Canad. Math. Bull. Vol. 61 (4), 2018 pp. 717–737
http://dx.doi.org/10.4153/CMB-2017-057-6

Periodic Solutions of Second Order Degenerate Differential Equations with Delay in Banach Spaces

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Abstract. We give necessary and sufficient conditions of the L^p-well-posedness (resp. $B^s_{p,q}$-well-posedness) for the second order degenerate differential equation with finite delays

$$(Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), \quad \big(t \in [0, 2\pi]\big)$$

with periodic boundary conditions $(Mu)(0) = (Mu)(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, where A, B, and M are closed linear operators on a complex Banach space X satisfying $D(A) \cap D(B) \subset D(M)$, F and G are bounded linear operators from $L^p([-2\pi, 0]; X)$ (resp. $B^s_{p,q}([-2\pi, 0]; X)$) into X.

1 Introduction

A great number of partial differential equations with delays arising in physics and applied sciences have been extensively studied in recent years; see *e.g.*, [6,7,9–17] and the references therein. For example, Lizama [12] considered the first order differential equations with finite delay:

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{1.1}$$

with periodic condition $u(0) = u(2\pi)$, where A is a closed linear operator on a complex Banach X, $u_t(\cdot) = u(t + \cdot)$ is defined in $[-2\pi, 0]$ for $t \in \mathbb{T}$, $f \in L^p(\mathbb{T}; X)$, and $F: L^p([-2\pi, 0]; X) \to X$ is a bounded linear operator. He gave necessary and sufficient condition for (1.1) to be L^p-well-posed by using Fourier multiplier theorems on $L^p(\mathbb{T}; X)$. Moreover, Bu and Fang [6] obtained necessary and sufficient conditions for (1.1) to be well-posed in Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ under suitable assumptions on the Fourier transform of the delay operator F. Recently, Fu and Li [9] characterized the existence and uniqueness of periodic solutions of second-order differential equations with infinite delay

$$u''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), \quad (t \in \mathbb{T}), \tag{1.2}$$

Received by the editors June 30, 2017.
Published electronically January 17, 2018.
Gang Cai is the corresponding author. This work was supported by the NSF of China (No.11401063, 11571194, 11731010,11771063), the Natural Science Foundation of Chongqing (cstc2017jcyjAX0006), Science and Technology Project of Chongqing Education Committee (Grant No. KJ KJ1703041), the University Young Core Teacher Foundation of Chongqing (020603011714), and the Talent Project of Chongqing Normal University (Grant No. 02030307-00024).
AMS subject classification: **34G10, 34K30,** 43A15, 47D06.
Keywords: second order degenerate differential equation, Fourier multiplier theorem, well-posedness, Lebesgue-Bochner space, Besov space.

where A and B are closed linear operators on a complex Banach space X, $u(t)$ is the state function with values in X, $u_t\colon(-\infty,0] \to X$, defined by $u_t(s) = u(t+s)$ for $s \le 0$ and $t \in \mathbb{T}$, belongs to some abstract phase space $\mathcal{B}$, F and G are bounded linear operators from $\mathcal{B}$ into X. Under suitable assumptions on the space $\mathcal{B}$, they are able to characterize the well-posedness of (1.2) in Lebesgue-Bochner spaces $L^p(\mathbb{T};X)$, Besov spaces $B^s_{p,q}(\mathbb{T};X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T};X)$.

On the other hand, Lizama and Ponce [13] characterized the well-posedness of the first order degenerate differential equation

$$(Mu)'(t) = Au(t) + f(t), \quad (t \in \mathbb{T}), \tag{1.3}$$

with periodic boundary condition $(Mu)(0) = (Mu)(2\pi)$ in Lebesgue–Bochner spaces $L^p(\mathbb{T};X)$, Besov spaces $B^s_{p,q}(\mathbb{T};X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T};X)$ under suitable assumptions on the modified resolvent operator determined by (1.3), where A and M are closed linear operators on a complex Banach space X satisfying $D(A) \subset D(M)$.

Bu [4] considered the second order degenerate equations

$$(Mu')'(t) = Au(t) + f(t), \quad (t \in \mathbb{T}), \tag{1.4}$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$, where A and M are closed linear operators on a complex Banach space X satisfying $D(A) \subset D(M)$, f is an X-valued function. Necessary or sufficient conditions for (1.4) to be L^p-well-posed (resp. $B^s_{p,q}$-well-posed and $F^s_{p,q}$-well-posed) are obtained using suitable assumptions on the growth of the modified resolvent operator determined by (1.4). See the monographs by Favini and Yagi [8] and by Sviridyuk and Fedorov [18] for detailed discussions of abstract degenerate differential equations.

In this paper, we study the well-posedness of the second order degenerate differential equations with finite delays

$$(P_2) \qquad \begin{cases} (Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t) & (t \in \mathbb{T}), \\ (Mu)(0) = (Mu)(2\pi), \quad (Mu)'(0) = (Mu)'(2\pi), \end{cases}$$

where A, B, M are closed linear operators on a complex Banach space X satisfying $D(A) \cap D(B) \subset D(M)$, F and G are bounded linear operators from $L^p([-2\pi,0];X)$ (resp. $B^s_{p,q}([-2\pi,0];X)$) into X, and u_t and u'_t are defined on $[-2\pi,0]$ by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ when $t \in \mathbb{T}$.

The main results in this paper are necessary and sufficient conditions for (P_2) to be L^p-well-posed (resp. $B^s_{p,q}$-well-posed). Precisely, we show that when the underlying Banach space X is a UMD Banach space and $1 < p < \infty$, assume that the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is R-bounded, where $G_k \in \mathcal{L}(X)$ is defined by $G_k x = G(e_k x)$, $e_k(t) = e^{ikt} (t \in \mathbb{T})$, (P_2) is L^p-well-posed if and only if $\rho_p(P_2) = \mathbb{Z}$ and the sets $\{-k^2 MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, and $\{kBN_k : k \in \mathbb{Z}\}$ are Rademacher bounded (see Theorem 2.6), where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$ and $\rho_p(P_2)$ is the resolvent set associated with (P_2) in the L^p-well-posedness case (see the precise definition in the next section). We also consider the well-posedness of (P_2) in periodic Besov spaces

https://doi.org/10.4153/CMB-2017-057-6 Published online by Cambridge University Press

$B^s_{p,q}(\mathbb{T};X)$, and a similar necessary and sufficient condition for (P_2) to be $B^s_{p,q}$-well-posed is also obtained. Let $1 \le p,q \le \infty$, and $s > 0$; we assume that the sets

$$\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\},$$
$$\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\},$$
$$\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$$

are norm bounded. Then (P_2) is $B^s_{p,q}$-well-posed if and only if $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{-k^2MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, and $\{kBN_k : k \in \mathbb{Z}\}$ are norm bounded (see Theorem 3.7), where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$ and $\rho_{p,q,s}(P_2)$ is the resolvent set associated with (P_2) in the $B^s_{p,q}$-well-posedness case (see the definition in the third section). Our results can be regarded as generalizations of the previous known results in the simpler case when $B = \alpha I_X$ for some scalar $\alpha \in \mathbb{C}$ and $G = 0$ obtained in [5].

The main tools that we will use are operator-valued Fourier multipliers theorems obtained by Arendt and Bu [2, 3] in $L^p(\mathbb{T};X)$ and $B^s_{p,q}(\mathbb{T};X)$. In fact, we will transform the well-posedness of (P_2) to an operator-valued Fourier multiplier problem in the corresponding vector-valued function spaces. In general, a second order Marcinkiewicz type condition is needed for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier [3]. When the underlying Banach space is B-convex, then a first order Marcinkiewicz type condition is already sufficient for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier [3]. This implies that when X is B-convex, the characterization of the $B^s_{p,q}$-well-posedness of (P_2) remains valid under weaker conditions on F and G. Assume that X is B-convex and the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is norm bounded; then (P_2) is $B^s_{p,q}$-well-posed if and only if $\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{-k^2MN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, and $\{kBN_k : k \in \mathbb{Z}\}$ are norm bounded (see Corollary 3.8).

