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WHEN IS A NUMERICAL SEMIGROUP A QUOTIENT?

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Abstract

A natural operation on numerical semigroups is taking a quotient by a positive integer. If $\mathcal{S}$ is a quotient of a numerical semigroup with k generators, we call $\mathcal{S}$ a k-quotient. We give a necessary condition for a given numerical semigroup $\mathcal{S}$ to be a k-quotient and present, for each $k \geq 3$, the first known family of numerical semigroups that cannot be written as a k-quotient. We also examine the probability that a randomly selected numerical semigroup with k generators is a k-quotient.

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1. Introduction

We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and we define a *numerical semigroup* to be a set $\mathcal{S} \subseteq \mathbb{N}$ that is closed under addition and contains 0. A numerical semigroup can be defined by a set of generators,

$$\langle a_1, \ldots, a_n \rangle = \{a_1 x_1 + \cdots + a_n x_n : x_i \in \mathbb{N}\},$$

and if $a_1, \ldots, a_n$ are the minimal set of generators of $\mathcal{S}$, we say that $\mathcal{S}$ has *embedding dimension* $\mathsf{e}(\mathcal{S}) = n$. For example,

$$\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \ldots\}$$

has embedding dimension 2.

If $\mathcal{S}$ is a numerical semigroup, then an interesting way to create a new numerical semigroup is by taking the *quotient*

$$\frac{\mathcal{S}}{d} = \{t \in \mathbb{N} : dt \in \mathcal{S}\}$$

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by some positive integer d. Note that $\mathcal{S}/d$ is itself a numerical semigroup, one that in particular satisfies $\mathcal{S} \subseteq \mathcal{S}/d \subseteq \mathbb{N}$. For example,

$$\frac{\langle 3,5\rangle}{2} = \{0,3,4,5,\ldots\} = \langle 3,4,5\rangle.$$

Quotients of numerical semigroups have appeared in the literature over the past couple of decades [17, 21] as well as recently [1, 14]. See [18, Ch. 5] for a thorough overview.

DEFINITION 1.1. We say a numerical semigroup $\mathcal{S}$ is a *k-quotient* if $\mathcal{S} = \langle a_1,\ldots,a_k\rangle/d$ for some positive integers $d, a_1,\ldots,a_k$. The *quotient rank* of $\mathcal{S}$ is the smallest k such that $\mathcal{S}$ is a k-quotient, and we say $\mathcal{S}$ has *full quotient rank* if its quotient rank is $\mathsf{e}(\mathcal{S})$ (since $\mathcal{S} = \mathcal{S}/1$, its quotient rank is at most $\mathsf{e}(\mathcal{S})$).

Numerical semigroups of quotient rank 2 are precisely the *proportionally modular* numerical semigroups [22], which have been well studied [6, 20]. This includes arithmetical numerical semigroups (whose generators have the form $a, a+d,\ldots, a+kd$ with $\gcd(a,d)=1$), which have a rich history in the numerical semigroup literature [2, 4, 9]. In fact, generalised arithmetical numerical semigroups [16], whose generating sets have the form $a, ah+d,\ldots, ah+kd$, can also be shown to have quotient rank 3.

For quotient rank $k \geq 3$, much less is known. It is identified as an open problem in [8] that no numerical semigroup has been proven to have quotient rank at least 4. Since then, the only progress in this direction is [13], wherein it is shown that there exist infinitely many numerical semigroups with quotient rank at least 4, though no explicit examples are given.

With this in mind, we state the main question of the present paper.

MAIN PROBLEM. When is a given numerical semigroup $\mathcal{S}$ a k-quotient?

Our main structural results, which are stated in Section 2, are as follows.

- We prove a sufficient condition for full quotient rank (Theorem 2.1), which we use to obtain, for each k, a numerical semigroup of embedding dimension $k+1$ that is not a k-quotient (Theorem 3.1). When $k \geq 3$, this is the first known example of a numerical semigroup that is not a k-quotient. We also construct, for each k, a numerical semigroup that cannot be written as an intersection of k-quotients (Theorem 3.2), settling a conjecture posed in [13].
- We prove quotient rank is sub-additive whenever the denominators are coprime. This provides a new method of proving a given numerical semigroup is a quotient: partition its generating set and prove that each subset generates a quotient, for example,

$$\begin{aligned}\langle 11,12,13,17,18,19,20\rangle = \langle 11,12,13\rangle + \langle 17,18,19,20\rangle &= \frac{\langle 11,13\rangle}{2} + \frac{\langle 17,20\rangle}{3}\\ &= \frac{3\langle 11,13\rangle + 2\langle 17,20\rangle}{2\cdot 3} = \frac{\langle 33,34,39,40\rangle}{6}.\end{aligned}$$

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We use this result to prove that any numerical semgiroup with *maximal embedding dimension* (that is, the smallest generator equals the embedding dimension) fails to have full quotient rank (Theorem 4.5).

