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THE PERMUTATION LEMMA OF RICHARD BRAUER

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Dear Charlie,

I promised to write to you about the "characteristic-free" extension of Richard Brauer's Lemma: if a row permutation and a column permutation have the same effect on a nonsingular matrix, the two permutations must have the same number of cycles of any given length. Differently put: if two permutation matrices are conjugate in the general linear group, they are already conjugate in the symmetric group; or, if two permutation representations of a cyclic group are equivalent as matrix representations, they are also equivalent as permutation representations. The latter form was certainly known to Burnside: as Peter Neumann has kindly reminded me, §217 of (the second edition of) Burnside's book even discusses the failure of this for all noncyclic groups.

Both in Burnside and in Brauer, the matrices have complex entries. Although Brauer's footnote on page 934 of *Annals* 42 (1941) says that "a modification is necessary, if the field is modular, but the Lemma remains valid", I have not been able to adapt his proof (or Burnside's) to the case of nonzero characteristic. The best characteristic-free proof I know runs as follows. For a permutation P (of some finite set X), let $\omega(P)$ denote the number of cycles (or orbits) of P, not ignoring the fixed points. If P is regarded as a matrix acting on a vectorspace (with basis X, over any field of any characteristic), the dimension of the subspace formed by the vectors fixed by P is easily seen to be $\omega(P)$. Thus if Q is another permutation of X, conjugate to P in the automorphism group of this vectorspace, we must have $\omega(P) = \omega(Q)$, and indeed $\omega(P^k) = \omega(Q^k)$ for all k. Let $\omega_l(P)$ denote the number of cycles of P of length l. The Lemma is now a consequence of the following combinatorial fact (which no longer refers to matrices): if P and Q are permutations such that $\omega(P^k) = \omega(Q^k)$ for all k, then $\omega_l(P) = \omega_l(Q)$ for all l. This must be in print somewhere, but I have not been able to find a reference. To see it, consider first a single cycle C of length l (on a set of l elements): then $\omega(C^k)$ is the greatest common divisor (k, l) of k and l. It follows that

$$\omega(P^k) = \sum_l (k, l)\omega_l(P) \qquad \text{for all } k\,,$$

so that the $\omega_l(P) - \omega_l(Q)$ form a solution of the homogeneous system

$$\sum_l (k, l)x_l = 0\,, \qquad k = 1, 2, \ldots,$$

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of simultaneous linear equations. The n-by-n determinant with k, l entry (k, l) is apparently known as Smith's determinant. Its value is $\phi(1)\phi(2)\ldots\phi(n)$ where ϕ is Euler's function, or

$$n! \prod_p \left(1 - \frac{1}{p}\right)^{[n/p]}$$

if you prefer: at any rate, it is never 0, so our claim follows.

It was Kurt Mahler who found this determinant for me in the literature. The original paper is by Henry J. Stephen Smith: 'On the value of a certain arithmetical determinant', *Proc. London Math. Soc.* 7 (1875–1876), 208–212. There are treatments in various classical books on determinants, and §42 of Ernesto Pascal's *Die Determinanten* (Teubner, Leipzig, 1900) gives quite a few further journal references as well. The simple proof (not Smith's, but perhaps Cesaro's: I got it from Pascal) runs as follows. Let A be the n-by-n matrix with i, j entry 1 or 0 depending on whether i does or does not divide j; write A' for the transpose, and F for the diagonal matrix with $\phi(1), \phi(2), \ldots, \phi(n)$ down the diagonal. The k, l entry of $A'FA$ is (k, l), since

$$\sum_{i=1}^{n} a_{ik}\phi(i)a_{il} = \sum_{i \mid (k,l)} \phi(i) = (k, l).$$

On the other hand A is unitriangular, so the determinant of $A'FA$ is $\phi(1)\phi(2)\ldots\phi(n)$. [If you want a closed formula for the $\omega_i(P)$ in terms of the $\omega(P^k)$, use that the i, j entry of A^{-1} is the value $\mu(j/i)$ of the Möbius function (read as 0 when i does not divide j).]

I shall send copies of this letter to a number of people, in the hope that someone might supply further relevant references.

Best regards,

(*signed*) Laci

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