At the end of the paper, we give concrete examples showing that our abstract results can be applied: let M be the operator of multiplication by a non-negative bounded measurable function m on the Hilbert space $H^{-1}(\Omega)$, where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, if B is a bounded linear operator on $H^{-1}(\Omega)$ and A is the Laplacian Δ on $H^{-1}(\Omega)$ with Dirichlet boundary condition satisfying $D(\Delta) \subset D(M)$, then we obtain the L^p-well-posedness of the corresponding second order degenerate differential equations with finite delays under suitable assumption on F and G.

The results obtained in this paper recover the known results presented in Bu and Fang [7] in the non-degenerate case when $M = I_X$ and $B = 0$. Thus, our results may be regarded as generalizations of the previous known results for the L^p-well-posedness and the $B^s_{p,q}$-well-posedness when $M = I_X$ and $B = F = G = 0$ obtained in Arendt and Bu [2, 3].

This work is organized as follows. In Section 2, we study the well-posedness of (P_2) in $L^p(\mathbb{T};X)$. In Section 3, we consider the well-posedness of (P_2) in periodic Besov spaces $B^s_{p,q}(\mathbb{T};X)$. In the last section, we give some examples that our abstract results can be applied.

2 Well-Posedness in Lebesgue-Bochner Spaces

Let X and Y be two Banach spaces; we denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X to Y. It is denoted simply by $\mathcal{L}(X)$ if $X = Y$. Let $1 \le p < \infty$; $L^p(\mathbb{T};X)$ is the space of all equivalent class of X-valued measurable functions f defined on $\mathbb{T}$ such that

$$\|f\|_p := \Big(\int_0^{2\pi} \|f(t)\|^p \frac{dt}{2\pi} \Big)^{1/p} < \infty.$$

When $f \in L^1(\mathbb{T};X)$, we denote by

$$\widehat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt$$

the k-th Fourier coefficient of f, here $k \in \mathbb{Z}$ and $e_k(t) := e^{ikt}$ for $t \in \mathbb{T}$.

Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X,Y)$ is Rademacher bounded (R-bounded), if there exists $C > 0$ satisfying

$$\sum_{\epsilon_j = \pm 1} \Big\| \sum_{j=1}^n \epsilon_j T_j x_j \Big\| \le C \sum_{\epsilon_j = \pm 1} \Big\| \sum_{j=1}^n \epsilon_j x_j \Big\|$$

for all $T_1, \dots, T_n \in \mathbf{T}, x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$.

It is easy to see from the definition that if $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ are R-bounded, then the product $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and the sum $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded. The main tool for the study of L^p-well-posedness for (P_2) is the operator-valued L^p-Fourier multipliers.

Let X, Y be Banach space and $1 \le p < \infty$. The sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ is an L^p-Fourier multiplier, if for each $f \in L^p(\mathbb{T};X)$, there exists a unique $u \in L^p(\mathbb{T};Y)$ such that $\widehat{u}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

The following results are very useful in the proof of this section's main result.

Proposition 2.1 ([2, Proposition 1.11]) *Let X be a Banach space and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be an L^p-Fourier multiplier; then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.2 ([2, Theorem 1.3]) *Let X, Y be* UMD *Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ defines an L^p-Fourier multiplier whenever $1 < p < \infty$.*

In this section, we consider the L^p-well-posedness of the second order degenerate differential equation with finite delays

$$(P_2) \qquad \begin{cases} (Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t), \ (t \in \mathbb{T}), \\ (Mu)(0) = (Mu)(2\pi), \ (Mu)'(0) = (Mu)'(2\pi), \end{cases}$$

where A, B, M are closed linear operators on a Banach space X satisfying $D(A) \cap D(B) \subset D(M)$ and $F, G\colon L^p([-2\pi, 0]; X) \to X$ are fixed bounded linear operators. Furthermore, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $L^p([-2\pi, 0]; X)$ defined by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ for $-2\pi \le s \le 0$, where we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

https://doi.org/10.4153/CMB-2017-057-6 Published online by Cambridge University Press

Let $F, G \in \mathcal{L}(L^p(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. We define the linear operators F_k, G_k on X by

$$F_k x := F(e_k x), \quad G_k x := G(e_k x), \quad (x \in X). \tag{2.1}$$

It can be seen easily that $F_k, G_k \in \mathcal{L}(X)$, $\|F_k\| \le \|F\|$, and $\|G_k\| \le \|G\|$, since $\|e_k\|_p = 1$. Furthermore, if $u \in L^p(\mathbb{T}; X)$, then

$$\widehat{Fu_\cdot}(k) = F_k \widehat{u}(k), \quad \widehat{Gu_\cdot}(k) = G_k \widehat{u}(k), \quad (k \in \mathbb{Z}), \tag{2.2}$$

which implies that $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, as

$$\|Fu_t\| \le \|F\| \|u_\cdot\|_p = \|F\| \|u\|_p, \quad (t \in \mathbb{T}),$$

and thus $Fu_\cdot, Gu_\cdot \in L^p(\mathbb{T}; X)$.

Now we define the resolvent set of (P_2) in the L^p-well-posedness setting by

$$\rho_p(P_2) := \Big\{ k \in \mathbb{Z} \ : \ -k^2 M + ikB + A - ikG_k - F_k \text{ is invertible from } D(A) \cap D(B) \text{ onto } X \text{ and } (-k^2 M + ikB + A - ikG_k - F_k)^{-1} \in \mathcal{L}(X) \Big\}.$$

If $k \in \rho_p(P_2)$, then $M(-k^2M + ikB + A - ikG_k - F_k)^{-1}$, $A(-k^2M + ikB + A - ikG_k - F_k)^{-1}$, and $B(-k^2 M + ikB + A - ikG_k - F_k)^{-1}$ make sense, as $D(A) \cap D(B) \subset D(M)$ by assumption, and they belong to $\mathcal{L}(X)$ by the closedness of A, B, and M.

For $1 \le p < \infty$, the periodic "Sobolev" space of order 1 is defined by

$$W^{1,p}_{\mathrm{per}}(\mathbb{T}; X) := \Big\{ u \in L^p(\mathbb{T}; X) \ : \ \text{there exists } v \in L^p(\mathbb{T}; X) \text{ such that } \widehat{v}(k) = ik\widehat{u}(k) \text{ for all } k \in \mathbb{Z} \Big\}.$$

Let $u \in L^p(\mathbb{T}; X)$; then $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X)$ if and only if u is differentiable almost everywhere on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}; X)$. Thus, u is actually continuous and $u(0) = u(2\pi)$ [2, Lemma 2.1].

Let $1 \le p < \infty$; the solution space of the L^p-well-posedness for (P_2) is defined by

$$S_p(A, B, M) := \Big\{ u \in L^p(\mathbb{T}; D(A)) \cap W^{1,p}_{\mathrm{per}}(\mathbb{T}; X) \ : \ u' \in L^p(\mathbb{T}; D(B)), \ Mu \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X), (Mu)' \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; X) \Big\}.$$

Here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with their graph norms. If $u \in S_p(A, B, M)$, then $Fu_\cdot$, $Gu'_\cdot \in L^p(\mathbb{T}; X)$ as

$$\|Fu_t\| \le \|F\| \|u\|_p, \quad \|Fu'_t\| \le \|F\| \|u'\|_p$$

when $t \in \mathbb{T}$. Moreover, $S_p(A, B, M)$ is a Banach space equipped with the norm

$$\|u\|_{S_p(A,B,M)} := \|u\|_p + \|u'\|_p + \|Au\|_p + \|Bu'\|_p + \|Mu\|_p + \|(Mu)'\|_p + \|(Mu)''\|_p.$$

By virtue of [2, Lemma 2.1], if $u \in S_p(A, B, M)$, then u and Mu' have continuous representatives, and $u(0) = u(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$.

