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ON THE COHEN-MACAULAYFICATION OF CERTAIN BUCHSBAUM RINGS

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§1. Introduction

Let A be a Noetherian local ring of dimension d and with maximal ideal m. Then A is called Buchsbaum if every system of parameters is a weak sequence. This is equivalent to the condition that, for every parameter ideal q, the difference $\ell_A(A/q) - e_A(q)$ is an invariant $I(A)$ of A not depending on the choice of q. (See Section 2 for the detail.) The concept of Buchsbaum rings was introduced by Stückrad and Vogel [8], and the theory of Buchsbaum singularities is now developing (c.f. [6], [7], [9], [10], and [12]).

Recently the author and Shimoda [1] have discovered that certain Buchsbaum rings are characterized by the behaviour of the Rees algebras of parameter ideals. The purpose of our paper is to ask for another criterion of such kind of Buchsbaum rings.

Together with that of [1] our result is stated as follows.

THEOREM (1.1). *Let $Q(A)$ be the total quotient ring of A. Then the following conditions are equivalent.*

(1) *A is a Buchsbaum ring and $H_m^i(A) = (0)$ for $i \neq 1, d$.*

(2) *The Rees algebra $R(q) = \bigoplus_{i \geqq 0} q^i$ is a Cohen-Macaulay ring for every parameter ideal q of A.*

(3) *There is a Cohen-Macaulay intermediate ring B between A and $Q(A)$ such that* (a) *B is of finite type as an A-module,* (b) *$\dim B_n = d$ for every maximal ideal n of B, and* (c) *$mB \subset A$.*

In this case, if $d \geqq 2$, B is uniquely determined and $H_m^1(A) = B/A$. Here $H_m^i(*)$ denotes the local cohomology functor. The equivalence of the statements (1) and (2) is the main result of [1]. The last assertion and the equivalence of the statements (1) and (3) are new results of the present

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paper, which we prove in Section 2.

Section 3 is devoted to some examples. We will give, in case the ring A appears as the local ring at the irrelevant maximal ideal of an affine semigroup ring, a criterion for A to satisfy the conditions of Theorem (1.1) and an explicit description of B in terms of the corresponding semigroup. In the final section it will be proved that for every parameter ideal q of a local ring A which satisfies the conditions of Theorem (1.1) the ring $(G_q^{\cdot}(A))_M$ is again a Buchsbaum local ring with $I((G_q^{\cdot}(A))_M) = I(A)$, where $G_q^{\cdot}(A)$ denotes the associated graded ring of A relative to q and M is the unique graded maximal ideal of $G_q^{\cdot}(A)$. This is an application of our main result.

Throughout this paper (A, m) will always denote a Noetherian local ring of dimension d.

§ 2. Proof of Theorem (1.1)

The concept of Buchsbaum rings was given by Stückrad and Vogel [8].

DEFINITION (2.1) ([8]). A local ring A is called Buchsbaum if every system $a_1, a_2, \cdots, a_d$ of parameters is a weak sequence, i.e., $(a_1, a_2, \cdots, a_{i-1})$: $a_i = (a_1, a_2, \cdots, a_{i-1})$: m for every $1 \leqq i \leqq d$. This is equivalent to the condition that, for every parameter ideal q, the difference $\ell_A(A/q) - e_A(q)$ is an invariant $I(A)$ of A not depending on the particular choice of q, where $e_A(q)$ denotes the multiplicity of A relative to q. (See [8], Satz 10. Notice that they used the term of I-rings instead of Buchsbaum rings.) Clearly Definition (2.1) may be extended to the case of modules, and it is a routine work to generalize the results on Buchsbaum rings given by [8] to the case of Buchsbaum modules.

EXAMPLES (2.2). (1) Every Cohen-Macaulay local ring is Buchsbaum.

(2) Let k be a field and $A = k[|X_1, X_2, \cdots, X_d, Y_1, Y_2, \cdots, Y_d|]/a$, where $k[|X_1, X_2, \cdots, X_d, Y_1, Y_2, \cdots, Y_d|]$ is a formal power series ring and $a = (X_1, X_2, \cdots, X_d) \cap (Y_1, Y_2, \cdots, Y_d)$. Then A is a Buchsbaum ring of $\dim A = d$ and depth $A = 1$. Moreover $I(A) = d - 1$ and $H_m^i(A) = (0)$ for $i \neq 1$, d. Of course, if $d \geqq 2$, A is not a Cohen-Macaulay ring (c.f. [5], p. 469, Beispiel).

