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A NOTE ON HECKE OPERATORS AND THETA-SERIES

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As typical examples of modular forms there are theta series and Eisenstein series. We can easily see the behaviour of the Hecke operator on the Eisenstein series but it is very difficult to know that on the theta series. In this paper we give the necessary and sufficient condition in order that $\vartheta(\tau, A)|T_n$[1] $-\lambda\vartheta(\tau, A)$ is a cusp form, where A is an even positive $2k\times 2k$ matrix, $\vartheta(\tau, A) = \sum_{\xi\in Z^{2k}} e^{\pi i A[\xi]\tau}$, and λ is some constant. Combining this result with a theorem of Siegel, we investigate furthermore a relation between the Eisenstein series and the theta series for prime level, and prove that the space spanned by the certain linear combinations of the theta series is closed with respect to all Hecke operators. At the same time we extend the theorem of Siegel to the case of $k=1, 2$, and by using this we characterize analytically imaginary quadratic fields with a single class in each genus.

Notations. Let A be an even[2] positive $2k\times 2k$ matrix with level N and determinant D; then we denote by $F(\tau, A)$ the function

$$F(\tau, A) = \frac{1}{M(A)} \sum \frac{\vartheta(\tau, A_l)}{E(A_l)},$$

where A_l runs over all representatives of the classes in the genus of A, $E(A_l)$ is the order of the unit group of A_l, $M(A) = \sum \frac{1}{E(A_l)}$, and $\vartheta(\tau, A_l) = \sum_{\xi\in Z^{2k}} e^{\pi i A_l[\xi]\tau}$. It is shown in [3] that $\vartheta(\tau, A_l)$ is a modular form of type $(-k, N, \varepsilon)$ in the sense of Hecke:

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}, A_l\right) = \varepsilon(d)(c\tau+d)^k\vartheta(\tau, A_l) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

where $\varepsilon(d) = \left(\frac{(-1)^k D}{d}\right)$, $\left(\frac{*}{*}\right)$ is Kronecker symbol). In this paper, the symbols A, N, D and ε will always have same meanings as here.

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1) The definition of the Hecke operator T_n used here is as in [1].

2) all entries are integrers and all diagonal entries are even integers.

On the other hand we denote by $G(a,b;A)$ the generalized Gaussian sum $\sum_{\xi \bmod b} e^{\pi i \frac{a}{b} A[\xi]}$, where ξ runs over all representative vectors in $\boldsymbol{Z}^{2k}$ mod $b\boldsymbol{Z}^{2k}$ and it should be recalled that $(b\tau+d)^{-k}\vartheta\left(\frac{a\tau+c}{b\tau+d}, A_t\right) = i^{-k}b^{-k}\sqrt{D}^{-1} \times$ $\times\ G(a,b;A)$ at $\tau = i\infty$ for $\begin{pmatrix} a & c \\ d & d \end{pmatrix} \in \Gamma(1)$.

THEOREM 1. *Let A be an even positive $2k \times 2k$ matrix with level N and determinant D, and put $\varepsilon(m) = \left(\frac{(-1)^k D}{m}\right)$. Then, for a natural number n relatively prime to N, the following two conditions are equivalent:*

(A) $\vartheta(\tau, A) \,|\, T_n - (\sum_{t|n} \varepsilon(t) t^{k-1}) \vartheta(\tau, A)$ *is a cusp form, where T_n is the Hecke operator (with level N),*

(B) *either* i) $\sum_{t|n} \varepsilon(t) t^{k-1} = 0$[3] *or* ii) $\varepsilon(n) = 1$ *and* $G(1,c;A) = G(n,c;A)$ *for any power c dividing N of a prime number.*