Definition 2.3 Let $1 \le p < \infty$ and $f \in L^p(\mathbb{T}; X)$. Then $u \in S_p(A, B, M)$ is called a strong L^p-solution of (P_2), if (P_2) is satisfied almost everywhere on $\mathbb{T}$. We say (P_2) is L^p-well-posed, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of (P_2).

If (P_2) is L^p-well-posed and $u \in S_p(A,B,M)$ is the unique strong L^p-solution of (P_2), then there exists a constant $C>0$ such that for each $f \in L^p(\mathbb{T};X)$,

$$\|u\|_{S_p(A,B,M)} \le C\,\|f\|_{L^p}. \tag{2.3}$$

This is an easy consequence of the Closed Graph Theorem by the closedness of A, B, and M.

In order to prove the main result of this section, we need the following preparation.

Proposition 2.4 *Let A, B, and M be closed linear operators defined on a* UMD *Banach space X satisfying $D(A)\cap D(B) \subset D(M)$, and let $F, G \in \mathcal{L}(L^p([-2\pi,0];X),X)$, where $1<p<\infty$. Assume that $\rho_p(P_2)=\mathbb{Z}$ and the sets $\{k^2MN_k : k\in\mathbb{Z}\}$, $\{kN_k : k\in\mathbb{Z}\}$, $\{kBN_k : k\in\mathbb{Z}\}$, and $\{k(G_{k+1}-G_k) : k\in\mathbb{Z}\}$ are R-bounded, where $N_k=(-k^2M+ikB+A-ikG_k-F_k)^{-1}$, F_k, and G_k are defined by (2.1) when $k\in\mathbb{Z}$. Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.*

Proof Put $M_k=k^2MN_k$, $S_k=kBN_k$, and $T_k=kN_k$ when $k\in\mathbb{Z}$. It follows from [12, Proposition 3.2] that the sets $\{G_k : k\in\mathbb{Z}\}$ and $\{F_k : k\in\mathbb{Z}\}$ are R-bounded. By the R-boundedness of the set $\{I_X/k : k\in\mathbb{Z}\setminus\{0\}\}$, the set $\{N_k : k\in\mathbb{Z}\}$ is R-bounded, as the product of R-bounded sets is still R-bounded. Moreover, we observe that

$$\begin{aligned}
N_{k+1}-N_k &= N_{k+1}(N_k^{-1}-N_{k+1}^{-1})N_k\\
&= N_{k+1}\big[-k^2M+ikB+A-ikG_k-F_k+(k+1)^2M-i(k+1)B-A\\
&\qquad +i(k+1)G_{k+1}+F_{k+1}\big]N_k\\
&= N_{k+1}[(2k+1)M-iB+iG_{k+1}+ik(G_{k+1}-G_k)+(F_{k+1}-F_k)]N_k\\
&= (2k+1)N_{k+1}MN_k-iN_{k+1}BN_k+iN_{k+1}G_{k+1}N_k\\
&\quad +ikN_{k+1}(G_{k+1}-G_k)N_k+N_{k+1}(F_{k+1}-F_k)N_k.
\end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned}
M_{k+1}-M_k &= (k+1)^2MN_{k+1}-k^2MN_k\\
&= k^2M(N_{k+1}-N_k)+(2k+1)MN_{k+1}\\
&= k^2(2k+1)MN_{k+1}MN_k-ik^2MN_{k+1}BN_k\\
&\quad +ik^2MN_{k+1}G_{k+1}N_k+ik^3MN_{k+1}(G_{k+1}-G_k)N_k\\
&\quad +k^2MN_{k+1}(F_{k+1}-F_k)N_k+(2k+1)MN_{k+1},
\end{aligned}$$

$$\begin{aligned}
S_{k+1}-S_k &= kB(N_{k+1}-N_k)+BN_{k+1}\\
&= k(2k+1)BN_{k+1}MN_k-ikBN_{k+1}BN_k\\
&\quad +ikBN_{k+1}G_{k+1}N_k+ik^2BN_{k+1}(G_{k+1}-G_k)N_k\\
&\quad +kBN_{k+1}(F_{k+1}-F_k)N_k+BN_{k+1},
\end{aligned}$$

and

$$\begin{aligned}
T_{k+1}-T_k &= k(2k+1)N_{k+1}MN_k-ikN_{k+1}BN_k+ikN_{k+1}G_{k+1}N_k\\
&\quad +ik^2N_{k+1}(G_{k+1}-G_k)N_k+kN_{k+1}(F_{k+1}-F_k)N_k+N_{k+1}.
\end{aligned}$$

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This implies that the sets

$$\{k(N_{k+1}-N_k):k\in\mathbb{Z}\},\qquad \{k(M_{k+1}-M_k):k\in\mathbb{Z}\},$$
$$\{k(S_{k+1}-S_k):k\in\mathbb{Z}\},\qquad \{k(T_{k+1}-T_k):k\in\mathbb{Z}\}$$

are R-bounded by the R-boundedness of the sets $\{k^2MN_k : k\in\mathbb{Z}\}$, $\{kN_k : k\in\mathbb{Z}\}$, $\{kBN_k : k\in\mathbb{Z}\}$, $\{k(G_{k+1}-G_k):\ k\in\mathbb{Z}\}$, $\{F_k : k\in\mathbb{Z}\}$, and $\{G_k : k\in\mathbb{Z}\}$. We obtain that $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by Theorem 2.2. This completes the proof. ∎

First, we give a necessary condition for the L^p-well-posedness of (P_2).

Theorem 2.5 *Let X be a Banach space, $1\le p<\infty$ and let A, B, M be closed linear operators on X satisfying $D(A)\cap D(B)\subset D(M)$. Let $F, G\in\mathcal{L}(L^p([-2\pi,0];X),X)$. Assume that (P_2) is L^p-well-posed. Then $\rho_p(P_2)=\mathbb{Z}$ and the sets $\{k^2MN_k : k\in\mathbb{Z}\}$, $\{kN_k : k\in\mathbb{Z}\}$, and $\{kBN_k : k\in\mathbb{Z}\}$ are R-bounded, where*

$$N_k=(-k^2M+ikB+A-ikG_k-F_k)^{-1}.$$

Proof Let $k\in\mathbb{Z}$ and $y\in X$. Let $f(t)=e^{ikt}y$ $(t\in\mathbb{T})$. Then $f\in L^p(\mathbb{T};X)$, $\widehat{f}(k)=y$ and $\widehat{f}(n)=0$ when $n\ne k$. Since (P_2) is L^p-well-posed, there exists $u\in S_p(A,B,M)$ such that

$$(Mu)''(t)+Bu'(t)+Au(t)=Gu'_t+Fu_t+f(t)\text{ a.e. on }\mathbb{T}. \tag{2.5}$$

We have $\widehat{u}(n)\in D(A)\cap D(B)$ when $n\in\mathbb{Z}$ by [2, Lemma 3.1], as $u\in L^p(\mathbb{T};D(A))$ and $u'\in L^p(\mathbb{T};D(B))$. Taking Fourier transforms on both sides of (2.5), we have

$$(-k^2M+ikB+A-ikG_k-F_k)\widehat{u}(k)=y \tag{2.6}$$

and $(-n^2M+inB+A-inG_n-F_n)\widehat{u}(n)=0$ when $n\ne k$. Thus, we obtain that $-k^2M+ikB+A-ikG_k-F_k$ is surjective. Next, we show that it is also injective. Let $x\in D(A)\cap D(B)$ be such that

$$(-k^2M+ikB+A-ikG_k-F_k)x=0,$$

and let $u(t)=e^{ikt}x$ when $t\in\mathbb{T}$. Then it is clear that $u\in S_p(A,B,M)$ and (P_2) holds almost everywhere on $\mathbb{T}$ when taking $f=0$. Therefore u is a strong L^p-solution of (P_2) when $f=0$. We obtain $u=0$ by the uniqueness assumption, hence $x=0$. We have shown that $-k^2M+ikB+A-ikG_k-F_k$ is also injective. Consequently $-k^2M+ikB+A-ikG_k-F_k$ is a bijection from $D(A)\cap D(B)$ onto X.

