Determinantal Systems of Points.

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The following interesting system of points presented itself while I was investigating certain properties of the cubic curve. What I have called a "Determinantal System of Nine Points" is really a generalisation of the nine points defined by two systems of three parallel lines crossing one another.

The following definitions will be required in the sequel :—

Let ABC be a given triangle, and let it be used as the triangle of reference. Let $P \equiv (X, Y, Z)$ be any assigned point. Then the polar line of P with respect to the given triangle is $\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z} = 0$, and the polar conic of P with respect to the given triangle is $\frac{X}{x} + \frac{Y}{y} + \frac{Z}{z} = 0$. Also let the polar line and the polar conic of P with respect to ABC intersect in the two points P' and P'', whose coordinates are easily shewn to be $(X, \omega Y, \omega^2 Z)$ and $(X, \omega^2 Y, \omega Z)$. We shall take $P' \equiv (X, \omega Y, \omega^2 Z)$ and $P'' \equiv (X, \omega^2 Y, \omega Z)$, with a similar notation for other points throughout the discussion.

Consider now any three assigned points $P_1 Q_2 R_3$. Let P_1 be isolated, and let $Q_2 P_1'$ and $R_3 P_1''$ intersect in Q_3. Let also $R_3 P_1'$ and $Q_2 P_1''$ intersect in R_2. Let R_1 and P_3 be similarly defined by the isolation of Q_2; also P_2 and Q_1 by the isolation of R_3. We thus obtain the following scheme of points :—

$$\begin{matrix} P_1 & P_2 & P_3 & & \\ Q_1 & Q_2 & Q_3 & \rightarrow & u' \\ R_1 & R_2 & R_3 & & \\ & \downarrow & & & \\ & u'' & & & \end{matrix} \qquad \text{(I.)}$$

defined by the leading term $P_1 Q_2 R_3$ (following the determinantal notation) along with the triangle ABC. The arrowheads and the index letters u' and u'' signify that given $P_1 Q_2 R_3$ and isolating P_1; $Q_2 Q_3$ and $R_2 R_3$ meet in P_1', while $Q_2 R_2$ and $Q_3 R_3$ meet in P_1''. Similarly with the other intersections. The following schemes follow at once, where in each case $P_1' Q_2' R_3'$ and $P_1'' Q_2'' R_3''$ are the leading terms :—

$$\begin{array}{ccc} P_1' & P_2' & P_3' \\ Q_1' & Q_2' & Q_3' \\ R_1' & R_2' & R_3' \\ & \downarrow & \\ & u & \end{array} \rightarrow u'' \quad \ldots\ldots\ldots\ldots\ldots \text{(II.)}$$

$$\begin{array}{ccc} P_1'' & P_2'' & P_3'' \\ Q_1'' & Q_2'' & Q_3'' \\ R_1'' & R_2'' & R_3'' \\ & \downarrow & \\ & u' & \end{array} \rightarrow u \quad \ldots\ldots\ldots\ldots\ldots \text{(III.)}$$

We are now going to shew that if we had begun with any other triad of points defined by a term of the above determinantal expansion (I.), the same set of nine points would have been obtained. To establish this result we require the two following typical theorems involving the "assigned" points P_1, Q_2, R_3 and the "derived" points P_2, P_3, Q_3, Q_1, R_1, R_2 :—

(1) P_1, P_2, Q_3' are collinear.

(2) P_2, P_3, Q_1' are collinear.

It is to be noted that (1) involves an "assigned" point P_1', while (2) does not.

To prove P_1, P_2, Q_3' collinear is tantamount to proving P_1'', P_2'', Q_3 collinear, and this is true from the construction of Scheme (III.) above.

To prove (2), we shall prove that P_2, P_3, Q_1', R_1' are collinear.

Let $P_1 \equiv (x_1, y_1, z_1)$; $Q_2 \equiv (x_2, y_2, z_2)$; $R_3 \equiv (x_3, y_3, z_3)$.

Let also $p \equiv y_1 z_2 - y_2 z_1$, $q \equiv z_1 x_2 - z_2 x_1$, $r \equiv x_1 y_2 - x_2 y_1$

$$\Delta \equiv \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

and $S \equiv x_1 y_3 z_3 + y_1 z_3 x_3 + z_1 x_3 y_3$.

Then regarding P_2 as the intersection of the lines $P_1 R_3'$ and $Q_2 R_3''$, we obtain

$$P_2 \equiv (X_1 + \omega X_2 + \omega^2 X_3 \,;\; Y_1 + \omega Y_2 + \omega^2 Y_3 \,;\; Z_1 + \omega Z_2 + \omega^2 Z_3)$$

where

$$\begin{aligned} X_1 &\equiv p\, x_3^2 \\ X_2 &\equiv -(x_2 S - q\, x_3 y_3) \\ X_3 &\equiv (x_2 S + r\, z_3 x_3) \end{aligned}$$

$$\begin{aligned} Y_1 &\equiv q\, y_3^2 \\ Y_2 &\equiv -(y_2 S - r\, y_3 z_3) \\ Y_3 &\equiv (y_2 S + p\, x_3 y_3) \end{aligned}$$

$$\begin{aligned} Z_1 &\equiv r\, z_3^2 \\ Z_2 &\equiv -(z_2 S - p\, z_3 x_3) \\ Z_3 &\equiv (z_2 S + q\, y_3 z_3). \end{aligned}$$

