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MAXIMAL d-IDEALS IN A RIESZ SPACE

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1. Introduction. We recall that the ideal I in an Archimedean Riesz space L is called a d-ideal whenever it follows from $f \in I$ that $\{f\}^{dd} \subset I$. Several authors (see **[4]**, **[5]**, **[6]**, **[12]**, **[13]**, **[15]** and **[18]**) have considered the class of all d-ideals in L, but the set $\mathscr{J}_d$ of all maximal d-ideals in L has not been studied in detail in the literature. In **[12]** and **[13]** the present authors paid some attention to certain aspects of the theory of maximal d-ideals, however neglecting the fact that $\mathscr{J}_d$, equipped with its hull-kernel topology, is a structure space of the underlying Riesz space L.

The main purpose of the present paper is to investigate the topological properties of $\mathscr{J}_d$ and to compare $\mathscr{J}_d$ to other structure spaces of L, such as the space $\mathscr{M}$ of minimal prime ideals and the space $\mathscr{Q}^e$ of all e-maximal ideals in L (where $e > 0$ is a weak order unit). This study of $\mathscr{J}_d$ makes it for instance possible to place some recent results of F. K. Dashiell, A. W. Hager and M. Henriksen [7], concerning the order completion of $C(X)$ in the general setting of Riesz spaces.

After establishing some preliminary results in Section 2, it is proved in Section 3 that for any uniformly complete Riesz space L with weak order unit the space $\mathscr{J}_d$ is a compact quasi-F-space. This result makes it possible to use $\mathscr{J}_d$ for the description of the order completion of L, which is done in Section 4. Furthermore we consider what conditions have to be imposed on L in order that $\mathscr{J}_d$ is an F-space, basically disconnected or extremally disconnected respectively. In Section 5 it is investigated what effect certain homeomorphisms between $\mathscr{J}_d$, $\mathscr{M}$, and $\mathscr{Q}^e$ have on L.

Finally in Section 6, we interpret some of the results in the case that L is the Riesz space $C(X)$. It turns out that $\mathscr{J}_d$ is precisely the minimal quasi-F-cover $K(X)$ of X, as introduced in [7]. We can apply therefore the results of Sections 4 and 5 to the space $K(X)$. Moreover, it is shown that the representation of the order completion of L which we have obtained in Section 4 is in the case $L = C(X)$ precisely the realization of the order completion as obtained in [7].

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2. Some preliminary results. All Riesz spaces considered in this paper are supposed to be Archimedean. For the terminology and unproved properties of Riesz spaces we refer to [**16**]. Throughout this paper relatively uniformly complete (closed) is called uniformly complete (closed).

Let L be an Archimedean Riesz space.

Definition 2.1 (see [**12**], Section 4). The ideal I in L is called a *d-ideal* if it follows from $f \in I$ that $\{f\}^{dd} \subset I$.

It is immediate from the definition that the ideal I is a d-ideal if and only if $f \in I$, $g \in L$ and $\{f\}^{dd} = \{g\}^{dd}$ imply $g \in I$. All σ-ideals in L (and hence all bands) are d-ideals. Moreover, every minimal prime ideal is a d-ideal. For any non-empty subset D of L, the d-ideal $E(D)$ generated by D is the intersection of all d-ideals containing D. The proof of the next lemma is routine.

LEMMA 2.2.

$$E(D) = \{f \in L: \exists d_1, \ldots, d_{n_f} \in D \text{ such that } f \in \{\,|d_1| \vee \ldots \vee |d_{n_f}|\,\}^{dd}\}.$$

In particular, every principal d-ideal is of the form $\{f\}^{dd}$.

The proper d-ideal J in L is called a maximal d-ideal if L is the only d-ideal in which J is properly contained. We denote the set of all maximal d-ideals by $\mathcal{J}_d$. Every maximal d-ideal is a (proper) prime ideal (see [**12**], Section 6). If L has in addition a weak order unit $e > 0$, then, by a Zorn argument, every proper d-ideal is contained in some maximal d-ideal.

We equip $\mathcal{J}_d$ with its hull-kernel topology by choosing as a base for the open sets all subsets of the form $\{J\}_u = \{J \in \mathcal{J}_d : u \notin J\}$ with $0 \leqq u \in L$ ([**16**], Section 36). The closed sets in this topology are precisely the hulls of subsets of L, i.e., sets of the form $h(D) = \{J \in \mathcal{J}_d : D \subset J\}$. Clearly, we can replace D in this formula by the d-ideal $E(D)$ generated by D. From the next proposition it follows that $\mathcal{J}_d$ is a structure space of L.

PROPOSITION 2.3. *Let L be an Archimedean Riesz space with weak order unit* $e > 0$.

(i) $\mathcal{J}_d$ *is a compact Hausdorff space.*

(ii) $\cap \{J : J \in \mathcal{J}_d\} = \{0\}$.

Proof. (i) This is similar to the proof of [**16**], Theorem 36.4.

(ii) For any $0 < u \in L$ there exists $n \in \mathbf{N}$ such that $(nu - e)^+ > 0$.

Since

$$(nu - e)^+ \notin I = \{ (nu - e)^- \}^{dd},$$

the ideal I is, as a proper d-ideal, contained in some maximal d-ideal J. We claim that $u \notin J$. Indeed, $u \in J$ would imply that $(nu - e)^+ \in J$. Combined with $(nu - e)^- \in J$ this would result in $e \in J$, a contradiction.

For future purposes we collect finally some simple properties of maximal d-ideals. The proof is omitted.

PROPOSITION 2.4. *In any Archimedean Riesz space L with a weak order unit $e > 0$ the following statements hold.*

(i) $\{u\}^{dd} = \cap \{J \in \mathscr{J}_d : u \in J\}$ *for all* $0 \leqq u \in L$.

(ii) $\{J\}_u \subset \{J\}_v$ *if and only if* $\{u\}^{dd} \subset \{v\}^{dd}$ $(0 \leqq u, v \in L)$.

In [**13**], Remark 7.6 (iii) it is observed that in any Archimedean Riesz space with a weak order unit every σ-ideal is an intersection of maximal d-ideals, so Proposition 2.4 (i) is a special case of the former result.

3. The quasi-F-space $\mathscr{J}_d$. The main purpose of this section is to prove that for any uniformly complete Archimedean Riesz space L with a weak order unit the space $\mathscr{J}_d$ is a quasi-F-space.

We recall that a completely regular Hausdorff space X is said to be a quasi-F-space whenever every bounded continuous function on a dense cozero-set has a continuous extension to the whole of X, i.e., every dense cozero-set in X is C_b-embedded. Quasi-F-spaces were orginally defined in [7], Definition 3.6. The following lemma gives a simple characterization of quasi-F-spaces.

LEMMA 3.1 ([**18**], Lemma 11.9). *Let X be a completely regular Hausdorff space. The following are equivalent.*

(i) *X is a quasi-F-space.*

(ii) *For any two zero-sets Z_1 and Z_2 in X with disjoint interiors, the sets* int Z_1 *and* int Z_2 *are completely separated by a continuous function.*

The next theorem gives more information about the topological structure of $\mathscr{J}_d$.