Now we prove $(-k^2M+ikB+A-ikG_k-F_k)^{-1}\in\mathcal{L}(X)$. For $f(t)=e^{ikt}y$, let $u\in S_p(A,B,M)$ be the unique strong L^p-solution of (P_2). Then

$$\widehat{u}(n)=\begin{cases}0, & n\ne k,\\ (-k^2M+ikB+A-ikG_k-F_k)^{-1}y, & n=k,\end{cases}$$

by (2.6). This implies that $u(t)=e^{ikt}(-k^2M+ikB+A-ikG_k-F_k)^{-1}y$. By (2.3), there exists a constant $C>0$, independent from y and k, such that

$$\|u\|_p+\|u'\|_p+\|Au\|_p+\|Bu'\|_p+\|Mu\|_p+\|(Mu)'\|_p+\|(Mu)''\|_p\le C\|f\|_p.$$

In particular, we have $\|u\|_p \le C\|f\|_p$. This implies that

$$\left\|(-k^2M + ikB + A - ikG_k - F_k)^{-1}y\right\| \le C\|y\|$$

for all $y \in X$. Hence,

$$\left\|(-k^2M + ikB + A - ikG_k - F_k)^{-1}\right\| \le C.$$

We have shown that $k \in \rho_p(P_2)$. Therefore, $\rho_p(P_2) = \mathbb{Z}$.

Let

$$M_k = -k^2M(-k^2M + ikB + A - ikG_k - F_k)^{-1},$$
$$S_k = kB(-k^2M + ikB + A - ikG_k - F_k)^{-1},$$
$$T_k = k(-k^2M + ikB + A - ikG_k - F_k)^{-1}$$

for $k \in \mathbb{Z}$. We are going to show that $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Indeed, let $f \in L^p(\mathbb{T}; X)$ be fixed. Then there exists a unique strong L^p-solution of (P_2) by assumption, which we denote by $u \in S_p(A, B, M)$. Taking Fourier transforms on both sides of (P_2), we get that $\widehat{u}(k) \in D(A) \cap D(B)$ by [2, Lemma 3.1], and

$$(-k^2M + ikB + A - ikG_k - F_k)\widehat{u}(k) = \widehat{f}(k)$$

when $k \in \mathbb{Z}$. Since $-k^2M + ikB + A - ikG_k - F_k$ is invertible, we obtain

$$\widehat{u}(k) = (-k^2M + ikB + A - ikG_k - F_k)^{-1}\widehat{f}(k)$$

when $k \in \mathbb{Z}$. We have

$$\widehat{u'}(k) = ik\widehat{u}(k), \quad \widehat{Bu'}(k) = ikB\widehat{u}(k), \quad \text{and} \quad \widehat{(Mu)''}(k) = -k^2M\widehat{u}(k)$$

by [2, Lemmas 2.1 and 3.1]. Therefore,

$$\widehat{u'}(k) = iT_k\widehat{f}(k), \quad \widehat{Bu'}(k) = iS_k\widehat{f}(k), \quad \widehat{(Mu)''}(k) = M_k\widehat{f}(k)$$

when $k \in \mathbb{Z}$. This implies that $(M_k)_{k\in\mathbb{Z}}$ and $(S_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers, as $u', Bu', (Mu)'' \in L^p(\mathbb{T}; X)$ by the assumption that $u \in S_p(A, B, M)$. It follows from Proposition 2.1 that the sets $\{M_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, and $\{T_k : k \in \mathbb{Z}\}$ are R-bounded. This finishes the proof. ∎

The next result gives a necessary and sufficient condition for the L^p-well-posedness of (P_2) when X is a UMD Banach space and $1 < p < \infty$.

Theorem 2.6 *Let X be a* UMD *Banach space, and let A, B, M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M)$. Let $F, G \in \mathcal{L}(L^p([-2\pi, 0]; X), X)$, where $1 < p < \infty$. We assume that $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is R-bounded, where G_k is defined by (2.1). Then the following assertions are equivalent.*

(i) *(P_2) is L^p-well-posed.*

(ii) *$\rho_p(P_2) = \mathbb{Z}$, and the sets $\{-k^2MN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$.*

Proof The implication (i)⇒(ii) is just Theorem 2.5. We only need to show that the implication (ii)⇒(i) remains true. Assume that $\rho_p(P_2) = \mathbb{Z}$ and the sets $\{-k^2MN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$. Let $M_k = -k^2MN_k$, $S_k = kBN_k$ and $T_k = kN_k$ when $k \in \mathbb{Z}$. It follows from Proposition 2.4 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(T_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. Then for all $f \in L^p(\mathbb{T};X)$, there exists u, v, w, $x \in L^p(\mathbb{T};X)$ satisfying

$$\widehat{u}(k) = M_k\widehat{f}(k), \quad \widehat{v}(k) = iS_k\widehat{f}(k), \quad \widehat{w}(k) = N_k\widehat{f}(k), \quad \widehat{x}(k) = iT_k\widehat{f}(k) \tag{2.7}$$

when $k \in \mathbb{Z}$. Consequently, $\widehat{x}(k) = ik\widehat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w \in W^{1,p}_{\text{per}}(\mathbb{T};X)$ and $w' = x$ by [2, Lemma 2.1]. Now by (2.7), we have $\widehat{v}(k) = ikB\widehat{w}(k) = B\widehat{w'}(k)$ when $k \in \mathbb{Z}$. This implies that $w' \in L^p(\mathbb{T};D(B))$ [2, Lemma 3.1]. We note that $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers by (2.2). Thus, $(ikG_kN_k)_{k\in\mathbb{Z}}$ and $(F_kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers as the product of L^p-Fourier multipliers is still an L^p-Fourier multiplier. We observe

$$AN_k = I_X - M_k - iS_k + ikG_kN_k + F_kN_k,$$

when $k \in \mathbb{Z}$. It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also an L^p-Fourier multiplier as the sum of L^p-Fourier multipliers is still an L^p-Fourier multiplier. Then there exists $y \in L^p(\mathbb{T};X)$ such that

$$\widehat{y}(k) = AN_k\widehat{f}(k) = A\widehat{w}(k),$$

when $k \in \mathbb{Z}$. This implies that $w \in L^p(\mathbb{T};D(A))$ [2, Lemma 3.1].

It is easy to see that the sequence $(\frac{1}{k}I_X)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier by Theorem 2.2, then $(ikMN_k)_{k\in\mathbb{Z}}$ is L^p-Fourier multiplier as the product of L^p-Fourier multipliers is still an L^p-Fourier multiplier. Therefore, there exists $h \in L^p(\mathbb{T};X)$ such that

$$\widehat{h}(k) = ikMN_k\widehat{f}(k) = ik\widehat{Mw}(k),$$

when $k \in \mathbb{Z}$. Consequently, $Mw \in W^{1,p}_{\text{per}}(\mathbb{T};X)$ by [2, Lemma 2.1]. By (2.7),

$$\widehat{u}(k) = -k^2MN_k\widehat{f}(k) = ik\widehat{(Mw)'}(k)$$

when $k \in \mathbb{Z}$. Thus, $(Mw)' \in W^{1,p}_{\text{per}}(\mathbb{T};X)$ by [2, Lemma 2.1]. We have shown that $w \in S_p(A,B,M)$. Again by (2.7), we have

$$\widehat{(Mw)''}(k) + ikB\widehat{w}(k) + A\widehat{w}(k) = ikG_k\widehat{w}(k) + F_k\widehat{w}(k) + \widehat{f}(k)$$

when $k \in \mathbb{Z}$. This together with the facts $\widehat{Fw_.}(k) = F_k\widehat{w}(k)$ and $\widehat{Gw'_.}(k) = ikG_k\widehat{w}(k)$ implies that

$$(Mw)''(t) + Bw'(t) + Aw(t) = Gw'_t + Fw_t + f(t)$$

almost everywhere on $\mathbb{T}$ by the uniqueness of Fourier coefficients [2, p. 314]. We have shown that w is a strong L^p-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_p(A,B,M)$ satisfying

$$(Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t \text{ a.e. on } \mathbb{T}.$$