(3) For arbitrary integers d, s with $d \geqq s \geqq 0$ there exists a Buchsbaum ring A such that $\dim A = d$ and depth $A = s$ (c.f. [9], Theorem 3).

DEFINITION (2.3) ([8]). Let a be an ideal of A and $a = \bigcap_{p \in \mathrm{Ass}_A(A/a)} a(p)$ a primary decomposition. We put $\mathrm{Assh}_A(A/a) = \{p \in \mathrm{Ass}_A(A/a); \dim A/p = \dim A/a\}$ and $U_A(a) = \bigcap_{p \in \mathrm{Assh}_A(A/a)} a(p)$.

Since $\dim A_p/aA_p = 0$ for every $p \in \mathrm{Assh}_A(A/a)$, this definition of $U_A(a)$ does not depend on the particular choice of a primary decomposition $a = \bigcap_{p \in \mathrm{Ass}_A(A/a)} a(p)$. We will often denote $U_A(a)$ simply by $U(a)$.

For a moment we assume that A is a Buchsbaum ring of depth $A > 0$. Let $a \in m$ and suppose that $\dim A/aA = d - 1$.

First we note

LEMMA (2.4). *a is A-regular.*

This follows from the fact $\mathrm{Ass}_A(A) = \{p \in \mathrm{Spec}\, A; \dim A/p = d\}$ (c.f. [1], (3.2) (1)).

LEMMA (2.5). (1) *Suppose $d \geqq 2$ and let a, b be a part of a system of parameters for A. Then $U(aA) = aA\colon b = aA\colon m$.*

(2) $U(aA)^2 = aU(aA)$.

Proof. (1) $aA\colon b = aA\colon m$ as b is weakly regular on A/aA. On the other hand, as $U(aA)/aA = U_{A/aA}((0))$ and as $U_{A/aA}((0)) = [0\colon m]_{A/aA}$ (c.f. [1], (3.2) (3)), we see $U(aA) = aA\colon m$. Hence the result follows.

(2) It suffices to show that $U(aA)^2 \subset aU(aA)$. If $d = 1$, $U(aA) = aA$ by definition and we have nothing to prove. Suppose $d \geqq 2$ and choose $b \in m$ so that a, b forms a part of a system of parameters for A. Let f, $g \in U(aA)$. Then we may express $bf = ax$ and $bg = ay$ $(x, y \in A)$. On the other hand, we have $fg = az$ for some $z \in A$. Hence $a(b^2z) = b^2(fg) = a^2(xy)$, and so we see $b^2z = a(xy)$ as a is A-regular by (2.4). Thus $z \in aA\colon b^2$, and consequently $z \in U(aA)$ as $U(aA) = aA\colon b^2$ by (1). Therefore $fg \in aU(aA)$, and so we have $U(aA)^2 \subset aU(aA)$ as required.

DEFINITION (2.6). Let $Q(A)$ denote the total quotient ring of A. We put $\tilde{A} = \{x/a; x \in U(aA)\}$ $(=a^{-1}U(aA))$ in $Q(A)$. Then

(1) $\tilde{A}$ is an intermediate ring between A and $Q(A)$.

(2) $\tilde{A} \cong U(aA)$ as A-modules, and $\tilde{A} \cong \mathrm{End}_A\, U(aA)$ as A-algebras.

(3) $U(aA) = a\tilde{A}$, and $m\tilde{A} \subset A$.

(4) $\tilde{A}$ does not depend on the choice of an element a and is uniquely determined by A.

Proof. (1) This follows from (2.5) (2).

(2) Let f: $U(aA) \to \tilde{A}$ be the map defined by $f(x) = x/a$. Then f is a required isomorphism. Let $y \in \tilde{A}$, then $yU(aA) \subset U(aA)$ clearly. We denote by $\hat{y}$ the endomorphism of $U(aA)$ induced by the multiplication of y. Then it is easy to check that the map g: $\tilde{A} \to \mathrm{End}_A U(aA)$, $g(y) = \hat{y}$, is an isomorphism of A-algebras.

(3) The first assertion is trivial. The second one follows from (2.5) (1).