THEOREM 3.2. *Let L be an Archimedean uniformly complete Riesz space with weak order unit $e > 0$. Then the space $\mathscr{J}_d$ is a quasi-F-space.*

Proof. We use the criterion of Lemma 3.1, so suppose that Z_1 and Z_2 are zero-sets in $\mathscr{J}_d$ such that

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$$(\text{int } Z_1) \cap (\text{int } Z_2) = \emptyset.$$

Following [22], Lemma 6.2, there exist $0 \leqq u_n \in L$, $0 \leqq v_n \in L$ $(n = 1, 2, \dots)$ such that

$$Z_1 = \bigcap_{n=1}^{\infty} \{J\}_{u_n}, \quad Z_2 = \bigcap_{n=1}^{\infty} \{J\}_{v_n}.$$

Since Z_1 and Z_2 are closed, $Z_1 = h(I_1)$ and $Z_2 = h(I_2)$ for appropriate d-ideals I_1 and I_2 in L. From $h(I_1) \subset \{J\}_{u_n}$ it follows that $E(I_1, u_n) = L$ $(n = 1, 2, \dots)$. There exists, therefore, an element $0 \leqq w_n \in I_1$ such that $e \in \{u_n \vee w_n\}^{dd}$ (see Lemma 2.2), i.e., $u_n \vee w_n$ is a weak order unit. Without loss of generality we may assume $0 < w_n \leqq e$ $(n = 1, 2, \dots)$. Analogously, there exist elements $z_n \in I_2$, $0 \leqq z_n \leqq e$ such that $v_n \vee z_n$ are weak order units in L $(n = 1, 2, \dots)$.

Now define

$$p = \sum_{n=1}^{\infty} 2^{-n} w_n, \quad q = \sum_{n=1}^{\infty} 2^{-n} z_n$$

(e-uniformly convergent series). Clearly $p \in I_1^{dd}$ and $q \in I_2^{dd}$. We assert that $p \vee q$ is a weak order unit in L. To this end, take $0 \leqq y \in L$ such that $y \wedge (p \vee q) = 0$. It follows that

$$y \wedge w_n = y \wedge z_n = 0 \quad (n = 1, 2, \dots).$$

Since $u_n \vee w_n$ is a weak order unit we have $y \in \{u_n \vee w_n\}^{dd}$. Hence,

$$y \wedge \{k(u_n \vee w_n)\} \uparrow_k y,$$

so

$$y \wedge (k\, u_n) \uparrow_k y$$

which implies $y \in \{u_n\}^{dd}$. By Proposition 2.4 (ii), $\{J\}_y \subset \{J\}_{u_n}$. Similarly, it is shown that $\{J\}_y \subset \{J\}_{v_n}$. It follows easily that

$$\{J\}_y \subset \text{int } (Z_1) \cap \text{int } (Z_2),$$

so $\{J\}_y = \emptyset$, i.e., $y = 0$ (Proposition 2.3 (ii)). Therefore $p \vee q$ is a weak order unit.

Now it follows that the sets $G_1 = h(p)$ and $G_2 = h(q)$ are disjoint closed sets in $\mathcal{J}_d$. We claim that int $Z_1 \subset G_1$. In order to prove this inclusion, take $J_0 \in \text{int } Z_1$. There exists $0 < z \in L$ such that $J_0 \in \{J\}_z \subset Z_1 = h(I_1)$. For any $0 \leqq x \in I_1$ and any $J \in \mathcal{J}_d$ we have $z \wedge x \in J$. Indeed, if $J \in \{J\}_z$, then $x \in I_1 \subset J$ implies $z \wedge x \in J$. On the other hand, if $J \notin \{J\}_z$ then $z \in J$ and so $z \wedge x \in J$. By Proposition 2.3 (ii), $z \wedge x = 0$. This holds for all $0 \leqq x \in I_1$, thus $z \in I_1^d$. Since $p \in I_1^{dd}$ we derive $z \wedge p = 0$. Therefore, $z \notin J_0$ implies $p \in J_0$, i.e., $J_0 \in h(p) = G_1$. The inclusion int $Z_1 \subset G_1$ has been proved. In like manner int $Z_2 \subset G_2$.

The space $\mathscr{J}_d$, being Hausdorff and compact, is a normal topological space, so, by Urysohn's lemma, G_1 and G_2 are completely separated. Hence the sets int Z_1 and int Z_2 are completely separated as well and the proof is complete.

As is well known, any continuous mapping f from a topological space X into the extended real number system $\mathbf{R}^\infty$ is called an extended continuous function on X if the set $\{x \in X: |f(x)| < \infty\}$ is dense in X. The set of all such functions is denoted by $C^\infty(X)$. With respect to pointwise operations $C^\infty(X)$ is a lattice, but in general not an algebra. In [**11**], Proposition 2.2, M. Henriksen and D. G. Johnson proved that in case X is a compact topological space, $C^\infty(X)$ is an algebra if and only if each open dense F_σ-set is C_b-embedded. Hence if X is a compact quasi-F-space, then $C^\infty(X)$ is a uniformly complete f-algebra. In fact, $C^\infty(X)$ is even order complete in this case. We recall the definition.

Definition 3.3. The Archimedean Riesz space L is called *order complete* (also *order Cauchy complete*) whenever every order Cauchy sequence $\{f_n\}_{n=1}^\infty$ in L (i.e., a sequence for which there exist $p_n \downarrow 0$ with $|f_n - f_{n+k}| \leqq p_n$ for all n, k) has an order limit f (i.e., $|f_n - f| \leqq p_n$ for all n, which is denoted by $f_n \rightarrow f$).

Papangelou's criterion ([**19**], Lemma 2.1) states that L is order complete if and only if for all sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ for which $f_n \uparrow \leqq g_n \downarrow$ and $\inf (g_n - f_n) = 0$, there exists $h \in L$ such that $f_n \leqq h \leqq g_n$ for all n. Evidently, $f_n \uparrow h$ and $g_n \downarrow h$ in this case. Note that any order complete Riesz space is uniformly complete. For any completely regular Hausdorff space X the Riesz space $C(X)$ is order complete if and only if X is a quasi-F-space ([**7**], Theorem 3.7; for a different proof see [**18**], Theorem 11.8).

Using the above mentioned criterion and the fact that $C(X)$ is order complete whenever X is a quasi-F-space, the next theorem follows easily.

THEOREM 3.4. *For any compact quasi-F-space X, the f-algebra* $C^\infty(X)$ *is order complete.*

Applying this result to $\mathscr{J}_d$ we get the following.

COROLLARY 3.5. *For any Archimedean uniformly complete Riesz space L with a weak order unit $C^{\infty}(\mathcal{J}_d)$ is an order complete f-algebra with unit element.*

Remark 3.6. The Riesz space L is said to have the σ-order continuity property (σ-o.c.p.) whenever every positive linear mapping from L into an arbitrary Archimedean Riesz space is σ-order continuous (the σ-o.c.p. has been introduced, under a different name, by C. T. Tucker in [**23**]). It is proved in [**8**], Theorem 1.3 A that L has the σ-o.c.p. if and only if every uniformly closed ideal in L is a σ-ideal. Moreover, every uniformly complete Riesz space with the σ-o.c.p. is order complete (see [**18**], Corollary 11.5). It can be shown that $C^{\infty}(X)$ has the σ-o.c.p. for every compact quasi-F-space X. Since the proof is beyond the scope of the paper, we omit it.