Our remaining results are probabilistic in nature. We examine two well-studied models for ‘randomly selecting’ a numerical semigroup: the ‘box’ model, where the number of generators and a bound on the generators are fixed [3, 5, 7]; as well as a model where the smallest generator and the number of gaps are fixed [10], whose prior study has yielded connections to enumerative combinatorics [19] and polyhedral geometry [11, 12]. We prove that under the first model, asymptotically all semigroups have full quotient rank (Theorem 4.1), while under the second model, asymptotically no semigroups have full quotient rank (Theorem 4.5).

Our results also represent partial progress on the following question, which has proved difficult.

PROBLEM 1.2. Given a numerical semigroup $\mathcal{S}$ and a positive number k, is there an algorithm to determine whether $\mathcal{S}$ is a k-quotient?

REMARK 1.3. Some texts require that the generators of a numerical semigroup be relatively prime, so that $\mathbb{N} \setminus \mathcal{S}$ is finite. This assumption is harmless, since any numerical semigroup can be written as $m\mathcal{S}$, where the generators of $\mathcal{S}$ are relatively prime, and it does not affect k-quotientability: given a positive integer d, one can readily check that

$$\frac{m\mathcal{S}}{d} = m'\Big(\frac{\mathcal{S}}{d'}\Big),$$

where $m' = m/\gcd(m,d)$ and $d' = d/\gcd(m,d)$.

2. When is $\mathcal{S}$ not a k-quotient?

In this section, we give two structural results. The first (Theorem 2.1) is a necessary condition for a given numerical semigroup $\mathcal{S}$ to be a k-quotient, which forms the backbone of the constructions in Section 3 and the probabilistic results in Section 4. The second (Theorem 2.3) is a constructive proof that quotient rank is sub-additive, provided the denominators are relatively prime.

In what follows, we write $[p] = \{1, 2, \ldots, p\}$ for any positive integer p, and given a collection of vectors $\{\mathbf{v}_i\}$ and a set of indices I, we define $\mathbf{v}_I = \sum_{i \in I} \mathbf{v}_i$.

THEOREM 2.1. *Suppose*

$$\mathcal{S} = \frac{\langle b_1, \ldots, b_k \rangle}{d}$$

for some $b_i \in \mathbb{N}$ *and positive integer* d*. Given any elements* $s_1, \ldots, s_p \in \mathcal{S}$ *with* $p > k$*, there exists a nonempty subset* $I \subseteq [p]$ *such that* $s_I/2 \in \mathcal{S}$*.*

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PROOF. Let $\mathbf{b} = (b_1, \ldots, b_k)$. For $1 \le i \le p$, let $\mathbf{c}_i = (c_{i1}, \ldots, c_{ik}) \in \mathbb{N}^k$ be such that

$$s_i = d(c_{i1}b_1 + \cdots + c_{ik}b_k),$$

which exist since $s_i \in \mathcal{S}$. For a vector $\mathbf{v} \in \mathbb{Z}^k$, define $\mathbf{v} \bmod 2 \in \mathbb{Z}_2^k$ to be the coordinate-wise reduction of $\mathbf{v}$ modulo 2. For $J \subseteq [p]$, examine $\mathbf{c}_J \bmod 2$. There are 2^p possible J and 2^k possible values for $\mathbf{c}_J \bmod 2$, with $p > k$, so there must be two distinct J_1 and J_2 such that

$$\mathbf{c}_{J_1} \bmod 2 = \mathbf{c}_{J_2} \bmod 2.$$

Let $I = (J_1 \setminus J_2) \cup (J_2 \setminus J_1)$ be their symmetric difference, which is nonempty. Then,

$$\mathbf{c}_I \bmod 2 = \mathbf{c}_{J_1} + \mathbf{c}_{J_2} - 2\mathbf{c}_{J_1 \cap J_2} \bmod 2 = \mathbf{0},$$