Taking the Fourier transforms on both sides, we have

$$(-k^2M + ikB + A - ikG_k - F_k)\widehat{u}(k) = 0$$

when $k \in \mathbb{Z}$. Since $\rho_p(P_2) = \mathbb{Z}$, we deduce that $\widehat{u}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u = 0$. We have shown that (P_2) is L^p-well-posed. This completes the proof. ∎

Remark 2.7 When $M = I_X$, we have $k^2MN_k = k^2N_k$. Check the proof of Proposition 2.4, the condition that the set $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is R-bounded can be removed. Thus, Theorem 2.5 and Theorem 2.6 recover the known results presented in Fu and Li[9] in the non degenerate case when $M = I_X$. Theorems 2.5 and 2.6 together also recover the previous known results for the L^p-well-posedness when $M = I_X$ and $B = F = G = 0$ obtained in Arendt and Bu[2].

3 Well-posedness in Periodic Besov Spaces

In this section, we study the $B^s_{p,q}$-well-posedness of (P_2). Now we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [3]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$ and let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms

$$\|f\|_\alpha = \sup_{x\in\mathbb{T}} \left|f^{(\alpha)}(x)\right|$$

for $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all continuous linear operators from $\mathcal{D}(\mathbb{T})$ to X. We consider the dyadic-like subsets of $\mathbb{R}$,

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\}, I_k = \left\{t \in \mathbb{R} : 2^{k-1} < |t| \le 2^{k+1}\right\} \text{ for } k \in \mathbb{N}.$$

Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$, and

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1, \quad (x \in \mathbb{R}),$$

$$\sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha} \left|\phi_k^{(\alpha)}(x)\right| < \infty, \quad (\alpha \in \mathbb{N}_0).$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \phi(\mathbb{R})$ be fixed. For $1 \le p, q \le \infty$, and $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}; X) := \Big\{ f \in \mathcal{D}'(\mathbb{T}, X) : \|f\|_{B^s_{p,q}} := \Big(\sum_{j\ge0} 2^{sjq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k) \Big\|_p^q \Big)^{1/q} < \infty \Big\}$$

with the usual modification if $q = \infty$.

The space $B^s_{p,q}(\mathbb{T}; X)$ is independent of the choice of ϕ, and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}; X)$. Then $B^s_{p,q}(\mathbb{T}; X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. See [3, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}; X)$. We know that if $s_2 \le s_1$, then $B^{s_1}_{p,q}(\mathbb{T}; X) \subset B^{s_2}_{p,q}(\mathbb{T}; X)$ and the embedding is continuous [3]. When $s > 0$, it was shown in [3] that $B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$, $f \in B^{s+1}_{p,q}(\mathbb{T}; X)$ if and only if f is differentiable almost everywhere on $\mathbb{T}$

https://doi.org/10.4153/CMB-2017-057-6 Published online by Cambridge University Press

and $f' \in B^s_{p,q}(\mathbb{T};X)$. This implies that if $u \in B^s_{p,q}(\mathbb{T};X)$ is such that there exists $v \in B^s_{p,q}(\mathbb{T};X)$ satisfying $\widehat{v}(k) = ik\widehat{u}(k)$ when $k \in \mathbb{Z}$, then $u \in B^{s+1}_{p,q}(\mathbb{T};X)$ and $u' = v$ [3, Lemma 2.1].

Let $1 \le p, q \le \infty$, $s > 0$ be fixed. We study the second order degenerate differential equation with finite delays

$$(P_2) \qquad \begin{cases} (Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t) & (t \in \mathbb{T}), \\ (Mu)(0) = (Mu)(2\pi), \quad (Mu)'(0) = (Mu)'(2\pi). \end{cases}$$

Here A, B, M are closed linear operators on a Banach space X such that $D(A) \cap D(B) \subset D(M)$, and $F, G : B^s_{p,q}([-2\pi, 0]; X) \to X$ are bounded linear operators. Furthermore, for fixed $t \in \mathbb{T}$, u_t and u'_t are elements of $B^s_{p,q}([-2\pi, 0]; X)$ defined by $u_t(s) = u(t+s)$, $u'_t(s) = u'(t+s)$ for $-2\pi \le s \le 0$ and $t \in \mathbb{T}$. Here we identify a function u on $\mathbb{T}$ with its natural 2π-periodic extension on $\mathbb{R}$.

Let $F, G \in \mathcal{L}(B^s_{p,q}(-2\pi, 0); X), X)$ and $k \in \mathbb{Z}$. Let the linear operators F_k, $G_k \in \mathcal{L}(X)$ be defined by $F_k x := F(e_k \otimes x)$, $G_k x := G(e_k \otimes x)$ for all $x \in X$. It is clear that there exists a constant $C > 0$ satisfying $\|e_k \otimes x\|_{B^s_{p,q}} \le C\|x\|$ for all $k \in \mathbb{Z}$. Thus,

$$(3.1) \qquad \|F_k\| \le C\|F\|, \quad \|G_k\| \le C\|G\|, \quad (k \in \mathbb{Z}).$$

We can verify that if $u \in B^s_{p,q}(\mathbb{T};X)$, then

$$\widehat{Fu_\cdot}(k) = F_k\widehat{u}(k) \quad \text{and} \quad \widehat{Gu_\cdot}(k) = G_k\widehat{u}(k),$$

$k \in \mathbb{Z}$. In contrast with the L^p-well-posedness case, we remark that the functions $Fu_\cdot$ and $Gu'_\cdot$ are only uniformly bounded on $\mathbb{T}$, and they are not necessarily in $B^s_{p,q}(\mathbb{T};X)$, even when $u \in W^{1,p}_{\text{per}}(\mathbb{T};X)$. The resolvent set of (P_2) in the $B^s_{p,q}$-well-posedness setting is defined by

$$\rho_{p,q,s}(P_2) := \big\{ k \in \mathbb{Z} : -k^2M + ikB + A - ikG_k - F_k \text{ is a bijection from } D(A) \cap D(B) \text{ onto } X, \text{ and } (-k^2M + ikB + A - ikG_k - F_k)^{-1} \in \mathcal{L}(X) \big\}.$$

When $k \in \rho_{p,q,s}(P_2)$, the operators $M(-k^2M + ikB + A - ikG_k - F_k)^{-1}$, $A(-k^2M + ikB + A - ikG_k - F_k)^{-1}$, and $B(-k^2M + ikB + A - ikG_k - F_k)^{-1}$ are well defined, as $D(A) \cap D(B) \subset D(M)$, and they belong to $\mathcal{L}(X)$ by the closedness of A, B, M and the Closed Graph Theorem.

Let $1 \le p, q \le \infty$, $s > 0$. The solution space of the $B^s_{p,q}$-well-posedness for (P_2) is defined by

$$S_{p,q,s}(A,B,M) := \big\{ u \in B^s_{p,q}(\mathbb{T}; D(A)) \cap B^{s+1}_{p,q}(\mathbb{T};X) : u' \in B^s_{p,q}(\mathbb{T}; D(B)), \; Mu \in B^{s+2}_{p,q}(\mathbb{T};X) \text{ and } Fu_\cdot, Gu'_\cdot \in B^s_{p,q}(\mathbb{T};X) \big\}.$$

Here again we consider $D(A)$ and $D(B)$ as Banach spaces equipped with their graph norms.