4. The order completion of Archimedean Riesz spaces. Let L be an Archimedean uniformly complete Riesz space with weak order unit $e > 0$. Since $\mathcal{J}_d$ is a structure space of L, we can consider the Johnson-Kist representation of L in $C^{\infty}(\mathcal{J}_d)$, i.e., L is Riesz isomorphic to some Riesz subspace ${}^{\wedge}L$ of the order complete Riesz space $C^{\infty}(\mathcal{J}_d)$. In addition, ${}^{\wedge}e(J) = 1$ for all $J \in \mathcal{J}_d$ (see e.g. [**16**], Section 44). Let $L^{\#}$ be the ideal generated by ${}^{\wedge}L$ in $C^{\infty}(\mathcal{J}_d)$, i.e.,

$$L^{\#} = \{g \in C^{\infty}(\mathcal{J}_d): |g| \leqq |{}^{\wedge}f| \text{ for some } f \in L\}.$$

The main purpose of this section is to show that $L^{\#}$ is a realization of the order completion of L. We recall the definition.

Definition 4.1. Let L be an Archimedean Riesz space. The order complete Riesz space K is called an *order completion of* L if

(i) L is Riesz isomorphic to a Riesz subspace $\tilde{L}$ of K,

(ii) for every $f \in K$ there exist sequences $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ in $\tilde{L}$ such that $g_n \uparrow f$ and $h_n \downarrow f$.

In the following we identify L and $\tilde{L}$, so we embed L as a Riesz subspace in K. It follows immediately from (ii) that L is order dense in K, i.e., for every $0 < u \in K$ there exists $0 < v \in L$ such that $v \leqq u$. This implies among other things that for any sequence $\{f_n\}_{n=1}^{\infty}$ in L satisfying $f_n \downarrow 0$ in L we also have $f_n \downarrow 0$ in K. Two order completions of L are Riesz isomorphic via a Riesz isomorphism which is the identity on L. Hence, it is justified to refer to K as "the" order completion of L. The order completion of L exists. Indeed, let $L^{\wedge}$ be the Dedekind completion of L and define

$$L_\sigma = \{f^\wedge \in L^\wedge : \exists \, \{h_n\}_{n=1}^\infty \text{ in } L \text{ and } v_n \downarrow 0 \text{ in } L \text{ such that } |f^\wedge - h_n| \leqq v_n, \, n \in \mathbf{N}\} = \{f^\wedge \in L^\wedge : \exists \, \{f_n\}_{n=1}^\infty, \{g_n\}_{n=1}^\infty \text{ in } L \text{ such that } f_n \uparrow f^\wedge, \, g_n \downarrow f^\wedge\},$$

as introduced by J. Quinn ([**20**], Section 3). In Theorem 4.1 (i) of the same paper it is shown that L_σ is the order completion of L. For any non-empty subset D of L_σ we shall denote the double disjoint complement in L_σ by $D^{dd(L_\sigma)}$. In order to avoid confusion we denote momentarily the space of all maximal d-ideals in L and L_σ by $\mathcal{J}_d(L)$ and $\mathcal{J}_d(L_\sigma)$ respectively.

LEMMA 4.2. *For any Archimedean uniformly complete Riesz space L with weak order unit $e > 0$ the spaces $\mathcal{J}_d(L)$ and $\mathcal{J}_d(L_\sigma)$ are homeomorphic.*

Proof. For any $J \in \mathcal{J}_d(L)$, let J_σ be the d-ideal generated by J in L_σ, i.e.,

$$J_\sigma = \{f^\wedge \in L_\sigma : \exists \, g \in J \text{ such that } f^\wedge \in \{g\}^{dd(L_\sigma)}\}.$$

We have $\{h\}^{dd(L_\sigma)} \cap L = \{h\}^{dd}$, as L is order dense in L_σ, and hence $J_\sigma \cap L = J$. We claim that J_σ is a maximal d-ideal in L_σ. To this end let N be a proper d-ideal in L containing J_σ. Obviously, $N \cap L$ is a d-ideal in L not containing e, so $J = N \cap L$. Take any $0 \leqq f^\wedge \in N$. There exist $u_n \in L (n = 1, 2, \dots)$ and $v \in L$ such that $0 \leqq u_n \uparrow f^\wedge \leqq v$. Obviously

$$u = \sum_{n=1}^\infty 2^{-n} u_n$$

(v-uniformly convergent series in L) is an element of J. Since $f^\wedge \in \{u\}^{dd(L_\sigma)}$ we get $f^\wedge \in J_\sigma$. This shows that $N = J_\sigma$ and thus $J_\sigma \in \mathcal{J}_d(L_\sigma)$.

The mapping $\alpha : \mathcal{J}_d(L) \to \mathcal{J}_d(L_\sigma)$ defined by $\alpha(J) = J_\sigma$ is a bijection. For any $0 \leqq g^\wedge \in L_\sigma$ there exist $w_n \in L (n = 1, 2, \dots)$ and $z \in L$ such that $0 \leqq w_n \uparrow g^\wedge \leqq z$. Using Proposition 2.4 (ii), it is not hard to show that the element

$$w = \sum_{n=1}^\infty 2^{-n} w_n$$

(z-uniformly convergent series in L) satisfies $\alpha^{-1}(\{J_\sigma\}_{g^\wedge}) = \{J\}_w$. Hence α is continuous. Since both $\mathcal{J}_d(L)$ and $\mathcal{J}_d(L_\sigma)$ are compact Hausdorff spaces, α is a homeomorphism.

We now consider the Johnson-Kist representation ${}^\wedge L_\sigma$ of L_σ in $C^\infty(\mathcal{J}_d(L_\sigma))$. In the next lemma we show that the Riesz subspace ${}^\wedge L_\sigma$ of $C^\infty(\mathcal{J}_d(L_\sigma))$ is even an ideal.

LEMMA 4.3 (compare [**3**], Proposition 2.8.). *$^\wedge L_\sigma$ is an ideal in $C^\infty(\mathcal{J}_d(L_\sigma))$.*

Proof. Observe first that $^\wedge L_\sigma$ separates the points of $\mathcal{J}_d(L_\sigma)$. Indeed, take $J_\sigma, J'_\sigma \in \mathcal{J}_d(L_\sigma)$ with $J_\sigma \neq J'_\sigma$. Since L_σ is order complete, the sum of two d-ideals in L_σ is again a d-ideal (see [**18**], Theorem 11.2), so $J_\sigma + J'_\sigma = L_\sigma$. Hence, $e = f + f'$ with $0 \leqq f \in J_\sigma$, $0 \leqq f' \in J'_\sigma$. Now $^\wedge f(J_\sigma) = 0$ implies $^\wedge(f')(J_\sigma) = 1$. On the other hand, $^\wedge(f')(J'_\sigma) = 0$, so $^\wedge(f')$ separates J_σ and J'_σ.