Proof. Put $f(\tau) = \vartheta(\tau, A) \,|\, T_n - (\sum_{t|n} \varepsilon(t) t^{k-1}) \vartheta(\tau, A)$. When $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\Gamma(1)$, $f|\sigma(\tau) = (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \vartheta(\tau, A) \left| \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right. T_n - (\sum_{t|n} \varepsilon(t) t^{k-1})$ $\vartheta(\tau, A)|\sigma$, where $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ is an element of $\Gamma(1)$ with $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} a & bn^{-1} \\ cn & d \end{pmatrix} \bmod N$. If σ is an element of $\Gamma_0(N)$, $f|\sigma(\tau) = \varepsilon(d) f(\tau)$. Hence $f = 0$ at all cusps equivalent to $i\infty$ with respect to $\Gamma_0(N)$. Since a set of left coset representatives $L_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ for $\Gamma_0(N)$ in $\Gamma(1)$ can be chosen so that γ_i divides N, we can assume $c|N$ and $c \neq N$. Then $f|\sigma(i\infty) = (\sum_{t|n} \varepsilon(t) t^{k-1}) \times i^{-k}\sqrt{D}^{-1} \times$ $(c_1^{-k} G(a_1, c_1; A) - c^{-k} G(a, c; A))$. If $\sum_{t|n} \varepsilon(t) t^{k-1} = 0$, the theorem is clear and so we can assume $\sum_{t|n} \varepsilon(t) t^{k-1} \neq 0$. Here we can take an integer $a_0 \equiv a \bmod N$ with $(a_0, cn) = 1$ since $(a, N, cn) = 1$, and then $c_1^{-k} G(a_1, c_1; A) = (cn)^{-k} G(a_0, cn; A)$ and $c^{-k} G(a, c; A) = c^{-k} G(a_0, c; A)$ follow from the periodiciy of the Gaussian sum, that is, if $l \equiv l_1$, $m \equiv m_1 \bmod N$, $(l, m) = (l_1, m_1) = 1$ and $mm_1 \neq 0$, then $m^{-k} G(l, m; A) = m_1^{-k} G(l_1, m_1; A)$. Put $S(a_0, c) = (cn)^{-k} G(a_0, cn; A) - c^{-k} G(a_0, c; A)$; then $f|\sigma(i\infty) = (\sum_{t|n} \varepsilon(t) t^{k-1}) i^{-k}\sqrt{D}^{-1} S(a_0, c)$ and $S(a_0, c) = 0$ if and only if $S(1, c) = 0$ since $(a_0, cn) = 1$. Since here $(c, n) = 1$ and $n^{-k} G(c, n; A) = \varepsilon(n)$,

$$\begin{aligned} S(1,c) &= c^{-k} G(n,c;A) n^{-k} G(c,n;A) - c^{-k} G(1,c;A) \\ &= \varepsilon(n) c^{-k} G(n,c;A) - c^{-k} G(1,c;A). \end{aligned}$$

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3) For $k \geq 2$, $\sum_{t|n} \varepsilon(t) t^{k-1} \neq 0$.

If the condition (A) holds, then $S(1,c)=0$ so that especially $S(1,1)=0$, i.e. $\varepsilon(n)=1$ and for $c|N$, $S(1,c)=c^{-k}G(n,c;A)-c^{-k}G(1,c;A)=0$, that is, the condition (B) holds, Conversely let the condition (B) hold; then $S(1,c)=0$ if c is a power dividing N of a prime number. Moreover, noting that if $(l_1,l_2)=1$, then $(l_1l_2)^{-k}G(m,l_1l_2;A)=l_1^{-k}G(ml_2,l_1;A)l_2^{-k}G(ml_1,l_2;A)$, we can see $S(1,c)=0$ if $c|N$. This completes the proof.

COROLLARY 1. i) *If N is a prime number, then the assertions* (A) *and* (B) *hold if and only if either* $\sum_{t|n}\varepsilon(t)t^{k-1}=0$ *or* $\varepsilon(n)=1$.

ii) *If $N=p_1p_2\cdots p_r$, where $p_1,p_2,\cdots,p_r$ are different prime numbers and the character ε is trivial, then the assertions* (A) *and* (B) *hold.*

iii) *If n is a quadratic residue* mod N, *then the assertions* (A) *and* (B) *hold.*

Proof. The assertion i) follows from $G(a,N;A)=i^kN^k\sqrt{D}\,\varepsilon(a)$, where a is relatively prime to N.