so $\mathbf{c}_I$ has even coordinates. Let $\mathbf{c}_I = (2q_1, \ldots, 2q_k)$ where $q_i \in \mathbb{N}$. Then,

$$s_I/2 = \sum_{i \in I} (d \cdot (c_{i1}b_1 + \cdots + c_{ik}b_k))/2 = d \sum_{j=1}^{k} b_j \sum_{i \in I} c_{ij}/2 = d \sum_{j=1}^{k} q_j b_j$$

is an element of $\mathcal{S}$, as desired. □

COROLLARY 2.2. *Let $\mathcal{S} = \langle a_1, \ldots, a_n \rangle$ be a numerical semigroup. If $\mathcal{S}$ does not have full quotient rank, then there exists $I \subseteq [n]$ such that*

$$a_I \in \langle a_j : j \notin I \rangle.$$

PROOF. By applying Theorem 2.1 to the generating set $\{a_1, \ldots, a_n\}$, we see that $a_J/2 \in \mathcal{S}$ for some $J \subseteq [n]$. So there exist $c_r \in \mathbb{N}$ such that

$$\sum_{j \in J} a_j = \sum_{r \in R} 2c_r a_r,$$

where $R = \{r : c_r > 0\}$. Letting $I = J \setminus R$ and subtracting each a_j with $j \in J \cap R$ from both sides, we have

$$a_I = \sum_{i \in I} a_i = \sum_{r \in J \cap R} (2c_r - 1)a_r + \sum_{r \in R \setminus J} 2c_r a_r$$

is an element of $\langle a_j : j \notin I \rangle$, as desired. Note that I is nonempty, as otherwise

$$0 = a_I = \sum_{r \in J \cap R} (2c_r - 1)a_r + \sum_{r \in R \setminus J} 2c_r a_r \ge \sum_{r \in J \cap R} a_r = \sum_{r \in J} a_r > 0$$

since J is nonempty, which is a contradiction. □

THEOREM 2.3. *If $\mathcal{S}$ and $\mathcal{T}$ are numerical semigroups and $\gcd(c, d) = 1$, then*

$$\frac{\mathcal{S}}{c} + \frac{\mathcal{T}}{d} = \frac{d\mathcal{S} + c\mathcal{T}}{cd}.$$

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PROOF. First suppose that $x \in \mathcal{S}/c + \mathcal{T}/d$. Then $x = s + t$, where $cs \in \mathcal{S}$ and $dt \in \mathcal{T}$, so

$$cdx = d(cs) + c(dt) \in d\mathcal{S} + c\mathcal{T},$$

which implies $x \in (d\mathcal{S} + c\mathcal{T})/cd$. Note this containment does not require $\gcd(c, d) = 1$.

However, suppose $cdx \in d\mathcal{S} + c\mathcal{T}$, so

$$cdx = ds + ct \quad \text{for some } s \in \mathcal{S}, t \in \mathcal{T}. \tag{2.1}$$

In particular, $ct = d(cx - s)$ is a multiple of d. Since c and d are relatively prime, this implies that t is a multiple of d, say $t = bd$. Since $t \in \mathcal{T}$, we conclude that $b \in \mathcal{T}/d$. Similarly, we can write $s = ac$ for some a and so $a \in \mathcal{S}/c$.

Substituting $t = bd$ and $s = ac$ into (2.1), we obtain

$$cdx = dac + cbd = cd(a + b).$$

By cancellation, we obtain $x = a + b$ with $a \in \mathcal{S}/c$ and $b \in \mathcal{T}/d$, as desired. □

Given the ease of proving Theorem 2.3, it is surprisingly more difficult when the denominators do have a common factor. In a follow-up to this current paper, we will translate the quotient operation into a geometric setting, which will allow us to generalise Theorem 2.3 to drop the 'coprime denominators' hypothesis. Intriguingly, the translation can cause a large blow-up in the numbers, for example,

$$\frac{\langle 11, 13\rangle}{2} + \frac{\langle 17, 19\rangle}{2} = \frac{\langle 2416656, 2894591, 3441983, 3869571\rangle}{25357536}.$$

Based on experimentation, this blow-up seems necessary.