Consider the uniformly closed Riesz subspace

$$^\wedge L_{\sigma,b} = \{^\wedge f \in {}^\wedge L_\sigma : |^\wedge f| \leqq n\, ^\wedge e \text{ for some } n \in \mathbf{N}\}$$

of $C(\mathcal{J}_d(L_\sigma))$. Since $^\wedge L_{\sigma,b}$ separates the points, the Stone-Weierstrass theorem yields

$$^\wedge L_{\sigma,b} = C(\mathcal{J}_d(L_\sigma)).$$

In order to prove that $^\wedge L_\sigma$ is an ideal, it suffices to show that $0 \leqq g \leqq {}^\wedge f$, $^\wedge f \in {}^\wedge L_\sigma$, $g \in C^\infty(\mathcal{J}_d(L_\sigma))$ implies $g \in {}^\wedge L_\sigma$. For this purpose note that

$$0 \leqq g - g \wedge (n\, ^\wedge e) \leqq {}^\wedge u_n$$

with

$$^\wedge u_n = {}^\wedge f - (^\wedge f) \wedge (n\, ^\wedge e) \quad (n = 1, 2, \dots)$$

and that $^\wedge u_n \downarrow 0$ in L_σ. By the above $g \wedge (n\, ^\wedge e) \in {}^\wedge L_{\sigma,b}$. For $m \geqq n$ we have

$$0 \leqq g \wedge (m\, ^\wedge e) - g \wedge (n\, ^\wedge e) \leqq 2\, ^\wedge u_n$$

and hence $\{g \wedge (n\, ^\wedge e)\}_{n=1}^\infty$ is an order Cauchy sequence in $^\wedge L_\sigma$. There exists $^\wedge h \in {}^\wedge L_\sigma$ such that

$$g \wedge (n\, ^\wedge e) \rightarrow {}^\wedge h \quad \text{in } ^\wedge L_\sigma,$$

as L_σ is order complete. Observing that $^\wedge L_\sigma$ is order dense in $C^\infty(\mathcal{J}_d(L_\sigma))$, it follows readily that

$$g \wedge (n\, ^\wedge e) \rightarrow {}^\wedge h \quad \text{in } C^\infty(\mathcal{J}_d(L_\sigma))$$

as well. Uniqueness of order limits implies $g = {}^\wedge h$, hence $g \in {}^\wedge L_\sigma$.

As before, L is an Archimedean uniformly complete Riesz space with weak order unit $e > 0$. Consider the Johnson-Kist representation $^\wedge L_\sigma$ of

L_σ in $C^\infty(\mathcal{J}_d(L_\sigma))$. By identifying $\mathcal{J}_d(L)$ and $\mathcal{J}_d(L_\sigma)$ via the homeomorphism α of Lemma 4.2, we may consider ${}^\wedge L_\sigma$ as an ideal of $C^\infty(\mathcal{J}_d(L))$. For all $f \in L$ and $J \in \mathcal{J}_d(L)$ we have $(\beta e - f)^+ \in J$ whenever $(\beta e - f)^+ \in J_\sigma = \alpha(J)$. Hence

$$\begin{aligned}{}^\wedge f(J) &= \sup\{\beta:(\beta e - f)^+ \in J\} = \sup\{\beta:(\beta e - f)^+ \in J_\sigma\} \\ &= {}^\wedge f(J_\sigma).\end{aligned}$$

It follows that the representation of L, induced by the Johnson-Kist representation ${}^\wedge L_\sigma$ of L_σ (regarded as an ideal in $(C^\infty(\mathcal{J}_d(L)))$) is precisely the Johnson-Kist representation ${}^\wedge L$ of L in $C^\infty(\mathcal{J}_d(L))$. Clearly, Lemma 4.3 implies that the ideal $L^\#$ generated by ${}^\wedge L$ in $C^\infty(\mathcal{J}_d(L))$ is equal to ${}^\wedge L_\sigma$. We summarize the results.

THEOREM 4.4. *Let L be an Archimedean uniformly complete Riesz space with weak order unit $e > 0$. Then the order ideal $L^\#$ generated by the Johnson-Kist representation ${}^\wedge L$ of L in $C^\infty(\mathcal{J}_d)$ is the order completion of L. In particular, the order completion of the principal ideal I_e generated by e is $C(\mathcal{J}_d)$.*

This theorem can be compared with the result that the ideal generated by the Ogasawara-Maeda representation of L in $C^\infty(\Omega)$ ([**16**], Sections 49 and 50) is the Dedekind completion $L^\wedge$ of L.

We proceed by investigating the effect upon the Riesz space L if we impose some topological conditions on $\mathcal{J}_d$. Recall that the completely regular Hausdorff space X is called an F-space whenever every cozero-set is C_b-embedded, equivalently, whenever any two disjoint cozero-sets in X are completely separated (see [**9**], Theorem 14.25). The Archimedean Riesz space L is said to have the σ-interpolation property (σ-i.p.) if it follows from $f_n \uparrow \leqq g_n \downarrow$ in L that there exists $h \in L$ such that $f_n \leqq h \leqq g_n$ $(n = 1, 2, \dots)$. It is shown in [**13**], Theorem 10.5 (see also [**21**], Theorem 1.1) that $C(X)$ has the σ-i.p. if and only if X is an F-space.

Definition 4.5. The Archimedean Riesz space L is said to have *almost property* (P) if L_σ has property (P).

Almost Dedekind complete Riesz spaces were introduced in [**20**], and the almost Dedekind σ-complete Riesz spaces in [**1**] and [**20**]. Characterizations of the almost σ-i.p. can be found in [**18**]. Another equivalence is stated in the next theorem.

THEOREM 4.6. *Let L be an Archimedean uniformly complete Riesz space with a weak order unit $e > 0$. Then L has the almost* σ-i.p. *if and only if $\mathcal{J}_d$ is an F-space.*

Proof. If L has the almost σ-i.p., then $L^{\#}$ (see Theorem 4.4) has the σ-i.p., hence $C(\mathscr{J}_d)$, being an ideal in $L^{\#}$, has the σ-i.p. Therefore $\mathscr{J}_d$ is an F-space.

Conversely, suppose that $C(\mathscr{J}_d)$ has the σ-i.p. A standard argument shows that $C^{\infty}(\mathscr{J}_d)$ has the σ-i.p. as well, and so $L^{\#}$ has the σ-i.p. Therefore L has the almost σ-i.p.

As is well known, the completely regular Hausdorff space X is basically disconnected if and only if $C(X)$ is Dedekind σ-complete. Moreover, X is extremally disconnected if and only if $C(X)$ is Dedekind complete. Using these facts, the proof of the next theorem goes along the same lines as the proof of the previous theorem.