(4) If $d = 1$, $U(aA) = aA$ by definition and so $\tilde{A} = A$. Thus we may assume $d \geqslant 2$. Let $b \in m$ such that $\dim A/bA = d - 1$. First suppose that a, b is a part of a system of parameters for A and let $x \in U(aA)$. Then $bx = ay$ for some $y \in A$ by (2.5) (1). Of course, as b, a is also a part of a system of parameters for A, we have $y \in U(bA)$ again by (2.5) (1). Thus $x/a = y/b \in b^{-1}U(bA)$, and so we have $a^{-1}U(aA) \subset b^{-1}U(bA)$. By the symmetry between a and b we get $a^{-1}U(aA) = b^{-1}U(bA)$ as required.

Now consider the general case, and choose $c \in m$ so that both $\{a, c\}$ and $\{b, c\}$ are parts of systems of parameters for A. Then $a^{-1}U(aA) = c^{-1}U(cA)$ and $b^{-1}U(bA) = c^{-1}U(cA)$ by the result in the special case above. Hence we have $a^{-1}U(aA) = b^{-1}U(bA)$, and this completes the proof of the assertion (4).

LEMMA (2.7). *Suppose* $d \geqq 2$. *Then* $H_m^1(A) = \tilde{A}/A$.

Proof. Apply the functor $H_m^i(*)$ to the exact sequence $0 \to A \to \tilde{A} \to \tilde{A}/A \to 0$, and we have the assertion $H_m^1(A) = \tilde{A}/A$ since $\mathrm{depth}_A \tilde{A} = \mathrm{depth}_A U(aA)$ by (2.6) (2) and since $\mathrm{depth}_A U(aA) \geqq 2$ by [1], Theorem (3.1) (3). (Recall that $H_m^0(\tilde{A}/A) = \tilde{A}/A$ as $\tilde{A}/A$ is a vector space over A/m.)

PROPOSITION (2.8). *Suppose that* $\dim A = d \geqq 2$. *Then* $\tilde{A}$ *is a Buchsbaum* A*-module with* $I(\tilde{A}) = I(A) - (d-1)\cdot \dim_{A/m} \tilde{A}/A$ *and* $\mathrm{depth}_A \tilde{A} = \min\{2 \leqq i \leqq d / H_m^i(A) \neq (0)\}$.

Proof. This follows at once from (2.6) (2), (2.7), and [1], Theorem (3.1).

Proof of Theorem (1.1). (1) $\Rightarrow$ (3) If $d \leqq 1$, A is a Cohen-Macaulay ring and we have nothing to prove. Suppose $d \geqq 2$. Then we have depth $A > 0$. Thus let $B = \tilde{A}$, and B has the required properties (a), (b), and (c) (c.f. (2.6) and (2.8)).

(3) $\Rightarrow$ (1) Notice that B is a Cohen-Macaulay A-module of dimension d. Hence, if $d \leqq 1$, A itself is a Cohen-Macaulay ring. Thus we may assume $d \geqq 2$. Consider the exact sequence $0 \to A \to B \to B/A \to 0$ of A-

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modules. Then, applying the functor $H_m^i(*)$ to it, we have $H_m^1(A) = B/A$ and $H_m^i(A) = (0)$ for $i \neq 1, d$ as $m(B/A) = (0)$ by the property (c). Of course $mH_m^1(A) = (0)$. Thus A is a Buchsbaum ring (c.f. [9], Corollary 1.1).

Now let us prove the last assertions. It suffices to show $B = \tilde{A}$. As $B \subset Q(A)$ and as B is of finite type as an A-module, we may choose a non-zerodivisor a of A so that $aB \subset A$. If a is a unit of A, $B = A$ and so A is a Cohen-Macaulay ring. In this case $\tilde{A} = A$ by (2.7), as $H_m^1(A) = (0)$. Hence $B = \tilde{A}$. Now assume that $a \in m$ and let $x \in B$. Then, as $mx \subset A$, $m(ax) \subset aA$. Hence $ax \in U(aA)$ because $U(aA) = aA\colon m$ by (2.5) (1), and this implies that $x \in a^{-1}U(aA) = \tilde{A}$. Thus $B \subset \tilde{A}$. Now consider the exact sequence $0 \to B \to \tilde{A} \to \tilde{A}/B \to 0$ of A-modules, and we have $\tilde{A}/B = (0)$ since $\operatorname{depth}_A B = \operatorname{depth}_A \tilde{A} = d \geqq 2$ and since $m(\tilde{A}/B) = (0)$. Therefore we have $B = \tilde{A}$ as claimed. This completes the proof of Theorem (1.1).

