A))^{tt})$, where $(u(A))^{tt}$ is the bipolar of $u(A)$ in the dual system $\langle E/H^0,H'\rangle$. One has $(u(A))^{tt}=u((A^0\cap H')^0)\supset u(A^{00})$ and $A^{00}+H^0\subset u^{-1}(u(A^{00}))$ which implies that $A^{00}+H^0\subset W$, whence $W^0\subset A^0\cap\bar{H}$. Finally $H'\cap W^0\subset H'\cap A^0\cap\bar{H}\cap W^0\subset A^0\cap H\cap W^0=A^0\cap M\cap W^0$, hence $u((A^0\cap M\cap W^0)^0)\subset u((H'\cap W^0)^0)=(u(W))^{tt}$ which is also the closure of $u(W)$ for the

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topology t. On the other hand one has $u(A) \subset u((A^0 \cap M \cap W^0)^0)$ and u is nearly open according to $u(A)$.

Now from (i) u is open according to $u(A)$, so that $A^0 \cap u'(H')$ is closed by Proposition 1.5(iv). We have $u'(H') = H'$ and $H'^0 = H^0$ [6, Cor. IV. 2.3, p. 129]. It follows that $\overline{A^0 \cap H} = (A^0 \cap H)^{00} = (A^0 \cap H')^{00} = \overline{A^0 \cap H'} = A^0 \cap H' \subset A^0 \cap H$. Thus $A^0 \cap H = A^0 \cap M$ is closed. Q.E.D.

1.7 Remark. Although the above proof is inspired in its second part from that of Pták [5, p. 54], is is noteworthy that the topology t is no longer compatible with the pairing $\langle E/H^0, H\rangle$.

It should be more intutitive to obtain a similar theorem with an openness notion defined on A instead of $u(A)$. It is our purpose in the next section.

2. **Open mappings according to a subset in the definition space.** We keep the same notation as in the previous section.

2.1 Definition. Let A be a *subset of* E. We say that u is *open* (resp. nearly open) *according to* A, if for every closed, convex 0-neighborhood W which contains $A + \ker u$, there exists a closed, convex 0-neighborhood V containing $u(A)$ such that:

$$u(E) \cap V \subset u(W) \qquad (\text{resp. } u(E) \cap V \subset \overline{u(W)}).$$

2.2 Convention. In what follows, when we shall say that a linear mapping u from E into F is open (or nearly open) according to a subset A, we shall refer to Definition 1.1 if $A \subset F$ and to Definition 2.1 if $A \subset E$.

2.3 Remarks. a) Applying Definition 2.1 to $A = \{0\}$ we obtain the usual definition of an open (or nearly open) mapping (see [4, Prop. 2.5, p. 87]).

b) It is equivalent to say that u is open (or nearly open) according to $A \subset E$ or according to $A + \ker u$.

c) Assume that u is continuous. Then, since the 0-neighborhoods W and V in Definition 2.1 are closed and convex, it does not matter to replace A by A^{00} in the Definition 2.1. (It follows from the equalities $\overline{A^{00} + \ker u} = (A + \ker u)^{00}$ and $(u(A))^{00} = \overline{u(A^{00})}$).

2.4 Proposition. *Assume that u is a continuous linear mapping from E into F. If u is open* (*resp. nearly open*) *according to a subset A of E, then u is open* (*resp. nearly open*) *according to* $u(A)$, *or equivalently to* $u(A^{00})$.

Proof. Let W be a closed, convex 0-neighborhood containing $u^{-1}((u(A))^{00})$. Since we have $(u(A))^{00} = \overline{u(A^{00})}$ and $\overline{A^{00} + \ker u} \subset u^{-1}(\overline{u(A^{00})})$ we obtain $W \supset \overline{A^{00} + \ker u}$. It follows that there exists a closed, convex 0-neighborhood V containing $u(A)$ such that $u(E) \cap V \subset u(W)$. Observing that $(u(A^{00}))^{00} = (u(A))^{00}$ the same proof holds for $u(A^{00})$. Q.E.D.

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2.5 Theorem. *Let A be a* **subset of E,** *and u be a continuous, linear mapping from E into F. Then the following assertions are equivalent*:

(i) *u is nearly open according to A (or A^{00})*;

(ii) *u is nearly open according to $u(A)$ (or $u(A^{00})$)*;

(iii) *For every closed, convex, equicontinuous subset B in E′, there exists a closed, convex, equicontinuous subset B′ contained in $u'^{-1}(A^0)$ such that*:

$$A^0 \cap u'(F') \cap B \subset u'(B').$$

Proof. We have (i) ⇒ (ii) (Prop. 2.4). Moreover it follows from Proposition 1.5 that (ii) and (iii) are equivalent. At last let us show that (iii) ⇒ (i). Let W be a closed, convex, 0-neighborhood containing $A^{00} + \ker u$. Then W^0 is equicontinuous and there exists $B' \subset u'^{-1}(A^0)$ such that $W^0 \cap A^0 \cap u'(F') \subset u'(B')$. Setting $V = B'^0$ it follows that $u^{-1}(V) \subset (W^0 \cap A^0 \cap u'(F'))^0$ or $u(E) \cap V \subset u((W^0 \cap A^0 \cap u'(F'))^0) \subset (u'^{-1}(W^0 \cap A^0 \cap u'(F')))^0$. But since $A^{00} \subset W$ we have $W^0 \cap u'(F') = W^0 \cap A^0 \cap u'(F')$. Hence $u'^{-1}(W^0 \cap A^0 \cap u'(F')) = u'^{-1}(W^0)$ which implies $u(E) \cap V \subset (u'^{-1}(W^0))^0 = \overline{u(W)}$. Noticing that $V = B'^0$ is a closed, convex 0-neighborhood which contains $(u'^{-1}(A^0))^0 = \overline{u(A^{00})}$, the proof is completed.