THEOREM 4.7. *Let L be an Archimedean uniformly complete Riesz space with a weak order unit.*

(i) *$\mathscr{J}_d$ is basically disconnected if and only if L is almost Dedekind σ-complete.*

(ii) *$\mathscr{J}_d$ is extremally disconnected if and only if L is almost Dedekind complete (equivalently, $L^{\wedge} = L_{\sigma}$).*

Characterizations of almost Dedekind σ-completeness can be found in [**1**], [**8**] and [**20**]. We mention one important criterion. Recall that an Archimedean Riesz space L is said to be d-regular (see [**13**], Definition 9.1) whenever every proper prime ideal which is a d-ideal as well is a minimal prime ideal. Note that if L has, in addition, a weak order unit, then L is d-regular if and only if the sets of maximal d-ideals and minimal prime ideals coincide. A combination of [**17**], Theorem 7 and [**13**], Remark 9.6 yields the following theorem.

THEOREM 4.8. *The Archimedean uniformly complete Riesz space L is almost Dedekind σ-complete if and only if L is d-regular.*

Characterizations of almost Dedekind completeness do not seem to occur in the literature. (According to Math. Rev. 81m:06041, the result of our Theorem 4.11 also appears in the paper 'Conditions for coincidence of the K-completion of an Archimedean l-group with its o-completion' by A. V. Koldunov, in Modern Algebra, pp. 50-57, Leningrad. Gos. Ped. Inst., Leningrad, 1980 (Russian).) In order to prove such a criterion, we first collect some facts on the set $\mathscr{M}$ of all minimal prime ideals in an Archimedean Riesz space L. The set $\mathscr{M}$, equipped with its hull-kernel topology, is a Hausdorff space and a structure space of L. For future references we list some properties equivalent to the compactness of $\mathscr{M}$ (for the proof we refer to [**16**], Theorem 37.4 and [**13**], Remark 9.6 and Theorem 9.8 (i)).

PROPOSITION 4.9. *In an Archimedean Riesz space the following statements are equivalent.*

(i) $\mathscr{M}$ *is compact.*

(ii) L *is d-regular and has a weak order unit.*

(iii) $\mathscr{J}_d = \mathscr{M}$ *and* L *has a weak order unit.*

(iv) *For every* $0 \leqq u \in L$ *there exists* $0 \leqq v \in L$ *such that* $\{u\}^{dd} = \{v\}^d$.

The proof of the next proposition is a duplicate of the corresponding proof of a theorem by T. P. Speed on distributive lattices (see [**22**], Corollary 6.6 and also [**10**], Theorem 4.4).

PROPOSITION 4.10. *For any Archimedean Riesz space* L *the following are equivalent.*

(i) $\mathscr{M}$ *is compact and extremally disconnected.*

(ii) *For any non-empty* $D \subset L$ *there exists* $0 \leqq v \in L$ *such that* $D^{dd} = \{v\}^d$ *(equivalently, every band in* L *is principal).*

We are now able to prove the following result on almost Dedekind completeness.

THEOREM 4.11. *Let* L *be an Archimedean uniformly complete Riesz space with a weak order unit. Then* L *is almost Dedekind complete if and only if every band in* L *is principal.*

Proof. First assume that L is almost Dedekind complete. By Theorem 4.7 (ii), $\mathscr{J}_d$ is extremally disconnected. By Theorem 4.8, L is d-regular and so $\mathscr{J}_d = \mathscr{M}$. Hence, $\mathscr{M}$ is compact and extremally disconnected, and therefore, by Proposition 4.10, every band in L is principal.

Conversely, suppose that every band in L is principal. Using Speed's result, $\mathscr{M}$ is compact and extremally disconnected. It follows from Proposition 4.9 that $\mathscr{M} = \mathscr{J}_d$ and therefore $\mathscr{J}_d$ is extremally disconnected. Hence, by Theorem 4.8 (ii), L is almost Dedekind complete.

5. Some homeomorphism problems. Throughout this section L denotes an Archimedean Riesz space with weak order unit $e > 0$. It is common knowledge that the compact Hausdorff space $\mathscr{Q}^e$ of all e-maximal ideals (equipped with its hull-kernel topology) is a structure space of L ([**16**], Theorem 36.4). The main object of the present section is to investigate what consequences homeomorphisms between $\mathscr{J}_d$, $\mathscr{M}$ and $\mathscr{Q}^e$ have for L.

First of all we compare $\mathscr{J}_d$ and $\mathscr{M}$. Since a minimal prime ideal is a proper d-ideal, any such ideal M is contained in a unique maximal d-ideal $\tau(M)$. Clearly this mapping $\tau:\mathscr{M} \to \mathscr{J}_d$ is surjective. Similarly to the proof of [**13**], Lemma 8.1, it can be shown that τ is continuous. Moreover, τ is injective (and hence bijective) if and only if every maximal d-ideal contains a unique minimal prime ideal. Since L has a weak order unit, it follows from [**18**], Theorem 12.9, that in the case L is uniformly complete the latter property is equivalent to the almost σ-i.p. Combining these observations with Theorem 4.6 we have proved the next proposition.

PROPOSITION 5.1. *Let L be an Archimedean uniformly complete Riesz space with a weak order unit. The following are equivalent.*

(i) τ *is injective.*

(ii) L *has the almost* σ-i.p.

(iii) $\mathscr{J}_d$ *is an F-space.*

The question when τ is a homeomorphism gets a complete answer in the next theorem.

THEOREM 5.2. *Let L be an Archimedean uniformly complete Riesz space with a weak order unit. The following are equivalent.*

(i) τ *is a homeomorphism.*

(ii) $\mathscr{M}$ *and* $\mathscr{J}_d$ *are homeomorphic.*

(iii) $\mathscr{M} = \mathscr{J}_d$.

(iv) L *is almost Dedekind* σ*-complete.*

Proof. First of all, the equivalence of (iii) and (iv) follows from a combination of Theorem 4.8 and Proposition 4.9. Assuming (ii), $\mathscr{M}$ is compact and hence, by Proposition 4.9, $\mathscr{M} = \mathscr{J}_d$. The remaining implications are evident.

The second homeomorphism problem we shall treat in this section is the homeomorphism of $\mathscr{J}_d$ and $\mathscr{Q}^e$. For every $J \in \mathscr{J}_d$ there exists a unique $Q \in \mathscr{Q}^e$ such that $J \subset Q$. This defines a mapping $\pi_e:\mathscr{J}_d \to \mathscr{Q}^e$ which is continuous (the proof is similar to the proof of [**13**], Lemma 8.1). Furthermore, π_e is surjective. Indeed, if $Q \in \mathscr{Q}^e$, then Q contains a minimal prime ideal M which is, as a proper d-ideal, contained in some maximal d-ideal J. But J and Q are comparable, so $e \notin J$ implies $J \subset Q$ and hence $\pi_e(J) = Q$. In general, π_e is not injective. The next theorem gives some criterions for the injectiveness of π_e.

THEOREM 5.3. *Let L be an Archimedean uniformly complete Riesz space with weak order unit* $e > 0$. *The following are equivalent.*

(i) π_e *is injective (equivalently,* π_e *is a homeomorphism).*

(ii) $\mathscr{J}_d$ *and* $\mathscr{Q}^e$ *are homeomorphic.*

(iii) *The principal ideal* I_e *generated by e is order complete.*

Proof. (i) $\Rightarrow$ (ii). This is evident.