DEFINITION (2.9). We call $\tilde{A}$ the Cohen-Macaulayfication of A in case A satisfies the conditions of Theorem (1.1) and $\dim A = d \geqq 2$.

EXAMPLE (2.11). Consider the example given by (2.2) (2) and suppose that $d \geqq 2$. Then $\tilde{A} = k[|X_1, X_2, \cdots, X_d|] \oplus k[|Y_1, Y_2, \cdots, Y_d|]$, and $\tilde{A}$ coincides with the normalization of A in $Q(A)$. This example shows that $\tilde{A}$ is not necessarily a local ring.

§ 3. Affine semigroup rings

In this section let k be a field, S a finitely generated (additive) submonoid of N^n, and L the subgroup of $H = Z^n$ generated by S. We put $d = \operatorname{rank}_Z L$. Let $k[H]$ denote the group algebra of H over k and let X^a denote the image of $a \in H$ in $k[H]$. For every subset V of H we put $k[V] = \sum_{a \in V} kX^a$. Of course $k[S]$ coincides with the monoid algebra of S over k and may be considered an H-graded subring of $k[H]$ (c.f. [2], Introduction). We put

$$\bar{S} = \{a \in L; ta \in S \text{ for some integer } t > 0\}\,.$$

$\bar{S}$ is called the normalization of S. It is known that $k[\bar{S}]$ coincides with the normalization of the ring $k[S]$ (c.f. [3], § 1). For simplicity we assume that $\bar{S} = L \cap N^n$. (Recall that $\bar{S}$, in general, is isomorphic to a monoid of this form. See [3], § 2.)

We put $F_i = \{(a_1, a_2, \cdots, a_n) \in S; a_i = 0\}$ and $S_i = S - F_i$ for $1 \leqq i \leqq n$. Let L_i denote the subgroup of L generated by F_i $(1 \leqq i \leqq n)$. We assume

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that $L_i \neq L_j$ for $i \neq j$. Let $\tilde{S} = \bigcap_{i=1}^{n} S_i$. Then $\tilde{S}$ is again a finitely generated submonoid of $\bar{S}$ containing S (c.f. [2], Proof of Lemma 3.3.8), and it is known that $k[\tilde{S}]$ is again a Cohen-Macaulay ring of dimension d (c.f. [2], Conclusion of the proof of 3.3.3).

The purpose of this section is to prove the following

THEOREM (3.1). *Let M be the unique H-graded maximal ideal of $k[S]$, i.e., $M = k[S \backslash \{0\}]$, and suppose that* $\operatorname{rank}_Z L = d \geqq 2$. *Then the following conditions are equivalent.*

(1) *$A = k[S]_M$ is a Buchsbaum ring.*

(2) *$(S \backslash \{0\}) + \tilde{S} \subset S$.*

In this case $H_m^i(A) = (0)$ for $i \neq 1$, d and $I(A) = (d-1) \cdot \sharp(\tilde{S} \backslash S)$. Moreover $\tilde{A} = k[\tilde{S}]_M$.

Proof. (2) $\Rightarrow$ (1) Let $B = k[\tilde{S}]_M$, and B has the properties required in (1.1) (3). Thus A is a Buchsbaum ring. Moreover the last assertions also follow from (1.1) (c.f. [5], Satz 2).

(1) $\Rightarrow$ (2) Applying the functor $H_m^i(*)$ to the exact sequence

$$0 \longrightarrow k[S] \longrightarrow k[\tilde{S}] \longrightarrow k[\tilde{S}]/k[S] \longrightarrow 0\,,$$