Then $S_{p,q,s}(A,B,M)$ is a Banach space with the norm

$$\|u\|_{S_{p,q,s}(A,B,M)} := \|u\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Bu'\|_{B^s_{p,q}} + \|Mu\|_{B^s_{p,q}} + \|(Mu)'\|_{B^s_{p,q}} + \|(Mu)''\|_{B^s_{p,q}} + \|Fu_\cdot\|_{B^s_{p,q}} + \|Gu'_\cdot\|_{B^s_{p,q}}.$$

By [2, Lemma 2.1], if $u \in S_{p,q,s}(A,B,M)$, then u and $(Mu)'$ are X-valued continuous functions on $\mathbb{T}$, and $u(0) = u(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$.

Definition 3.1 Let $1 \le p, q \le \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T}; X)$. Then $u \in S_{p,q,s}(A,B,M)$ is called a strong $B^s_{p,q}$-solution of (P_2), if (P_2) is satisfied almost everywhere on $\mathbb{T}$. We say that (P_2) is $B^s_{p,q}$-well-posed if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique strong $B^s_{p,q}$-solution of (P_2).

If (P_2) is $B^s_{p,q}$-well-posed and $u \in S_{p,q,s}(A,B,M)$ is the unique strong $B^s_{p,q}$-solution of (P_2), there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T}; X)$,

$$\|u\|_{S_{p,q,s}(A,B,M)} \le C\, \|f\|_{B^s_{p,q}}. \tag{3.2}$$

This can be obtained by the closedness of the operators A, B, M and the Closed Graph Theorem.

The main tool in the investigation of $B^s_{p,q}$-well-posedness of (P_2) is the operator-valued $B^s_{p,q}$-Fourier multiplier theory established in [3].

Definition 3.2 Let X, Y be Banach spaces, $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. Then $(M_k)_{k\in\mathbb{Z}}$ is called a $B^s_{p,q}$-Fourier multiplier, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in B^s_{p,q}(\mathbb{T}; Y)$, such that $\widehat{u}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

It is easy to see that when $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, then the set $\{M_k : k \in \mathbb{Z}\}$ must be bounded. The following result gives a sufficient condition for an operator-valued sequence to be a $B^s_{p,q}$-Fourier multiplier [3].

Theorem 3.3 *Let X, Y be Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$. We assume that*

$$\sup_{k\in\mathbb{Z}} \big(\|M_k\| + \|k(M_{k+1} - M_k)\| \big) < \infty, \tag{3.3}$$

$$\sup_{k\in\mathbb{Z}} \big\|k^2(M_{k+2} - 2M_{k+1} + M_k)\big\| < \infty. \tag{3.4}$$

Then for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. If X is B-convex, then the first order condition (3.3) is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-Fourier multiplier.

Recall that a Banach space X is B-convex if it does not contain l^n_1 uniformly. This is equivalent to saying that X has Fourier type $1 < p \le 2$, *i.e.,* the Fourier transform is a bounded linear operator from $L^p(\mathbb{T}; X)$ to $l^q(\mathbb{Z}; X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, $L^p(\mu)$ has Fourier type $\min\{p, \frac{p}{p-1}\}$.

Remark 3.4 (i) If $(M_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, then the product sequence $(M_kN_k)_{k\in\mathbb{Z}}$ and the sum sequence $(M_k + N_k)_{k\in\mathbb{Z}}$ are also $B^s_{p,q}$-Fourier multipliers.

(ii) If $c_k = \frac{1}{k}$ when $k \ne 0$ and $c_0 = 1$, then $(c_kI_X)_{k\in\mathbb{Z}}$ satisfies conditions (3.3) and (3.4). Thus, $(c_kI_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier by Theorem 3.3.

We need the following result for proving the main results of this section.

Proposition 3.5 *Let A, B, and M be closed linear operators defined on a Banach space X satisfying $D(A) \cap D(B) \subset D(M)$, and let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. Assume that $\rho_{p,q,s}(P_2) = \mathbb{Z}$, and that the sets*

$$\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}, \quad \{k(G_{k+1} - G_k) : k \in \mathbb{Z}\},$$
$$\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}, \quad \{-k^2 MN_k : k \in \mathbb{Z}\},$$
$$\{kN_k : k \in \mathbb{Z}\}, \quad \{kBN_k : k \in \mathbb{Z}\}$$

are norm bounded, where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$ when $k \in \mathbb{Z}$. Then $(-k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(kN_k)_{k\in\mathbb{Z}}$, $(kBN_k)_{k\in\mathbb{Z}}$, $(F_kN_k)_{k\in\mathbb{Z}}$, and $(kG_kN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers whenever $1 \le p, q \le \infty$, $s \in \mathbb{R}$.

Proof Let $M_k = -k^2MN_k$, $S_k = kBN_k$, $T_k = kN_k$, $P_k = F_kN_k$, and $Q_k = kG_kN_k$ when $k \in \mathbb{Z}$. We have that $(G_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are norm bounded by (3.1). This implies that the sequences $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, and $(Q_k)_{k\in\mathbb{Z}}$ are norm bounded by assumption. Using the same argument used in the proof of Proposition 2.4, we obtain

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|k(N_{k+1} - N_k)\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k(S_{k+1} - S_k)\| < \infty, \qquad \sup_{k\in\mathbb{Z}} \|k(T_{k+1} - T_k)\| < \infty.$$

Moreover, it is easy to see that we have the stronger estimations

$$\sup_{k\in\mathbb{Z}} \left\|k^2(N_{k+1} - N_k)\right\| < \infty, \quad \sup_{k\in\mathbb{Z}} \left\|k^3M(N_{k+1} - N_k)\right\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\|k^2B(N_{k+1} - N_k)\right\| < \infty, \tag{3.5}$$

by using the norm boundedness of $\{k(G_{k+} - G_k) : k \in \mathbb{Z}\}$. For P_k and Q_k, we have

$$P_{k+1} - P_k = F_{k+1}(N_{k+1} - N_k) + (F_{k+1} - F_k)N_k,$$
$$Q_{k+1} - Q_k = G_{k+1}N_{k+1} + k(G_{k+1} - G_k)N_k + kG_k(N_{k+1} - N_k)$$

when $k \in \mathbb{Z}$. This implies that

$$\sup_{k\in\mathbb{Z}} \|k(P_{k+1} - P_k)\| < \infty, \quad \sup_{k\in\mathbb{Z}} \|k(Q_{k+1} - Q_k)\| < \infty$$

by (3.5) and the boundedness of $(F_k)_{k\in\mathbb{Z}}$, $(G_k)_{k\in\mathbb{Z}}$, and $(k(G_{k+1} - G_k))_{k\in\mathbb{Z}}$.

By (2.4), we have

$$N_{k+1} - N_k = (2k+1)N_{k+1}MN_k - iN_{k+1}BN_k + iN_{k+1}G_{k+1}N_k$$
$$+ ikN_{k+1}(G_{k+1} - G_k)N_k + N_{k+1}(F_{k+1} - F_k)N_k$$
$$=: I^{(1)}_k + I^{(2)}_k + I^{(3)}_k + I^{(4)}_k + I^{(5)}_k.$$

We have

$$I_{k+1}^{(1)} - I_k^{(1)} = (2k+3)N_{k+2}MN_{k+1} - (2k+1)N_{k+1}MN_k$$
$$= 2N_{k+2}MN_{k+1} + (2k+1)(N_{k+2} - N_{k+1})MN_{k+1}$$
$$+ (2k+1)N_{k+1}M(N_{k+1} - N_k).$$