Also, interchanging ω and ω^2 in the coordinates of P_2, we obtain

$$Q_1 \equiv (X_1 + \omega^2 X_2 + \omega X_3 \,;\; Y_1 + \omega^2 Y_2 + \omega Y_3 \,;\; Z_1 + \omega^2 Z_2 + \omega Z_3).$$

Hence

$$Q_1' \equiv (X_1 + \omega^2 X_2 + \omega X_3 \,;\; \omega Y_1 + Y_2 + \omega^2 Y_3 \,;\; \omega^2 Z_1 + \omega Z_2 + Z_3).$$

Let us now form the line-coordinates of $P_2 Q'$, getting

$$(L_1 + \omega L_2 + \omega^2 L_3 \,;\; M_1 + \omega M_2 + \omega^2 M_3 \,;\; N_1 + \omega N_2 + \omega^2 N_3),$$

where

$$L_1 \equiv 0$$

$$L_2 = -L_3 \equiv \begin{vmatrix} Y_1 & Y_3 & Y_2 \\ Z_3 & Z_2 & Z_1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$M_2 \equiv 0$$

$$M_3 = -M_1 \equiv \begin{vmatrix} Z_1 & Z_3 & Z_2 \\ X_3 & X_2 & X_1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$N_3 \equiv 0$$

$$N_1 = -N_2 \equiv \begin{vmatrix} X_1 & X_3 & X_2 \\ Y_3 & Y_2 & Y_1 \\ 1 & 1 & 1 \end{vmatrix}$$

The line-coordinates of $P_2 Q'$ are therefore $(L_2,\ \omega M_3,\ \omega^2 N_1)$.

Evaluating L_2, M_3, N_1 in terms of the expressions given above for X_1, Y_1, etc., we obtain

$$L_2 \equiv S\left[\Delta(y_2 z_3 - y_3 z_2) + 3\{x_1 y_2 z_2 y_3 z_3 - y_1 z_2 z_3 (x_2 y_3 + x_3 y_2) - z_1 y_2 y_3 (z_2 x_3 + z_3 x_2)\}\right]$$

$$M_3 \equiv S\left[\Delta(z_2 x_3 - z_3 x_2) + 3\{y_1 z_2 x_2 z_3 x_3 - z_1 x_2 x_3 (y_2 z_3 + y_3 z_2) - x_1 z_2 z_3 (x_2 y_3 + x_3 y_2)\}\right]$$

$$N_1 \equiv S\left[\Delta(x_2 y_3 - x_3 y_2) + 3\{z_1 x_2 y_2 x_3 y_3 - x_1 y_2 y_3 (z_2 x_3 + z_3 x_2) - y_1 x_2 x_3 (y_2 z_3 + y_3 z_2)\}\right]$$

Now, on the assumption that R_3 does not lie on the polar conic of P_1, in which case S would vanish, we see that the line coordinates of $P_2 Q_1'$ are unaltered if we interchange the suffixes 2 and 3. Hence the line coordinates of $P_3 R_1'$ are proportional to those of $P_2 Q_1'$, proving that P_2, P_3, Q_1', R_1' are collinear.

Let us now consider the two triads of points $P_1 Q_3 R_2$ and $P_2 Q_3 R_1$, in the first of which P_1 is an "assigned" point, while Q_3 and R_2 are "derived" points, whereas in the second of which all three are "derived" points. We wish to shew that if we take either of these triads as the "assigned" triad, the "assigned" triad together with the six "derived" points form the same system of nine points as above

(1) $P_1 Q_3 R_2$.

If we isolate P_1, and we know that $Q_3 P_1'$ and $R_2 P_1''$ meet in Q_2, and that $R_2 P_1'$ and $Q_3 P_1''$ meet in R_3.

Again, if we isolate Q_3, we know that $P_1 Q_3'$ and $R_2 Q_3''$ intersect in P_2, since $P_1 P_2 Q_3'$ and $P_2 R_2 Q_3''$ are collinear by what has been proved above. The rest follows immediately.

(2) $P_2 Q_3 R_1$.

If we isolate P_2, we know that $Q_3 P_2'$ and $R_1 P_2''$ meet in Q_1, while $Q_3 P_2''$ and $R_1 P_2'$ meet in R_3, by the theorems above proved.

We have thus proved that *the scheme of points*

$$\begin{matrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{matrix}$$

form a closed system of nine points, and that the same system is *obtained from any of the six triads of points defined by the terms of the expansion of the above determinantal form by using the construction above defined.*

In virtue of the above property, we have called the (P, Q, R) system of points a "Determinantal system of nine points with respect to the base triangle ABC."

Many further interesting properties of such systems would certainly be found on further investigation.