we see that $H_M^i(k[\tilde{S}]/k[S]) = H_M^{i+1}(k[S])$ for every $0 \leqq i \leqq d-2$. Notice that $\operatorname{Supp}_{k[S]}(k[\tilde{S}]/k[S]) \subset \{M\}$. For, assume the contrary and put $r = \dim_{k[S]} k[\tilde{S}]/k[S]$. Then $0 < r \leqq d-2$ (c.f. [2], Conclusion of the proof of 3.3.3), and so $H_M^r(k[\tilde{S}]/k[S]) = H_M^{r+1}(k[S])$ by the remark above. But this is impossible as $H_M^{r+1}(k[S]) = H_m^{r+1}(A)$ and as $H_m^i(A)$ is a finite-dimensional vector space over A/m for every $0 \leqq i < d$ (c.f. [5], Hilfsatz 3). Thus we conclude $\operatorname{Supp}_{k[S]}(k[\tilde{S}]/k[S]) \subset \{M\}$, and so $H_M^1(k[S]) = k[\tilde{S}]/k[S]$. This implies $Mk[\tilde{S}] \subset k[S]$ as $mH_m^1(A) = (0)$. Of course this is equivalent to the condition that $(S \backslash \{0\}) + \tilde{S} \subset S$, and we have completed the proof of Theorem (3.1).

COROLLARY (3.2). *Under the same situation as* (3.1) *suppose that $(S \backslash \{0\}) + \bar{S} \subset S$. Then $A = k[S]_M$ is a Buchsbaum ring with $I(A) = (d-1) \cdot \sharp(\bar{S} \backslash S)$ and $H_m^i(A) = (0)$ for $i \neq 1$, d. In this case $\tilde{A}$ coincides with the normalization $\bar{A} = k[\bar{S}]_M$ of A.*

Proof. It suffices to show that $\tilde{S} = \bar{S}$. First notice that $Mk[\bar{S}] \subset k[\tilde{S}]$ as $Mk[\bar{S}] \subset k[S]$, and we have $\ell_{k[\tilde{S}]}(k[\bar{S}]/k[\tilde{S}]) < \infty$. Thus the assertion follows from the fact that $k[\tilde{S}]$ and $k[\bar{S}]$ are Cohen-Macaulay rings of dimension d. (See [3], Theorem 1 for $k[\bar{S}]$.)

EXAMPLE (3.3). Let d, r be integers with $d \geqq 2$ and $r \geqq 1$, and let $T = \{(a_1, a_2, \cdots, a_d) \in N^d; \sum_{i=1}^d a_i \equiv 0 \bmod r\}$. For a subset I of $\{(a_1, a_2, \cdots, a_d) \in N_d; \sum_{i=1}^d a_i = r\}$, we put $S = T \backslash I$. Then S is a finitely generated submonoid of N^d with $T = \bar{S}$, and $(S \backslash \{0\}) + T \subset S$. Thus, by (3.2), $A = k[S]_M$ is a d-dimensional Buchsbaum ring with $I(A) = (d-1) \cdot \sharp I$ and $H_m^i(A) = (0)$ for $i \neq 1, d$. Of course $\tilde{A} = k[T]_M$ in this case.

§4. The associated graded rings $G_q^{\cdot}(A)$

In what follows we suppose that A is a Buchsbaum ring of $\dim A = d \geqq 2$ and with $H_m^i(A) = (0)$ for $i \neq 1, d$. Let $q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal of A. We denote by $G_q^{\cdot}(A)$ the associated graded ring $\bigoplus_{i \geqq 0} q^i/q^{i+1}$ of A relative to q. The purpose of this section is to prove the following

THEOREM (3.1). *Let M be the unique graded maximal ideal of $G_q^{\cdot}(A)$. Then $(G_q^{\cdot}(A))_M$ is again a Buchsbaum ring of dimension d and with $I((G_q^{\cdot}(A))_M) = I(A)$. Moreover $(G_q^{\cdot}(A))_M$ satisfies the conditions of* (1.1).

For this purpose we need some notations and a few lemmas. Let $\tilde{A}$ be the Cohen-Macaulayfication of A, and $\tilde{q} = q\tilde{A}$. We denote by R (resp. $\tilde{R}$) the graded ring $\bigoplus_{i \geqq 0} q^i$ (resp. $\bigoplus_{i \geqq 0} \tilde{q}^i$), and by $G_{\tilde{q}}^{\cdot}(\tilde{A})$ the associated graded ring $\bigoplus_{i \geqq 0} \tilde{q}^i/\tilde{q}^{i+1}$ of $\tilde{A}$ relative to $\tilde{q}$. Recall that $G_q^{\cdot}(A) = R/qR$ and $G_{\tilde{q}}^{\cdot}(\tilde{A}) = \tilde{R}/\tilde{q}\tilde{R}$.