3. Some families of numerical semigroups with full quotient rank

In this section, we produce two families of numerical semigroups: those in the first have embedding dimension $k + 1$ but are not k-quotients, so in particular have full quotient rank (Theorem 3.1); and those in the second are not even *intersections* of k-quotients (Theorem 3.2).

THEOREM 3.1. *Given a positive integer k, let $a \geq 2^k$ be an integer. Define $a_i = 2a + 2^i$ for $i = 0, 1, \dots, k$. Then the numerical semigroup*

$$\mathcal{S} = \langle a_0, a_1, \dots, a_k\rangle$$

is not a k-quotient.

PROOF. For $1 \leq j \leq 2^k - 1$, let $b_j = \omega(j)a + j$, where $\omega(j)$ is the number of 1s in the binary representation of j. We first prove that if $\mathcal{T}$ is *any* k-quotient that contains $a_0, \dots, a_k$ (so $\mathcal{T} = \mathcal{S}$ will be an example), then there exists j $(1 \leq j \leq 2^k - 1)$ such that $b_j \in \mathcal{T}$. Indeed, we apply Theorem 2.1. We know that there exists a nonempty $I \subseteq \{0, 1, \dots, k\}$ such that $a_I/2 \in \mathcal{T}$. If $0 \in I$, then a_I is odd and $a_I/2$ is not an integer, so we know $I \subseteq \{1, \dots, k\}$. Let

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$$j = \sum_{i \in I} 2^{i-1}.$$

We have $1 \leq j \leq 2^k - 1$, and

$$a_I/2 = \sum_{i \in I}(2a + 2^i)/2 = |I|a + \sum_{i \in I} 2^{i-1} = \omega(j)a + j = b_j,$$

so $b_j \in \mathcal{T}$.

Now we apply this to $\mathcal{T} = \mathcal{S}$. Seeking a contradiction, suppose $\mathcal{S}$ is a k-quotient, so that some $b_j \in \mathcal{S}$, that is, $b_j = \sum_{i=0}^{k} a_i x_i$ with $x_i \in \mathbb{N}$. Examining this sum modulo a, and noting that $b_j = j \pmod{a}$ and $a_i = 2^i \pmod{a}$, we see that

$$\sum_{i=0}^{k} x_i \geq \omega(j).$$

However, a sum of $\omega(j)$ generators of $\mathcal{S}$ is too large:

$$\omega(j)a + j = b_j \geq \omega(j) \cdot a_0 = \omega(j)(2a + 1) \geq \omega(j)a + a \geq \omega(j)a + 2^k,$$

which is a contradiction. Therefore, $b_j \notin \mathcal{S}$ and so $\mathcal{S}$ cannot be a k-quotient. □

THEOREM 3.2. *Given a positive integer $k \geq 2$, let $a \geq k2^k$ be an integer. As before, define $a_i = 2a + 2^i$ and $b_j = \omega(j)a + j$, where $\omega(j)$ is the number of 1s in the binary representation of j. Let $N = (2k + 1)a$. Then,*

$$\mathcal{S} = \langle a_0, a_1, \ldots, a_k, N - b_1, N - b_2, \ldots, N - b_{2^k-1} \rangle$$

cannot be written as an intersection of k-quotients.

PROOF. Suppose, seeking a contradiction, that $\mathcal{S} = \bigcap_{\ell=1}^{p} \mathcal{S}_\ell$, where the $\mathcal{S}_\ell$ are k-quotients. Each $\mathcal{S}_\ell$ must contain $a_0, a_1, \ldots, a_k$, and we noted in the proof of Theorem 3.1 that this implies that $\mathcal{S}_\ell$ must contain b_j for some j. However, then $\mathcal{S}_\ell$ contains both b_j and $N - b_j$, and so additive closure implies that it contains N. This means $N \in \bigcap_{\ell=1}^{p} \mathcal{S}_\ell = \mathcal{S}$. Let

$$N = \sum_{i=1}^{k} a_i x_i + \sum_{j=1}^{2^k-1} (N - b_j) y_j, \tag{3.1}$$

where $x_i, y_j \in \mathbb{N}$. We break into three cases.

- If $\sum_j y_j \geq 2$, then (3.1) would be too large, as for some j_1, j_2,

$$\begin{aligned}(2k + 1)a = N &\geq (N - b_{j_1}) + (N - b_{j_2}) \\ &= 2N - (\omega(j_1) + \omega(j_2))a - (j_1 + j_2) \\ &> 2 \cdot (2k + 1)a - 2ka - 2 \cdot 2^k \\ &= (2k + 2)a - 2^{k+1},\end{aligned}$$

  which is impossible since $a \geq 2^{k+1}$.