2.6 Theorem. *Let A be a* **subset of E,** *and u a continuous linear mapping from E into F. Then the following assertions are equivalent*:

(i) *u is open according to A*;

(ii) *For every closed, convex, equicontinuous subset B in E′, there exists a closed, convex, equicontinuous subset B′ contained in $u'^{-1}(A^0)$ such that*:

$$A^0 \cap \overline{u'(F')} \cap B \subset u'(B');$$

(iii) *For every closed, convex, equicontinuous subset B in E′, there exists a closed, convex, equicontinuous subset B′ contained in $u'^{-1}(A^0)$ such that*:

$$A^0 \cap u'(F') \cap B \subset u'(B') \quad \textit{and} \quad A^0 \cap u'(F') = A^0 \cap \overline{u'(F')}.$$

Proof. Let us show that (i) ⇒ (ii). Let B be a closed, convex, equicontinuous subset, we set $W = (\overline{u'(F')} \cap A^0 \cap B^{00})^0$. It follows that $\overline{A^{00} + \ker u} = (A^0 \cap \overline{u'(F')})^0 \subset W$. Hence there exists $V \supset u(A)$ such that $u(E) \cap V \subset u(W)$ which is equivalent to $u^{-1}(V) \subset W + \ker u$. But $\ker u$ is contained in W which implies that $W^0 \subset \overline{u'(F')}$ and $(W + \ker u)^0 = W^0 \cap \overline{u'(F')} = W^0$, whence $W \subset W + \ker u \subset (W + \ker u)^{00} \subset W^{00} = W$. Thus we have $u^{-1}(V) \subset W$ which implies that $W^0 \subset \overline{u'(V^0)} = u'(V^0)$ because V^0 is weak *-compact and u' continuous. Then it follows that $\overline{u'(F')} \cap A^0 \cap B \subset W^0 \subset u'(V^0)$, and setting $B' = V^0$ we have our first implication, noticing that $\overline{u(A^{00})} \subset V$ entails $V^0 \subset u'^{-1}(A^0)$.

We attack now the reverse implication. Let W be a closed, convex 0-neighborhood containing $A + \ker u$. By duality it follows that $W^0 \subset A^0 \cap \overline{u'(F')}$; from (ii) we get $W^0 = A^0 \cap \overline{u'(F')} \cap W^0 \subset u'(B')$ with $B' \subset u'^{-1}(A^0)$. Setting $B'^0 = V$ we obtain $u^{-1}(V) \subset W^{00} = W$ whence $u(E) \cap V \subset u(W)$.

At last let us show that (ii) implies (iii), the reverse implication being obvious. It is clear that a given $\hat{y} \in A^0 \cap \overline{u'(F')}$ is contained in a closed, convex, equicontinuous subset of E'. Thus $\hat{y} \in u'(B') \subset u'(F')$ which implies that $\hat{y} \in A^0 \cap u'(F')$ and $A^0 \cap u'(F') = A^0 \cap \overline{u'(F')}$. Q.E.D.

2.7 Remark. It is the same property for u to be nearly open according to A as to $u(A)$ (Th. 2.5). However the situation is different for the openness. Proposition 1.5 and Theorem 2.6 show that if a set satisfies $A^0 \cap u'(F') = \overline{A^0 \cap u'(F')} \neq A^0 \cap \overline{u'(F')}$, then the linear map u is open according to $u(A)$ and is not open according to A. In other words the converse of Proposition 2.4 is not true. It also means that there exists, at least, one closed, convex 0-neighborhood W satisfying $\overline{A^{00} + \ker u} \subset W \subset u^{-1}\overline{(u(A^{00}))}$ (the last inclusion being a strict one).

We can now give a result which also subsumes, for $A = \{0\}$, the Pták theorem [5, Th. 4.1, p. 54]. One will notice that if it is more natural to confine ourself in the space E for the definition of the openness, it unfortunately reduces the number of the nearly open maps which are open.

2.8. Theorem. *Let E be a* L.C.T.V.S. *and A a nonempty* **subset of E**. *Then the following assertions are equivalent*:

(i) *Every continuous, nearly open according to A, linear mapping u from E into some* L.C.T.V.S. *F is open according to A*;

(ii) *For any vector subspace M of E′ such that* $A^0 \cap M \cap V^0$ *is closed for every 0-neighborhood V in E, one has* $A^0 \cap M = A^0 \cap \bar{M}$.

Proof. It suffices to follow the proof of Theorem 1.6 using the previous duality Theorems 2.5 and 2.6. Q.E.D.

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