Let X be an indeterminate over A. Then we may identify the Rees ring R (resp. $\tilde{R}$) with the graded subring $\sum_{i \geqq 0} q^i X^i$ (resp. $\sum_{i \geqq 0} \tilde{q}^i X^i$) of $A[X]$ (resp. $\tilde{A}[X]$) canonically. Of course, under these identifications, R is a graded subring of $\tilde{R}$. Notice that $\tilde{q}\tilde{R} \subset R$ as

$$\tilde{q}^{i+1} \subset q^i \subset \tilde{q}^i$$

for every integer $i \geqq 0$ (c.f. (2.6) (3)). We put $E = \tilde{R}/R$, $F = \tilde{q}\tilde{R}/qR$, and $E' = \sum_{n>0} E_n$.

For a given graded R-module $U = \bigoplus_{n \in Z} U_n$, we denote by $U(1)$ the graded R-module whose underlying R-module is the same as that of U and whose graduation is given by $[U(1)]_n = U_{n+1}$ $(n \in Z)$.

LEMMA (4.2). *$F \cong E'(1)$ as graded R-modules.*

Proof. Let $x \in F$ and assume that $x \equiv c \bmod qR$ for some $c \in \tilde{q}\tilde{R}$. We express $c = \sum_{i \geqq 0} c_i X^i$ $(c_i \in \tilde{A})$. Then $c_i \in \tilde{q}^{i+1}$ for every $i \geqq 0$ as $c \in \tilde{q}\tilde{R}$ by the assumption, and so we have $cX \in \tilde{R}$. Let $\overline{cX}$ denote the residue class of cX in $E = \tilde{R}/R$. Then $\overline{cX} \in E'(1)$, and it is easy to check that the map

$f\colon F \longrightarrow E'(1)$, $f(x) = \overline{cX}$, is well-defined. Of course f is an isomorphism of graded R-modules.

COROLLARY (4.3). *There is an exact sequence*

$$0 \longrightarrow E'(1) \longrightarrow G_q^{\cdot}(A) \longrightarrow G_{\tilde{q}}^{\cdot}(\tilde{A}) \longrightarrow E \longrightarrow 0$$

of graded R-modules.

Proof. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & qR \arrow[r, "i"] \arrow[d, "i"] & \tilde{q}\tilde{R} \arrow[r] \arrow[d, "i"] & F \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & R \arrow[r, "i"] & \tilde{R} \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

of graded R-modules with exact rows, where all the i's denote inclusion maps. Then $j = 0$ as $\tilde{q}\tilde{R} \subset R$. Thus, identifying $G_q^{\cdot}(A) = R/qR$ and $G_{\tilde{q}}^{\cdot}(\tilde{A}) = \tilde{R}/\tilde{q}\tilde{R}$ and using (4.2), we get the required exact sequence by virtue of the snake lemma.

Let V be an A-module. We denote A (resp. V) by A^* (resp. V^*) if we regard A (resp. V) as a graded ring (resp. a graded A^*-module) trivially, i.e., $[A^*]_0 = A$ (resp. $[V^*]_0 = V$) and $[A^*]_n = (0)$ (resp. $[V^*]_n = (0)$) for $n \neq 0$. Let $p\colon R \to A$ be the canonical projection. Then, as $p\colon R \to A^*$ is a homomorphism of graded rings, we may consider V^* via p a graded R-module, which we shall denote by ${}_pV$. Let N be the unique graded maximal ideal of R, i.e., $N = mR + R_+$. For every graded R-module U and for every integer i, we denote by $H_N^i(U)$ the i-th local cohomology module of U relative to N, which we consider a graded R-module. Recall that, for a non-zero graded R-module U of finite type, U is a Cohen-Macaulay R-module of dimension r if and only if U_N is a Cohen-Macaulay R_N-module of dimension r (c.f. [4], Theorem). Of course the latter is equivalent to the condition that $H_N^i(U) = (0)$ for $i \neq r$.