For the assertion ii), we can assume p_1 as c in (B) of Theorem 1. Put $\sigma=\begin{pmatrix} n & b \\ c & d\end{pmatrix}$ with $nd-bc=1$ and let $d\equiv 0 \bmod p_2\cdots p_r$; then $\sigma\begin{pmatrix} 1 & b \\ c & nd\end{pmatrix}^{-1}$ is an element of $\Gamma_0(N)$ and so $\vartheta(\tau,A)|\sigma=\vartheta(\tau,A)|\begin{pmatrix} 1 & b \\ c & nd\end{pmatrix}$. Comparing now the values at $i\infty$, we get $G(1,c;A)=G(n,c;A)$. From this the assertion ii) follows.

For the assertion iii), we have, by the reciprocity law (Hilfssatz 32 in [4]), $S(1,c)=0$ if and only if

$$\sum_{A^{-1}\xi \bmod 1} e^{\pi icnA^{-1}[\xi]}=\sum_{A^{-1}\xi \bmod 1} e^{\pi icA^{-1}[\xi]},$$

and now the later is true by our assumption and so the assertion (A) holds, too.

COROLLARY 2. *For a natural number n relatively prime to N, $F(\tau,A)$ is an eigenfunction for the Hecke operator T_n (level N) if and only if the assertion* (B) *holds, and if so, the eigenvalue is* $\sum_{t|n}\varepsilon(t)t^{k-1}$.

Proof. The assertion follows directly from Theorem 1 and Lemma 1 which is stated next to Theorem 2.

It should be remarked that Corollary 2 means that the Dirichlet series associated to the linear combination $F(\tau,A)$ of some theta series has an

Euler product relative to p which satisfies the condition (B) in Theorem 1 and Corollary 1 gives a sufficient condition in the special case.

THEOREM 2. *Assume that N is a prime number p. If $k \geq 2$ and $D \neq p^k$, then the two dimensional space spanned by $F(\tau, A)$ and $F(\tau, pA^{-1})$ is closed under the operations of all Hecke operators (level p). If either $k = 2$ and $D = p^2$ or $k = 1$, then $F(\tau, A)$ is an eigenfunction for all Hecke operators (level p).*

Proof. $F(\tau, A)$ and $F(\tau, pA^{-1})$ are modular forms of type $(-k, p, \varepsilon)$, where ε is the character defined by $\varepsilon(d) = \left(\frac{(-1)^k D}{d}\right)$. If $D \neq p^k$, then the values at the cusp 0 of $F(\tau, A)$ and $F(\tau, pA^{-1})$ are different. This shows that $F(\tau, A)$ and $F(\tau, pA^{-1})$ are linearly independent since $F(\tau, A) = F(\tau, pA^{-1}) = 1$ at $i\infty$. Therefore from Lemma 1 the space spanned by $F(\tau, A)$ and $F(\tau, pA^{-1})$ is the space spanned by Eisenstein series with the character ε and dimension $-k$ i.e. of type $(-k, p, \varepsilon)$. This space is mapped into itself by all Hecke operators. But, if either $k = 2$, and $D = p^2$, or $k = 1$, then Eisenstein series of type $(-k, p, \varepsilon)$ is unique up to constants. This completes the proof.

LEMMA 1. *When A is an even positive $2k \times 2k$ matrix with level N and determinant D, then $F(\tau, A)$ is an Eisenstein series.*

Proof. For $k \geq 3$ this lemma is proved by Siegel in [4]. So, we assume $k = 1$ or 2 and prove the lemma by using an idea of Maass in [2]. Put

$$F(\tau, s) = 1 + i^k \sqrt{D}^{-1} \sum_{\substack{(a, b)=1 \\ b>0}} \frac{G(2a, b; A)}{b^k} (b\tau - 2a)^{-k} |b\tau - 2a|^{-s},$$

where a runs over the set of integers relatively prime to b and b runs over all natural numbers. If $Re\, s > 1$, $F(\tau, s)$ is absolutely convergent. Similarly to the case $k \geq 3$ treated in [4] we can easily see that

$$F(\tau, s) = 1 - 2^{-s} + \frac{1}{2} \sum_{\substack{\alpha, \beta \bmod 4D \\ \beta>0}} \delta(\alpha, s) H\left(\frac{\alpha}{\beta}\right) \sum_{\substack{a \equiv \alpha \bmod 4D \\ b \equiv \beta \bmod 4D \\ (a, b)=1}} (b\tau - a)^{-k} |b\tau - a|^{-s},$$

where $H\left(\frac{\alpha}{\beta}\right) = \left(\frac{i}{\beta}\right)^k \sqrt{D}^{-1} G(\alpha, \beta; A)$, $\delta(\alpha, s) = 1$ if α is even and $\delta(\alpha, s) = 2^{-s}$ if α is odd. This shows that $F(\tau, s)$ is an Eisenstein series at $s = 0$ in the sense of Hecke.