This implies that

$$\sup_{k\in\mathbb{Z}} \left\| k^3(I_{k+1}^{(1)} - I_k^{(1)}) \right\| < \infty, \quad \sup_{k\in\mathbb{Z}} \left\| k^4 M(I_{k+1}^{(1)} - I_k^{(1)}) \right\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\| k^3 B(I_{k+1}^{(1)} - I_k^{(1)}) \right\| < \infty$$

using (3.5). A similar argument shows that

$$\sup_{k\in\mathbb{Z}} \left\| k^3(I_{k+1}^{(i)} - I_k^{(i)}) \right\| < \infty, \quad \sup_{k\in\mathbb{Z}} \left\| k^4 M(I_{k+1}^{(i)} - I_k^{(i)}) \right\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\| k^3 B(I_{k+1}^{(i)} - I_k^{(i)}) \right\| < \infty,$$

when $i = 2, 3, 4, 5$, using (3.5) and the norm boundedness of $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$, $\{k(G_{k+1} - G_k : k \in \mathbb{Z}\}$, and $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$. We have shown that

$$\sup_{k\in\mathbb{Z}} \left\| k^3(N_{k+2} - 2N_{k+1} + N_k) \right\| < \infty, \tag{3.6}$$

$$\sup_{k\in\mathbb{Z}} \left\| k^4 M(N_{k+2} - 2N_{k+1} + N_k) \right\| < \infty, \tag{3.7}$$

$$\sup_{k\in\mathbb{Z}} \left\| k^3 B(N_{k+2} - 2N_{k+1} + N_k) \right\| < \infty. \tag{3.8}$$

In particular,

$$\sup_{k\in\mathbb{Z}} \left\| k^2(N_{k+2} - 2N_{k+1} + N_k) \right\| < \infty.$$

By using an argument similar to that used in the proof of (3.6), we show that

$$\sup_{k\in\mathbb{Z}} \left\| k^2(M_{k+2} - 2M_{k+1} + M_k) \right\| < \infty, \quad \sup_{k\in\mathbb{Z}} \left\| k^2(S_{k+2} - 2S_{k+1} + S_k) \right\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\| k^2(T_{k+2} - 2T_{k+1} + T_k) \right\| < \infty, \quad \sup_{k\in\mathbb{Z}} \left\| k^2(P_{k+2} - 2P_{k+1} + P_k) \right\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\| k^2(Q_{k+2} - 2Q_{k+1} + Q_k) \right\| < \infty.$$

Therefore, $(N_k)_{k\in\mathbb{Z}}$, $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, and $(Q_k)_{k\in\mathbb{Z}}$ are $B_{p,q}^s$-Fourier multipliers, by Theorem 3.3. ■

Now we give a necessary condition for the $B_{p,q}^s$-well-posedness of (P_2).

Theorem 3.6 *Let X be a Banach space, $1 \le p, q \le \infty$, $s > 0$ and let A, B, M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M)$. Let*

$$F, G \in \mathcal{L}\big(B_{p,q}^s([-2\pi, 0]; X), X\big).$$

https://doi.org/10.4153/CMB-2017-057-6 Published online by Cambridge University Press

Assume that (P_2) is $B^s_{p,q}$-well-posed; then $\rho_{p,q,s}(P_2) = \mathbb{Z}$, and the sets

$$\{-k^2 MN_k : k \in \mathbb{Z}\}, \quad \{kBN_k : k \in \mathbb{Z}\}, \quad \text{and} \quad \{kN_k : k \in \mathbb{Z}\}$$

are norm bounded, where $N_k = (-k^2M + ikB + A - ikG_k - F_k)^{-1}$ when $k \in \mathbb{Z}$.

Proof Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$ $(t \in \mathbb{T})$. Then

$$f \in B^s_{p,q}(\mathbb{T}; X), \quad \widehat{f}(k) = y, \quad \text{and} \quad \widehat{f}(n) = 0$$

when $n \neq k$. Since (P_2) is $B^s_{p,q}$-well-posed, there exists $u \in S_{p,q,s}(A, B, M)$ such that

$$(Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t + f(t)$$

almost everywhere on $\mathbb{T}$. We have $\widehat{u}(n) \in D(A) \cap D(B)$ when $n \in \mathbb{Z}$ by [2, Lemmas 2.1 and 3.1], as $u \in B^s_{p,q}(\mathbb{T}; D(A))$ and $u' \in B^s_{p,q}(\mathbb{T}; D(B))$. Taking Fourier transforms on both sides, we get

$$(-k^2M + ikB + A - ikG_k - F_k)\widehat{u}(k) = y \tag{3.9}$$

and $(-n^2M + inB + A - inG_n - F_n)\widehat{u}(n) = 0$ when $n \neq k$. Thus, $-k^2M + ikB + A - ikG_k - F_k$ is surjective. To show that it is also injective, we let $x \in D(A) \cap D(B)$ be such that

$$(-k^2M + ikB + A - ikG_k - F_k)x = 0$$

and let $u(t) = e^{ikt}x$ for $t \in \mathbb{T}$. Then $u \in S_{p,q,s}(A, B, M)$ and (P_2) holds almost everywhere on $\mathbb{T}$ when taking $f = 0$. Therefore, u is a strong L^p-solution of (P_2) when $f = 0$. We obtain $u = 0$ by the uniqueness assumption, hence $x = 0$. We have shown that $-k^2M + ikB + A - ikG_k - F_k$ is also injective. Thus, $-k^2M + ikB + A - ikG_k - F_k$ is a bijection from $D(A)$ onto X.

Next we show that $(-k^2M + ikB + A - ikG_k - F_k)^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, let $u \in S_{p,q,s}(A, B, M)$ be the strong $B^s_{p,q}$-solution of (P_2). Then, taking Fourier transforms on both sides of (P_2), we have

$$\widehat{u}(n) = \begin{cases} 0 & n \neq k, \\ (-k^2M + ikB + A - ikG_k - F_k)^{-1}y & n = k, \end{cases}$$

by (3.9). This implies that $u(t) = e^{ikt}(-k^2M + ikB + A - ikG_k - F_k)^{-1}y$ when $t \in \mathbb{T}$. By (3.2), there exists a constant $C > 0$ independent from y and k such that

$$\|u\|_{B^s_{p,q}} + \|u'\|_{B^s_{p,q}} + \|(Mu)''\|_{B^s_{p,q}} \le C\,\|f\|_{B^s_{p,q}}.$$

We deduce that $\|u\|_{B^s_{p,q}} \le C\,\|f\|_{B^s_{p,q}}$. This implies that

$$\|(-k^2M + ikB + A - ikG_k - F_k)^{-1}y\| \le C\,\|y\|$$

for all $y \in X$. Therefore,

$$\|(-k^2M + ikB + A - ikG_k - F_k)^{-1}\| \le C.$$

We have shown that $k \in \rho_{p,q,s}(P_2)$. Therefore, $\rho_p(P_2) = \mathbb{Z}$.

Let

$$M_k = -k^2 M(-k^2 M + ikB + A - ikG_k - F_k)^{-1},$$
$$S_k = kB(-k^2 M + ikB + A - ikG_k - F_k)^{-1},$$
$$T_k = k(-k^2 M + ikB + A - ikG_k - F_k)^{-1}$$

when $k \in \mathbb{Z}$. We are going to show that $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(T_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers. Indeed, let $f \in B^s_{p,q}(\mathbb{T}; X)$ be fixed. There exists $u \in S_{p,q,s}(A, B, M)$, a strong $B^s_{p,q}$-solution of (P_2) by assumption. Taking Fourier transforms on both sides of (P_2), we get that $\widehat{u}(k) \in D(A) \cap D(B)$ by [2, Lemmas 2.1 and 3.1] and

$$(-k^2 M + ikB + A - ikG_k - F_k)\widehat{u}(k) = \widehat{f}(k)$$

when $k \in \mathbb{Z}$. Since $-k^2 M + ikB + A - ikG_k - F_k$ is invertible, we obtain

$$\widehat{u}(k) = (-k^2 M + ikB + A - ikG_k - F_k)^{-1}\widehat{f}(k)$$

when $k \in \mathbb{Z}$. We have

$$\widehat{u'}(k) = ik\widehat{u}(k), \quad \widehat{Bu'}(k) = ikB\widehat{u}(k), \quad \text{and} \quad \widehat{(Mu)''}(k) = -k^2 M\widehat{u}(k)$$

by [2, Lemmas 2.1 and 3.1]. Therefore,

$$\widehat{u'}(k) = iT_k\widehat{f}(k), \widehat{Bu'}(k) = iS_k\widehat{f}(k), \widehat{(Mu)''}(k) = M_k\widehat{f}(k)$$

when $k \in \mathbb{Z}$. This implies that $(M_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(T_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers as $u', Bu', (Mu)'' \in B^s_{p,q}(\mathbb{T}; X)$ by assumption. It follows that the sets $\{M_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, and $\{T_k : k \in \mathbb{Z}\}$ are norm bounded. This completes the proof. ∎

The following result gives a necessary and sufficient condition for (P_2) to be the $B^s_{p,q}$-well-posed.