- If $\sum_j y_j = 1$, then (3.1) uses exactly one $N - b_j$. However, then $N = (N - b_j) + b_j$ implies that $b_j \in \langle a_0, a_1 \dots, a_k \rangle$, which we saw was impossible in the proof of Theorem 3.1 since $a \geq 2^k$.
- If $\sum_j y_j = 0$, then $N = \sum_i a_i x_i$. If $\sum_i x_i \leq k$, then

$$(2k+1)a = N \leq k(2a + 2^k),$$

which is impossible since $a > k2^k$. However, if $\sum_i x_i > k$, then

$$(2k+1)a = N \geq (k+1)(2a+1) > (2k+1)a,$$

which is also impossible.

In each case, we obtain a contradiction. □

4. How often do numerical semigroups have full quotient rank?

In this section, we consider the question 'how likely is a randomly selected numerical semigroup to have full quotient rank?' We consider two sampling methods. The first is the 'box' method, wherein a fixed number of generators are selected uniformly and independently from an interval $[1, M]$. Numerical semigroups selected under this model have high probability (that is, approaching 1 as $M \to \infty$) of having full quotient rank.

THEOREM 4.1. *Fix a positive integer n. If $\mathcal{S} = \langle a_1, \dots, a_n \rangle$, where $a_1, \dots, a_n \in [M]$ are uniformly and independently chosen, then the probability that $\mathcal{S}$ has full quotient rank tends to 1 as $M \to \infty$. More precisely, this probability is $1 - O(M^{-1/n})$.*

PROOF. By Corollary 2.2, it suffices to bound the probability that there exists $I \subseteq [n]$ such that $a_I \in \langle a_j : j \notin I \rangle$. Let A be this event, and let B be the event where $a_i \leq M^{(n-1)/n}$ for some i. We will use the fact that

$$\Pr(A) = \Pr(B)\Pr(A \mid B) + \Pr(B^c)\Pr(A \mid B^c) \leq \Pr(B) + \Pr(A \mid B^c).$$

For the first term, the union bound gives

$$\Pr(B) \leq n\left(\frac{M^{(n-1)/n}}{M}\right) = \frac{n}{M^{1/n}}.$$

For the second term, fix a nontrivial subset $I \subsetneq [n]$ and $b_i \in \mathbb{N}$ for $i \notin I$. If $b_i > nM^{1/n}$ for some $i \notin I$, then since every a_i is greater than $M^{(n-1)/n}$, we have

$$\sum_{j \notin I} b_j a_j \geq b_i a_i > (nM^{1/n})M^{(n-1)/n} = nM.$$

However, a_I cannot be this large because it is the sum of at most $n-1$ integers that are each at most M. So we need only consider $b_i \leq nM^{1/n}$. Letting $i^* = \min(I)$ and $m = nM^{1/n}$,

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$$\begin{aligned}
\Pr(A \mid B^c) &\leq \sum_{\substack{I \subsetneq [n] \\ I \neq \emptyset}} \sum_{\substack{b_j \leq m \\ j \notin I}} \Pr\Big(\sum_{i \in I} a_i = \sum_{i \notin I} b_i a_i \;\Big|\; a_1, \ldots, a_n > M^{n/(n-1)}\Big) \\
&= \sum_{\substack{I \subsetneq [n] \\ I \neq \emptyset}} \sum_{\substack{b_j \leq m \\ j \notin I}} \Pr\Big(a_{i^*} = \sum_{i \notin I} b_i a_i - \sum_{i \in I \setminus \{i^*\}} a_i \;\Big|\; a_1, \ldots, a_n > M^{n/(n-1)}\Big) \\
&\leq \sum_{\substack{I \subsetneq [n] \\ I \neq \emptyset}} \sum_{\substack{b_j \leq m \\ j \notin I}} \frac{1}{M - M^{(n-1)/n}} \leq \frac{(2^n - 2)(nM^{1/n})^{n-1}}{M - M^{(n-1)/n}} = \frac{(2^n - 2)n^{n-1}}{M^{1/n} - 1},
\end{aligned}$$

where the second inequality comes from the fact that for any choice of the a_i with $i \neq i^*$, there is at most one choice of a_i^* that makes the linear equation hold. Thus,