On the other hand, Poisson's summation formula implies

$$F(\tau, s) = 1 + 2^{-k-s} i^k \sqrt{D}^{-1} \sum_{b=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{b^{2k+s}} \sum_{\substack{j \bmod b \\ (j,b)=1}} G(2j, b; A) e^{-2\pi i \frac{jn}{b}} A_n\left(s, \frac{\tau}{2}, k\right),$$

where

$$A_n(s, w, k) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n x}}{(x+w)^k |x+w|^s} dx.$$

Here, the following properties of $A_n(s, w, k)$ are obtained in the usual analytical way:

i) $A_n(s, w, k)$ is entire in s for $n \neq 0$.

ii) $A_n(s, w, k) = O(e^{-\delta|n|})$ for some $\delta > 0$, as $|n| \to \infty$, uniformly in s on every compact set.

iii) $$A_n(0, w, k) = \begin{cases} 0 & \text{for } n < 0, \\ i^{-k} \dfrac{(2\pi)^k}{\Gamma(k)} n^{k-1} e^{2\pi i w n} & \text{for } n > 0. \end{cases}$$

iv) $$A_0(0, w, k) = \begin{cases} -i\pi & \text{for } k = 1, \\ 0 & \text{for } k = 2. \end{cases}$$

Now we put $T_n(s) = \sum_{b=1}^{\infty} \frac{C_n(b)}{b^{2k+s}}$ with $C_n(b) = \sum_{\substack{j \bmod b \\ (j,b)=1}} G(2j, b; A) e^{-2\pi i \frac{nj}{b}}$. Since $|G(2j, b; A)| \leq (2b)^k \sqrt{D}$ (Hilfssatz 27 in [4]), we can show first

$$F(\tau, s) = 1 + 2^{-k-s} i^k \sqrt{D}^{-1} \sum_{n=-\infty}^{\infty} T_n(s) A_n\left(s, \frac{\tau}{2}, k\right)$$

for $Re\, s > 1$, where $T_n(s) = \prod_p U_p(n, s)$ with $U_p(n, s) = \sum_{l=0}^{\infty} \frac{C_n(p^l)}{p^{l(2k+s)}}$. Next, by using Hilfssatz 13 in [4], we see that $C_n(p^l) = 0$ if $p^{l-1} \nmid (2n)^2$ and $l \geq 1$, because $C_n(p^l) = p^{2kl}(p^{l(1-2k)} A_{p^l}(A, n) - p^{(l-1)(1-2k)} A_{p^{l-1}}(A, n))$ for $l \geq 1$ with the number $A_q(A, n)$ of solutions mod q of $A[g] \equiv n \bmod q$. Hence $T_n(s) = \prod_{p \nmid 2nN} \left(1 - \frac{\varepsilon(p)}{p^{k+s}}\right) \prod_{p|2nN} U_p(n, s)$, where $\varepsilon(p) = \left(\frac{(-1)^k D}{p}\right)$, and by a simple calculation we get $|\prod_{p|2nN} U_p(n, s)| \leq (2n)^4 \prod_{p|2nN} p^4$ and $\prod_{p|2nN} \left(1 - \frac{\varepsilon(p)}{p^{k+s}}\right)^{-1} = O(\sqrt{|n|})$ for $s > -\frac{1}{2}$, and so

$$T_n(s) = \prod_p \left(1 - \frac{\varepsilon(p)}{p^{k+s}}\right) \times O(|n|^{8+\frac{1}{2}}).$$

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This shows $T_n(s) = O(|n|^{8+\frac{1}{2}})$ for $s > -\varepsilon$ with some $\varepsilon > 0$. Noting $T_n(0) = \prod_p U_p(n, 0) = \prod_p \alpha_p(A, n)$ in the notation in [4], we have $T_n(0) = 2\pi^{-k}\sqrt{D}\, n^{1-k} a_n$ for $n \geq 1$, where $F(\tau, A) = \sum_{n=0}^{\infty} a_n e^{\pi i n \tau}$. Thus, from the properties i), ii), iii), iv) of $A_n(s, w, k)$, we get finally

$$\lim_{s\to\infty} F(\tau, s) = \frac{2}{\Gamma(k)} F(\tau, A).$$

This completes the proof of Lemma 1.