(ii) $\Rightarrow$ (iii). Denote the space of all maximal ideals in I_e by $\mathscr{J}(I_e)$. The mapping $\gamma:\mathcal{Q}^e \rightarrow \mathcal{J}(I_e)$ defined by $\gamma(Q) = Q \cap I_e$ is a homeomorphism. It follows from the hypothesis that $\mathcal{J}_d$ and $\mathcal{J}(I_e)$ are homeomorphic, and hence, by Theorem 3.2, $\mathcal{J}(I_e)$ is a quasi-F-space, so $C(\mathcal{J}(I_e))$ is order complete. An application of Yosida's representation theorem ([**16**], Theorem 45.4) yields that I_e is order complete.

(iii) $\Rightarrow$ (i). Suppose that there exist $J_1, J_2 \in \mathcal{J}_d$, $J_1 \neq J_2$ with $J_1 \subset Q$ and $J_2 \subset Q$ for some $Q \in \mathcal{Q}^e$. Since $E(J_1 + J_2) = L$, there exist, by Lemma 2.2, $0 \leqq p \in J_1$ and $0 \leqq q \in J_2$ such that $p + q$ is a weak order unit. Observe that

$$e \in I_e \cap \{p + q\}^{dd} = I_e \cap \{(p + q) \wedge e\}^{dd}$$
$$\subset I_e \cap (p \wedge e + q \wedge e\}^{dd} = \{p \wedge e + q \wedge e\}^{dd(I_e)}.$$

Since I_e is order complete, the sum of two principal bands in I_e is again principal ([**18**], Theorem 11.2). Hence

$$e \in \{p \wedge e + q \wedge e\}^{dd(I_e)} = \{p \wedge e\}^{dd(I_e)} + \{q \wedge e)^{dd(I_e)}$$
$$\subset \{p\}^{dd} + \{q\}^{dd} \subset J_1 + J_2 \subset Q,$$

a contradiction. Therefore π_e is injective.

Unfortunately, it is in general not true that in the above situation order completeness of I_e implies order completeness of L, even though I_e is order dense. We present a counterexample.

Example 5.4. Let $Y = \left\{\frac{1}{n}:n = 1, 2, \dots\right\}$ and $X = Y \cup \{0\}$ both equipped with the relative Euclidean topology. Then X is a compact Hausdorff space, whereas Y is discrete. The function e in $C(X)$ defined by $e(x) = x$ is a weak order unit in $C(X)$. The set Y, being the cozero-set of e, is dense. Clearly, Y is not C_b-embedded in X, and so X is not a quasi-F-space, i.e., $C(X)$ is not order complete. However, I_e is even Dedekind complete, as I_e is Riesz isomorphic to l_∞.

Remark 5.5. (i) In contrast to order completeness, d-regularity does carry over from an order dense principal ideal I_e to L (use the equivalence of (ii) and (iv) of Proposition 4.9). Hence, if L is in addition uniformly complete, then L is almost Dedekind σ-complete if and only if I_e is almost Dedekind σ-complete. (ii) Clearly, L is order complete if and only if I_e is order complete for all weak order units $0 < e \in L$. Therefore, if L is in addition uniformly complete, then L is order complete if and only if one of the conditions (i), (ii) of Theorem 5.3 holds for all weak order units simultaneously.

The question remains whether it is possible to impose on a single weak order unit $0 < e \in L$ such an extra condition that order completeness of I_e does result in order completeness of L. To this end we introduce the notion of a near order unit.

Definition 5.6 ([**13**], Definition 7.1). The element $0 < e \in L$ is called a *near order unit* if $\overline{I_e} = L$ (where $\overline{I_e}$ denotes the closure of I_e in the uniform topology).

Since every band is uniformly closed, each near order unit is a weak order unit. Near order units do occur 'in nature':

(i) every strong order unit is a near order unit,

(ii) a multiplicative unit e of an Archimedean f-algebra A is a near order unit ([**14**], Proposition 3.2 (i)). This holds in particular for $A = C(X)$ and $e(x) = 1$ for all $x \in X$,

(iii) every quasi-interior point in a Banach lattice is a near order unit.

PROPOSITION 5.7. *Let L be an Archimedean uniformly complete Riesz space with near order unit $e > 0$. Then L is order complete if and only if I_e is order complete.*

Proof. Suppose that I_e is order complete. By [**13**], Theorem 8.9, I_e and I_u are Riesz isomorphic for any near order unit $0 < u \in L$. Therefore I_u is order complete for all near order units $0 < u \in L$, and so L is order complete as well. The converse being trivial, we are done.

Combining Theorem 5.3 and Proposition 5.7, the following result is immediate.

THEOREM 5.8. *Let L be an Archimedean uniformly complete Riesz space with near order unit $e > 0$. The following are equivalent.*

(i) *π_e is a homeomorphism.*

(ii) *$\mathcal{J}_d$ and $\mathcal{Q}^e$ are homeomorphic.*

(iii) *L is order complete.*

The final homeomorphism problem to be considered is the homeomorphism between $\mathcal{M}$ and $\mathcal{Q}_e$. For every $M \in \mathcal{M}$ there exists a unique $Q \in \mathcal{Q}^e$ such that $M \subset Q$. As in the previous cases, the mapping $\sigma_e : \mathcal{M} \to \mathcal{Q}^e$, assigning Q to M, is continuous and surjective. Obviously $\sigma_e = \pi_e \circ \tau$. The question under what conditions σ_e is a homeomorphism has been studied before in the literature. The fact that the principal projection property in L implies that σ_e is a homeomorphism goes back to I. Amemiya ([**2**], Theorem 6.4 or [**16**], Corollary 37.12).

Suppose now that L is in addition uniformly complete. Evidently, σ_e is injective if and only if π_e and τ are both injective. By Proposition 5.1, τ is injective if and only if L has the almost σ-i.p. If e is, in addition, a near order unit, then π_e is injective if and only if L is order complete (Theorem 5.8). It is proved in [**18**], Corollary 12.6 that L has the σ-i.p. if and only if L has the almost σ-i.p. and L is order complete. We thus have proved the following result.

PROPOSITION 5.9. *Let L be an Archimedean uniformly complete Riesz space with a near order unit $e > 0$. Then σ_e is injective if and only if L has the* σ-i.p.

From this proposition the next theorem is easily deduced.

THEOREM 5.10. *For L as in Proposition 5.9. the following are equivalent.*

(i) *σ_e is a homeomorphism.*

(ii) *$\mathscr{M}$ and $\mathscr{Q}^e$ are homeomorphic.*

(iii) *L is Dedekind σ-complete.*

Proof. (i) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (iii). Since $\mathscr{M}$ and $\mathscr{Q}^e$ are homeomorphic, $\mathscr{M}$ is compact, so L is d-regular and $\mathscr{M} = \mathscr{J}_d$ (Proposition 4.9). By Theorem 5.8, L is order complete. By [**13**], Theorem 9.16, order completeness together with d-regularity implies Dedekind σ-completeness.

(iii) $\Rightarrow$ (i). Dedekind σ-completeness implies d-regularity, i.e., $\mathscr{M} = \mathscr{J}_d$, and so $\sigma_e = \pi_e$. Moreover, from Dedekind σ-completeness follows order completeness, hence π_e is a homeomorphism by Theorem 5.8.