LEMMA (4.4). (1) *E is a Cohen-Macaulay R-module of dimension d.*

(2) $$H_N^i(E') = \begin{cases} {}_p(\tilde{A}/A) & (i = 1) \\ (0) & (i \neq 1,\ d). \end{cases}$$

Proof. (1) As $E = \tilde{R}/R$ by definition it suffices to show that both of R and $\tilde{R}$ are Cohen-Macaulay R-modules of dimension $d+1$. For R this follows from (1.1), since A is a Buchsbaum ring with $H_m^i(A) = (0)$ for $i \neq 1$,

d by our standard assumption. For $\tilde{R}$ first recall that $a_1, a_2, \cdots, a_d$ is an $\tilde{A}$-sequence, as $a_1, a_2, \cdots, \cdots, a_d$ is a system of parameters for A and as $\tilde{A}$ is a Cohen-Macaulay A-module of dimension d. Hence $\tilde{R}$ is a Cohen-Macaulay R-module of dimension $d+1$ as $a_1, a_2 + a_1X, \cdots, a_d + a_{d-1}X, a_dX$ is an $\tilde{R}$-sequence (c.f., for example, [11]. Theorem 2.5). (Of course $\tilde{R}$ is of finite type as an R-module. This follows from the facts that $\tilde{R} = R\tilde{A}$ and that $\tilde{A}$ is of finite type as an A-module.)

(2) As $N_p(\tilde{A}/A) = (0)$ (c.f. (2.6) (3). Notice that $p(N) = m$.), we have $H_N^0({}_p(\tilde{A}/A)) = {}_p(\tilde{A}/A)$ and $H_N^i({}_p(\tilde{A}/A)) = (0)$ for $i > 0$. On the other hand $H_N^i(E) = (0)$ for $i \neq d$, because E is a Cohen-Macaulay R-module of dimension d by (1). Hence, applying the functor $H_N^i(*)$ to the exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow {}_p(\tilde{A}/A) \longrightarrow 0$ (Here we identify $E/E' = {}_p(\tilde{A}/A)$.), we have the assertion.

Proof of Theorem (4.1). First we split the exact sequence given by (4.3) into two short exact sequences

$$0 \longrightarrow E'(1) \longrightarrow G_q^{\cdot}(A) \longrightarrow U \longrightarrow 0\,,$$
$$0 \longrightarrow U \longrightarrow G_{\tilde{q}}^{\cdot}(\tilde{A}) \longrightarrow E \longrightarrow 0\,.$$

Then it follows from the second sequence that U is a Cohen-Macaulay R-module of dimension d because E and $G_{\tilde{q}}^{\cdot}(\tilde{A})$ are Cohen-Macaulay R-modules of dimension d. (See (4.4) (1) for E. For $G_{\tilde{q}}^{\cdot}(\tilde{A})$, recall that $G_{\tilde{q}}^{\cdot}(\tilde{A})$ is a polynomial ring with d variables over $\tilde{A}/\tilde{q}$ as $\tilde{q}$ is generated by an $\tilde{A}$-sequence $a_1, a_2, \cdots, a_d$ of length d.) Hence, applying the functor $H_N^i(*)$ to the first short exact sequence. we see by (4.4) (2) that

$$(*)\quad H_N^i(G_q^{\cdot}(A)) = \begin{cases} [{}_p(\tilde{A}/A)](1) & (i = 1) \\ (0) & (i \neq 1, d)\,. \end{cases}$$

Because $H_M^i(G_q^{\cdot}(A)) = H_N^i(G_q^{\cdot}(A))$ as graded $G_q^{\cdot}(A)$-modules, we have by [9], Corollary 1.1 that $(G_q^{\cdot}(A))_M$ is a Buchsbaum local ring. Notice that

$$\begin{aligned} I((G_q^{\cdot}(A))_M) &= (d-1)\cdot \dim_{G_q^{\cdot}(A)/M} H_M^1(G_q^{\cdot}(A)) && \text{(by [5], Satz 2)} \\ &= (d-1)\cdot \dim_{A/M} \tilde{A}/A && \text{(by (*))} \\ &= (d-1)\cdot \dim_{A/m} H_m^1(A) && \text{(by (2.7))} \\ &= I(A) && \text{(by [5], Satz 2).} \end{aligned}$$

This completes the proof of Theorem (4.1).

COROLLARY (4.5) (to the proof).

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$$H_M^i(G_q^{\cdot}(A)) = \begin{cases} [_p(\tilde{A}/A)](1) & (i = 1) \\ (0) & (i \neq 1, d)\,. \end{cases}$$

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