As an application of Lemma 1 we have lastly

THEOREM 3. *Let K be an imaginary quadratic field, then there is a single class in each genus in K if and only if $\vartheta(\tau) = \sum e^{2\pi i N(\alpha)\tau}$ is an Eisenstein series, where α runs over all integers in K and $N(\alpha)$ is the norm of α.*

Proof. Let A be the even positive matrix associated with the quadratic form $2N(x\omega_1 + y\omega_2)$, where $[\omega_1, \omega_2]$ is an integral basis of the maximal order in K. Since then $\vartheta(\tau)$ and $F(\tau, A)$ have the same values at all cusps, it is seen that $\vartheta(\tau) = F(\tau, A)$ holds if and only if $\vartheta(\tau)$ is an Eisenstein series. Therefore, the assertion of the theorem follows from the following Lemma 2, because the theta series in the definition of $F(\tau, A)$ are a part of $\vartheta(\tau, C_i)$, $(C_i \in S)$, in the notation of Lemma 2.

LEMMA 2. *Let S be a set of ideal classes in K such that*

i) *either each ideal class in K or its inverse class is contained in S,*

ii) *the product of any two different ideal classes in S is not the principal ideal class.*

On the other hand, put $\vartheta(\tau, C_i) = \sum_{\alpha\in J_i} e^{2\pi i \frac{N(\alpha)}{N(J_i)}\tau}$, where C_i is an ideal class in K and J_i is an ideal in C_i. Then $\vartheta(\tau, C_i)$, $(C_i \in S)$, are linearly independent.

Proof. We may take as J_i a prime ideal with the prime norm p_i. Since p_i is an element in J_i, $\frac{N(\alpha)}{N(J_i)} = p_i$ for $\alpha = p_i \in J_i$, but for any ideal class C_j in S different from C_i there is no α in J_j which satisfies $\frac{N(\alpha)}{N(J_j)} = p_i$. From the uniqueness of the Fourier-series expansion, this assures that $\vartheta(\tau, C_i)$, $(C_i \in S)$, are linearly independent.

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Remark 1. If either the ckaracter ε is not trivial and $k \geqq 2$ or if the level is not one and $k \geqq 3$, then the theta series $\vartheta(\tau, A)$ is not a common eigenfunction of all Hecke operators.

Remark 2. The condition ii) in (B) in Tehorem 1 can be stated in terms of those quantities which are used in the classical definition of genus of quadratic forms. For example, if $k = 1$ and D is the absolute value of the discriminant of an imaginary quadratic field, then ii) is equivalent to:

$$\left(\frac{n}{p}\right) = 1 \text{ for any odd prime factor } p \text{ of } D,$$

and $n \equiv 1 \mod 4$, $n \equiv 1$ or $3 \mod 8$, or $n \equiv \pm 1 \mod 8$, according as $D \equiv 4 \mod 16$, $D \equiv 8 \mod 32$, or $D \equiv 24 \mod 32$.

Remarks on Theorem 2. If $k \geqq 2$, $D \neq p^k$ and the character ε is not trivial, then $F(\tau, A) - p^k D^{-1} F(\tau, pA^{-1})$ is a common eigenfunction of all Hecke operators (level p). If $k \geqq 2$ and the character ε is trivial, then $F(\tau, A)$ is a common eigenfunction of all Hecke operators T_n with $(n, p) = 1$. In the classical case of $k = 1$ and $D = p \equiv 3 \mod 4$, the Dirichlet series associated to $F(\tau, A)$ is the Dedekind zeta function of $Q(\sqrt{-p})$ and it has Euler product and the above results may be considered as an extension of the classical case.

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