Remark 5.11. Let L be an Archimedean uniformly complete Riesz space with merely a weak order unit $e > 0$. Using similar methods as in the proof of Theorem 5.3 and using Remark 5.5 (i) it can be shown that σ_e is a homeomorphism if and only if $\mathscr{M}$ and $\mathscr{Q}^e$ are homeomorphic, which property is in its turn equivalent to Dedekind σ-completeness of I_e. Notice in this connection that in general Dedekind σ-completeness of I_e does not carry over to L (see Example 5.4).

We conclude this section with an application to f-algebras. Let A be an Archimedean uniformly complete f-algebra with multiplicative unit e (which is a near unit). Recall that every prime ring ideal in A is a prime (order) ideal (see e.g. [**14**], Corollary 4.8). We denote by $\mathscr{J}_r$ the set of all maximal ring ideals in A and by $\mathscr{M}_r$ the set of all minimal prime ring ideals in A, both equipped with their hull-kernel topologies. The structure spaces $\mathscr{J}_r$ and $\mathscr{M}_r$ are both Hausdorff; moreover, $\mathscr{J}_r$ is compact. It is shown in [**5**], Theorème 9.3.2 that $\mathscr{M}_r = \mathscr{M}$ (cf. [**13**], Lemma 10.1).

Since every $J \in \mathscr{J}_r$ is a prime (order) ideal not containing e, there exists a unique $Q \in \mathscr{Q}^e$ such that $J \subset Q$. The mapping $\rho:\mathscr{J}_r \to \mathscr{Q}^e$, defined by $\rho(J) = Q$ is a homeomorphism. Applying Theorem 5.10 we get the following result.

THEOREM 5.12. *The spaces $\mathscr{J}_r$ and $\mathscr{M}_r$ are homeomorphic if and only if A is Dedekind σ-complete.*

6. Applications to the Riesz space $C(X)$. In the present section we shall interpret some of the preceding results in the case that $L = C(X)$ for some completely regular Hausdorff space X. As observed before, the function e defined by $e(x) = 1$ for all $x \in X$ is a near order unit in $C(X)$.

In order to give a description of the space of maximal d-ideals of $C(X)$ we need another characterization of $\mathscr{J}_d$ for a Riesz space L with weak order unit. Let Λ denote the set of all weak order units in L. Obviously. Λ is a sublattice of the positive cone of L. As in Section 5, $\mathscr{Q}^u (u \in \Lambda)$ denotes the space of all u-maximal ideals in L. If $u, v \in \Lambda$ and $u \leqq v$, then any $Q \in \mathscr{Q}^u$ is contained in a unique $Q' \in \mathscr{Q}^v$. This defines a continuous surjective mapping $\pi_v^u:\mathscr{Q}^u \to \mathscr{Q}^v$. Note that $\pi_w^u = \pi_w^v \circ \pi_v^u$ whenever $u \leqq v \leqq w$ in Λ. Furthermore, for any $u \in \Lambda$ we have the continuous surjection $\pi_u:\mathscr{J}_d \to \mathscr{Q}^u$, as defined in Section 5. Obviously $u \leqq v$ in Λ implies $\pi_v = \pi_v^u \circ \pi_u$. The next theorem shows that $\mathscr{J}_d$ is a natural object for the spaces $\mathscr{Q}^u (u \in \Lambda)$.

THEOREM 6.1. *$\mathscr{J}_d$ is the projective limit of $\{\mathscr{Q}^u:u \in \Lambda\}$ (with respect to the above introduced mappings).*

Proof. It suffices to show that for any topological space X with continuous mappings $\xi_u:X \to \mathscr{Q}^u (u \in \Lambda)$ satisfying $\pi_v^u \circ \xi_u = \xi_v$ for $u \leqq v$ in Λ, there exists a unique continuous mapping $\xi:X \to \mathscr{J}_d$ such that $\xi_u = \pi_u \circ \xi$ for all $u \in \Lambda$. We denote $\xi_u(x)$ by Q_x^u for all $x \in X$ and all $u \in \Lambda$. For any $x \in X$ the collection $\{Q_x^u:u \in \Lambda\}$ of proper prime ideals in L is linearly ordered by inclusion and therefore

$$J_x = \cap \{Q_x^u:u \in \Lambda\}$$

is a proper prime ideal in L. Since J_x does not contain weak order units of L it follows that J_x is contained in some maximal d-ideal J. We assert that $J_x = J$. Indeed, if M is a minimal prime ideal contained in J_x then $M \subset Q_x^u$ for all $u \in \Lambda$. This implies that $J \subset Q_x^u$ for all $u \in \Lambda$, as Q_x^u and J are comparable, and so $J_x = J \in \mathscr{J}_d$. The mapping $\xi:X \to \mathscr{J}_d$ defined by $\xi(x) = J_x$ clearly satisfies the requirements.

Now consider the case that $L = C(X)$. We denote by $\mathscr{C}(X)$ the collection of all dense cozero-sets in X. For any $0 \leqq u \in C(X)$ we have $u \in \Lambda$ if and only if the cozero-set $\operatorname{coz}(u) \in \mathscr{C}(X)$. Take $u \in \Lambda$ and put $S = \operatorname{coz}(u)$. The mapping $\varphi: I_u \to C_b(S)$, defined by $\varphi(f) = fu^{-1}$ on S is a Riesz isomorphism and hence φ induces a homeomorphism between the space $\mathscr{J}(I_u)$ of maximal ideals in I_u and the Stone-Čech compactification βS of S (see e.g. [**13**], Corollary 8.5). Since $\mathscr{J}(I_u)$ is homeomorphic with $\mathscr{Q}^u$, we thus get a homeomorphism $\lambda_u: \mathscr{Q}^u \to \beta S$. Now suppose that $u \leqq v$ in Λ and put $S = \operatorname{coz}(u)$, $T = \operatorname{coz}(v)$. Then $S \subset T$ and the embedding of S into βT has a continuous Stone extension $\pi_T^S: \beta S \to \beta T$ (see [**9**], 6.5). Using the definition of λ_u and λ_v, it is not hard to see that

$$\pi_T^S = \lambda_v \circ \pi_v^u \circ \lambda_u^{-1}.$$

The directed system $\{\beta S: S \in \mathscr{C}(X)\}$ (with respect to the mappings π_T^S) has a projective limit $K(X)$, which is also a compact Hausdorff space. The space $K(X)$ is called the minimal quasi-F-cover of X (see [7], Sections 3 and 4). It follows from the above remarks and from Theorem 6.1 that $K(X)$ is homeomorphic to $\mathscr{J}_d$. In particular it follows from Theorem 3.2 that $K(X)$ is a quasi-F-space. Applying the results of Section 4 we are now able to prove the next theorem.

THEOREM 6.2. *For a completely regular Hausdorff space X the following statements hold.*

(i) *$K(X)$ is an F-space if and only if $C(X)$ has the almost σ-interpolation property (equivalently, for any two disjoint cozero-sets C_1 and C_2 in X there exist zero-sets Z_1 and Z_2 such that $C_1 \subset Z_1$, $C_2 \subset Z_2$ and* $\operatorname{int}(Z_1 \cap Z_2) = \emptyset$).