$$\Pr(A) \leq \frac{(2^n - 2)n^{n-1}}{M^{1/n} - 1} + \frac{n}{M^{1/n} - 1} = O(M^{-1/n}),$$

which completes the proof. □

REMARK 4.2. The 'minimally generated' and 'finite complement' conditions, which are often imposed on numerical semigroups, do not affect Theorem 4.1. Indeed, under this 'box' probability model, the chosen generators $a_1, \ldots, a_n$ need not form a minimal generating set. Since the quotient rank is at most the embedding dimension, the (asymptotically rare) event that the rank of $\mathcal{S}$ is less than n contains the event that the chosen generating set is not minimal. Additionally, the probability that $a_1, \ldots, a_n$ are relatively prime approaches the positive constant $1/\zeta(n)$ by [15], where $\zeta(n)$ is the Reimann zeta function $\sum_{i=1}^{\infty} 1/i^n$. Therefore, even if one restricts to those $a_1, \ldots, a_n$ that are relatively prime, the conditional probability that the quotient rank of the resulting numerical semigroup is less than n still tends to 0.

Under the second model, a numerical semigroup $\mathcal{S}$ is selected uniformly at random from among the (finitely many) with a fixed smallest generator m and number of gaps g. Such numerical semigroups have high probability (that is, tending to 1 as $g \to \infty$) of having embedding dimension m (such numerical semigroups are said to have *maximal embedding dimension*). We prove that maximal embedding dimension numerical semigroups never have full quotient rank, illustrating a stark contrast in asymptotic behaviour to the first model.

We first recall a characterisation of quotient rank 2 numerical semigroups, which appears in [18] as a characterisation of proportionally modular numerical semigroups in the case $\gcd(\mathcal{S}) = 1$. Our statement here is more general, thanks to Remark 1.3.

THEOREM 4.3. *A numerical semigroup $\mathcal{S}$ with $\gcd(\mathcal{S}) = D$ has quotient rank 2 if and only if there exists an ordering $b_1, \ldots, b_n$ of its minimal generators such that:*

(a) $\gcd(b_i, b_{i+1}) = D$ *for* $1 \leq i \leq n-1$; *and*
(b) $b_{i-1} + b_{i+1}$ *is divisible by* b_i *for* $2 \leq i \leq n-1$.

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LEMMA 4.4. *For any $a, b, m \geq 1$, the numerical semigroup $\mathcal{S} = \langle m, am - 1, bm + 1\rangle$ is a 2-quotient.*

PROOF. If $\mathsf{e}(\mathcal{S}) \leq 2$, then $\mathcal{S}$ is clearly a 2-quotient. Otherwise, letting $b_1 = am - 1$, $b_2 = m$ and $b_3 = bm + 1$, it is clear that $\gcd(b_1, b_2) = \gcd(b_2, b_3) = 1$ and that $b_2 \mid (b_1 + b_3)$. As such, $\mathcal{S}$ is a 2-quotient by Theorem 4.3. □

THEOREM 4.5. *If $m = \min(\mathcal{S} \setminus \{0\})$, then $\mathcal{S}$ is an $(m - 1)$-quotient. In particular, if $\mathsf{e}(\mathcal{S}) = m$, then $\mathcal{S}$ does not have full quotient rank.*

PROOF. If $\mathcal{S} = \mathcal{S}/1$ has embedding dimension less than m, then the proof is immediate. If not, then $\mathcal{S}$ has m minimal generators, and so they must all have distinct residues modulo m. That is,

$$\mathcal{S} = \langle m, b_1 m + 1, \ldots, b_{k-1} m + (m - 1)\rangle$$

for some positive integers $b_1, \ldots, b_{m-1}$. Write $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, where

$$\mathcal{S}_1 = \langle m, b_1 m + 1, b_{k-1} m + (m - 1)\rangle, \quad \mathcal{S}_2 = \langle b_2 m + 2, \ldots, b_{m-2} m + (m - 2)\rangle.$$

Now by Lemma 4.4, $\mathcal{S}_1$ as a 2-quotient, and $\mathcal{S}_2 = \mathcal{S}_2/1$ is trivially an $(m - 3)$-quotient. Since 1 is coprime to every integer, Theorem 2.3 implies $\mathcal{S}$ is an $(m - 1)$-quotient. □

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