# AUTOCLINISMS AND AUTOMORPHISMS OF FINITE GROUPS II

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In part I of this paper P. Hall's formula for finite stem groups was derived. Using results of C. R. Leedham-Green and S. McKay, a similar formula for isoclinic groups with arbitrary branch factor group is shown.

The main result of this paper is the following theorem, which appears without proof in [**1**, p. 203].

THEOREM. *Let* $\Gamma$ *be an isoclinism family of finite groups,* $Q$ *a finite abelian group and* $\mathrm{Acl}(\Gamma)$ *the autoclinism group of* $\Gamma$. *Then we have*

$$\frac{1}{|\mathrm{Acl}(\Gamma)\times \mathrm{Aut}(Q)|}=\sum \frac{1}{|\mathrm{Aut}(G)|},$$

*where* $G$ *runs through a complete system of non-isomorphic groups in* $\Gamma$ *with* $Q$ *as branch factor group.*

Let $G$ and $H$ be groups (not necessarily finite). We call $G$ and $H$ *strongly isoclinic*, if there exists an isomorphism $\alpha$ from $G/(Z(G)\cap G')$ to $H/(Z(H)\cap H')$, which induces an isomorphism $\gamma$ from $G'$ to $H'$; $\alpha$ is called a *strong isoclinism*, if $G=H$ a *strong autoclinism*. It can be easily verified that a strong isoclinism induces isomorphisms $\alpha_1$ from $G/Z(G)$ to $H/Z(H)$ and $\alpha_2$ from $G_{ab}$ to $H_{ab}$, where $\alpha_1$ and $\alpha_2$ "coincide" on $G/(G'Z(G))$, and which determine $\alpha$. The pair $(\alpha_1, \gamma)$ is an "ordinary" isoclinism from $G$ to $H$. The restriction of $\alpha_2$ to $(G'Z(G))/G'$ is an isomorphism onto $(H'Z(H))/H'$. These quotients are called the *branch factor groups* of $G$ and $H$, being invariant under strong isoclinism. In the terminology of P. Hall, strong isoclinism describes the "situation of the commutator quotients".

Let $\alpha$ be a strong autoclinism of $G$, $K=G/Z(G)$, $Q$ the branch factor group of $G$, and $\tau$ the restriction of $\alpha_2$ to $Q$. Then $\alpha$ determines an element $((\alpha_1, \gamma), \tau)$ of $\mathrm{Acl}(\Gamma)\times \mathrm{Aut}(Q)$. Let $\Phi$ denote the class of groups being strongly isoclinic to $G$, and $A(\Phi)$ the corresponding group of strong autoclinisms (which does not depend on the representatives of $\Phi$). Then we have a homomorphism from $A(\Phi)$ to $\mathrm{Acl}(\Gamma)\times\mathrm{Aut}(Q)$, and it is easy to see that the kernel of this homomorphism is isomorphic to $\mathrm{Hom}(K_{ab}, Q)$. Let $L=G/(Z(G)\cap G')$ and $B=Z(G)\cap G'$, and we consider the central extension

$$C: 1\to B\to G\to L\to 1.$$

Then $C$ determines an epimorphism from the Schur multiplier of $L$ onto $B$, which corresponds to a coset $\Omega$ of $\mathrm{Ext}(L_{ab}, B)$, regarded as a subgroup of $H^2(L, B)$, (in part I the

Ext-group was denoted by $\overline{\text{sym}}$). It follows from Theorem I.2.1 in [**2**] and similar observations as in part I that the isomorphism classes of groups in $\Phi$ correspond to the orbits of $A(\Phi)$ on $\Omega$, where $\alpha \in A(\Phi)$ acts on $H^2(L, B)$, resp. $\Omega$ via the action of $\alpha$ on $L$ and $\gamma$ on $B$. For finite groups we obtain in the same way as in part I the formula

$$\frac{1}{|A(\Phi)|} = \sum \frac{1}{|\mathrm{Aut}(H)|}, \tag{1}$$

where $H$ runs through the isomorphism classes of groups in $\Phi$.

Now we consider all groups in a family $\Gamma$ with a fixed branch factor group $Q$, which are divided into certain classes $\Phi$ of strongly isoclinic groups. Let $S$ be a (fixed) stem group in $\Gamma$ with $K = S/Z(S)$. We consider all abelian extensions $D$ of $Q$ by $K_{ab}$:

$$D: 1 \to Q \to \bar{D} \to K_{ab} \to 1,$$

and denote for each $D$ by $G(D)$ the direct product of $S$ and $\bar{D}$ with amalgamated quotient $K_{ab}$. From Theorem II.3.2 in [**2**] we obtain that each group in $\Gamma$ with $Q$ as branch factor group is strongly isoclinic to some $G(D)$. In order to determine the isomorphism classes, we only have to decide, which groups $G(D)$ are in the same class $\Phi$. The groups $G(D)$ are in one-to-one correspondence with the elements of $\mathrm{Ext}(K_{ab}, Q)$. As each autoclinism of $S$ induces an automorphism of $K_{ab}$, we have an action of $\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)$ on $\mathrm{Ext}(K_{ab}, Q)$, and it is not very difficult to see that groups of the form $G(D)$ are strongly isoclinic, if and only if the corresponding elements of $\mathrm{Ext}(K_{ab}, Q)$ are conjugate under $\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)$. The corresponding stabilizers are the homomorphic images of the groups $A(\Phi)$. For finite groups we obtain

$$|\mathrm{Ext}(K_{ab}, Q)| = \sum \frac{|\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)|}{|A(\Phi)|/|\mathrm{Hom}(K_{ab}, Q)|},$$

where the sum is taken over all classes $\Phi$ of strongly isoclinic groups in $\Gamma$ with $Q$ as branch factor group, which yields

$$\frac{1}{|\mathrm{Acl}(\Gamma) \times \mathrm{Aut}(Q)|} = \sum \frac{1}{|A(\Phi)|}. \tag{2}$$

Now the theorem follows from (1) and (2).

REMARKS. The theorem above can also be obtained by a dual procedure, using Hall's "situation of the centrals". It is also possible to "extend" the formulae and the results on the isomorphism classes of groups in a family to isoclinism classes of arbitrary central extensions without any further complications.

## REFERENCES

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