(ii) *$K(X)$ is basically disconnected if and only if $C(X)$ is almost Dedekind σ-complete (equivalently, for every cozero-set C in X there exists a zero-set Z such that* $C^- = (\operatorname{int} Z)^-$).

(iii) *$K(X)$ is extremally disconnected if and only if $C(X)$ is almost Dedekind complete (equivalently, for every closed set F in X there exists a zero-set Z such that* $(\operatorname{int} F)^- = (\operatorname{int} Z)^-$).

Proof. (i) The first equivalence follows from Theorem 4.6, the second one is proved in [**18**], Theorem 12.13.

(ii) The first equivalence follows from Theorem 4.7 (i), the second one is proved in [**13**], Theorem 10.4 (i).

(iii) The first equivalence follows from Theorem 4.7 (ii). Furthermore, by Theorem 4.11, $C(X)$ is almost Dedekind complete if and only if every band in $C(X)$ is principal. Any band in $C(X)$ is of the form

$$\{f \in C(X):(\text{int } F)^- \subset Z(f)\}$$

for some closed set F in X, whereas any principal band in $C(X)$ is of the form

$$\{f \in C(X):(\text{int } Z)^- \subset Z(f)\}$$

for some zero-set Z in X (see e.g. [**13**], Section 10). The second equivalence is now immediate.

Remark 6.3. In [7], Proposition 4.5, it is proved that $K(X)$ is extremally disconnected if and only if $C(X)$ is almost Dedekind complete. In Proposition 4.6 of the same paper it is shown that a sufficient condition for $K(X)$ to be extremally disconnected, is that every dense open set in X contains a dense cozero-set. The latter condition, however, is not necessary. By way of example, let D be an uncountable discrete topological space and take $X = \beta D$. In fact, it can be shown that for every compact Hausdorff space Y the condition that every open dense set contains a dense cozero-set is equivalent to the countable chain condition in Y, i.e., to order separability of $C(Y)$ ([**13**], Theorem 10.3).

Next we apply the results of Section 5 to $L = C(X)$.

THEOREM 6.4. *For a completely regular Hausdorff space X the following statements hold.*

(i) *$\mathcal{M}$ and $K(X)$ are homeomorphic if and only if $C(X)$ is almost Dedekind σ-complete.*

(ii) *$K(X)$ and βX are homeomorphic if and only if $C(X)$ is order complete (equivalently, X is a quasi-F-space).*

(iii) *$\mathcal{M}$ and βX are homeomorphic if and only if $C(X)$ is Dedekind σ-complete (equivalently, X is basically disconnected).*

Proof. (i) Since $K(X)$ is homeomorphic with $\mathcal{J}_d$, the result follows from Theorem 5.2.

(ii) The space $\mathcal{Q}^e$ is a model for the Stone-Čech compactification βX of X (see e.g. [**13**], Corollary 8.5), so the desired equivalence follows from Theorem 5.8.

(iii) Apply Theorem 5.10.

Remark 6.5. In [**10**], Theorem 5.3, M. Henriksen and M. Jerison proved that $C(X)$ is Dedekind σ-complete if and only if $\mathcal{M}_r$ and $\mathcal{J}_r = \beta X$ are homeomorphic via the mapping which assigns to every minimal prime ring ideal the unique maximal ring ideal in which it is contained. In statement (iii) of the above theorem such a mapping is not specified.

Finally, we apply the results on the order completion of Section 4 to $L = C(X)$. It follows from Theorem 4.4 that the order completion of $C(X)$ can be represented as an ideal in the space $C^\infty(K(X))$ of all extended real valued continuous functions on $K(X)$. In particular, the order completion of $C_b(X)$ is $C(K(X))$ (cf. [7], Theorem 3.9 (b)). Now we shall indicate how the realization of the order completion of $C(X)$ as the space $C^{\#}[\mathscr{C}_\delta(\beta X)]$ can be deduced from the representation of the order completion of $C(X)$ as an ideal in $C^\infty(\mathscr{J}_d)$ (for notations, see [7], Section 2). To this end, take $f \in C[\mathscr{C}(X)]$, i.e., $S \in \mathscr{C}(X)$ and $f \in C(S)$. There exists a weak order unit $0 \leqq u \in C(X)$ such that $S = \operatorname{coz}(u)$ and, as observed before, a homeomorphism $\lambda_u:\mathscr{Q}^u \to \beta S$. The function f has a Stone extension $\beta f:\beta S \to \mathbf{R}^\infty$, so we can define the continuous function $(\beta f)_u:\mathscr{Q}^u \to \mathbf{R}^\infty$ by

$$(\beta f)_u = \beta f \circ \lambda_u.$$

Now define $^\wedge f:\mathscr{J}_d \to \mathbf{R}^\infty$ by

$$^\wedge f = (\beta f)_u \circ \pi_u.$$

It is routine to prove that $^\wedge f$ is independent of the choice of the representative and the choice of the weak order unit u. Moreover, $^\wedge f \in C^\infty(\mathscr{J}_d)$. The mapping

$$\Phi:C[\mathscr{C}(X)] \to C^\infty(\mathscr{J}_d)$$

defined by $\Phi(f) = {}^\wedge f$ is an injective algebra and Riesz homomorphism which extends the Johnson-Kist representation of $C(X)$ in $C^\infty(\mathscr{J}_d)$. Note that the image $^\wedge C[\mathscr{C}(X)]$ of $C[\mathscr{C}(X)]$ under the mapping Φ separates the points of $\mathscr{J}_d$. Since $C[\mathscr{C}(X)]$ can clearly be identified with $C[\mathscr{C}(\beta X)]$, it follows from [7], Proposition 3.4, that $C[\mathscr{C}(X)]$ is an e-uniformly dense Riesz subspace of $C[\mathscr{C}_\delta(\beta X)]$. Hence, Φ can be uniquely extended to an injective algebra and Riesz homomorphism

$$\Phi:C[\mathscr{C}_\delta(\beta X)] \to C^\infty(\mathscr{J}_d),$$

the image of $C[\mathscr{C}_\delta(\beta X)]$ denoted by $^\wedge C[\mathscr{C}_\delta(\beta X)]$ again. Using that $^\wedge C[\mathscr{C}_\delta(\beta X)]$ is a uniformly complete, point separating Riesz subspace of $C^\infty(\mathscr{J}_d)$, and that every weak order unit in $C[\mathscr{C}_\delta(\beta X)]$ has an inverse, it is not hard to show that

$$^\wedge C[\mathscr{C}_\delta(\beta X)] = C^\infty(\mathscr{J}_d).$$

Applying Theorem 4.4, it follows immediately that the order ideal $C^{\#}[\mathscr{C}_\delta(\beta X)]$ generated by $C(X)$ in $C[\mathscr{C}_\delta(\beta X)]$ is the order completion of $C(X)$. Hence, Theorem 4.4 can properly be considered as a generalization of [7], Theorem 2.1.

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