Theorem 3.7 *Let X be a Banach space and $1 \le p, q \le \infty$, $s > 0$, let A, B, M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M)$. Let*

$$F, G \in \mathcal{L}\big(B^s_{p,q}([-2\pi, 0]; X), X\big).$$

Assume that the sets $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$, $\{k^2(G_{k+2} - 2G_{k+1} + G_k) : k \in \mathbb{Z}\}$, and $\{k(F_{k+2} - 2F_{k+1} + F_k) : k \in \mathbb{Z}\}$ are norm bounded. Then the following assertions are equivalent.

(i) *(P_2) is $B^s_{p,q}$-well-posed;*
(ii) *$\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{-k^2 MN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (-k^2 M + ikB + A - ikG_k - F_k)^{-1}$.*

Proof It follows from Theorem 3.6 that the implication (i)⇒(ii) is valid. To show that the implication (ii)⇒(i) remains true, we assume that $\rho_{p,q,s}(P_2) = \mathbb{Z}$. Let $M_k = -k^2 MN_k$, $S_k = kBN_k$, $T_k = kN_k$, $P_k = F_k N_k$, and $Q_k = kG_k N_k$ when $k \in \mathbb{Z}$. It follows from Proposition 3.5 that $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(T_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, and $(Q_k)_{k\in\mathbb{Z}}$

https://doi.org/10.4153/CMB-2017-057-6 Published online by Cambridge University Press

are $B^s_{p,q}$-Fourier multipliers. Then for all $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists $u, v, w, x \in B^s_{p,q}(\mathbb{T}; X)$ satisfying

$$\widehat{u}(k) = M_k \widehat{f}(k), \quad \widehat{v}(k) = iS_k \widehat{f}(k), \quad \widehat{w}(k) = N_k \widehat{f}(k), \quad \widehat{x}(k) = iT_k \widehat{f}(k) \tag{3.10}$$

when $k \in \mathbb{Z}$. This implies that $\widehat{x}(k) = ik\widehat{w}(k)$ for all $k \in \mathbb{Z}$. Hence, $w \in B^{s+1}_{p,q}(\mathbb{T}; X)$ and $w' = x$ as $x \in B^s_{p,q}(\mathbb{T}; X)$ by [2, Lemma 2.1]. Again by (3.10), we have $\widehat{v}(k) = ikB\widehat{w}(k)$ when $k \in \mathbb{Z}$. This implies that $w' \in B^s_{p,q}(\mathbb{T}; D(B))$ [2, Lemmas 2.1 and 3.1]. Since $(P_k)_{k\in\mathbb{Z}}$ and $(Q_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, then $Fw, Gw' \in B^s_{p,q}(\mathbb{T}; X)$ as $\widehat{Fw}$ and $\widehat{Gw'}$

$$\widehat{Fw}(k) = F_k \widehat{w}(k) = P_k \widehat{f}(k), \quad \widehat{Gw'}(k) = G_k \widehat{w'}(k) = ikG_k \widehat{w}(k) = iQ_k \widehat{f}(k)$$

when $k \in \mathbb{Z}$. We observe that

$$AN_k = I_X - M_k - iS_k + iQ_k + P_k$$

when $k \in \mathbb{Z}$. It follows that $(AN_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-Fourier multiplier, as the sum of $B^s_{p,q}$-Fourier multipliers is still a $B^s_{p,q}$-Fourier multiplier. Then there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{g}(k) = AN_k \widehat{f}(k) = A\widehat{w}(k),$$

when $k \in \mathbb{Z}$. We deduce that $w \in B^s_{p,q}(\mathbb{T}; D(A))$ [2, Lemma 3.1].

By Remark 3.4, the sequence $(\frac{1}{k} I_X)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, hence $(ikMN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier, since $(k^2MN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-Fourier multiplier. Therefore, there exists $h \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{h}(k) = ikMN_k \widehat{f}(k) = ik\widehat{Mw}(k),$$

when $k \in \mathbb{Z}$. Thus, $Mw \in B^{1+s}_{p,q}(\mathbb{T}; X)$ by [2, Lemmas 2.1 and 3.1]. By (3.10), we have

$$\widehat{u}(k) = -k^2 MN_k \widehat{f}(k) = ik\widehat{(Mw)'}(k)$$

when $k \in \mathbb{Z}$. Thus, we obtain $(Mw)' \in B^{1+s}_{p,q}(\mathbb{T}; X)$ by [2, Lemmas 2.1 and 3.1]. We have shown that $w \in S_{p,q,s}(A, B, M)$. Again by (3.10), we have

$$\widehat{(Mw)''}(k) + ikB\widehat{w}(k) + A\widehat{w}(k) = ikG_k \widehat{w}(k) + F_k \widehat{w}(k) + \widehat{f}(k)$$

when $k \in \mathbb{Z}$. It follows that

$$(Mw)''(t) + Bw'(t) + Aw(t) = Gw'_t + Fw_t + f(t)$$

almost everywhere on $\mathbb{T}$ by the uniqueness of Fourier coefficients [2, p. 314]. Thus, w is a strong $B^s_{p,q}$-solution of (P_2). This shows the existence.

To show the uniqueness, we let $u \in S_{p,q,s}(A, B, M)$ be such that

$$(Mu)''(t) + Bu'(t) + Au(t) = Gu'_t + Fu_t$$

almost everywhere on $\mathbb{T}$. Taking the Fourier transforms on both sides, we have

$$(-k^2 M + ikB + A - ikG_k - F_k)\widehat{u}(k) = 0$$

when $k \in \mathbb{Z}$. Since $\rho_p(P_2) = \mathbb{Z}$, this implies that $\widehat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. We have shown that (P_2) is $B^s_{p,q}$-well-posed. This completes the proof. ■

When the underlying Banach space X is B-convex, condition (3.3) is already sufficient for a sequence to be a $B^s_{p,q}$-Fourier multiplier. This, together with the proofs of Theorems 2.6 and 3.7, gives the following corollary.

Corollary 3.8 *Let X be a B-convex Banach space and $1 \le p,q \le \infty, s > 0$, let A, B, M be closed linear operators on X satisfying $D(A) \cap D(B) \subset D(M)$. Let $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi, 0]; X), X)$. We assume that the sets $\{k(G_{k+1} - G_k) : k \in \mathbb{Z}\}$ is norm bounded. Then the following assertions are equivalent.*

(i) *(P_2) is $B^s_{p,q}$-well-posed;*

(ii) *$\rho_{p,q,s}(P_2) = \mathbb{Z}$ and the sets $\{-k^2 MN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$ are norm bounded, where $N_k = (-k^2 M + ikB + A - ikG_k - F_k)^{-1}$ when $k \in \mathbb{Z}$.*

4 Applications

In this section, we give examples to which our abstract results (Theorems 2.6 and 3.7) can be applied.

Example 4.1 Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and m be a non-negative bounded measurable function defined on Ω. Let f be a given function on $[0, 2\pi] \times \Omega$ and $X = H^{-1}(\Omega)$. We consider the periodic degenerate differential equations with finite delay

$$(P) \qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}(m(x)u(t,x)) + B\dfrac{\partial}{\partial t}u(t,x) + \Delta u & \\ \qquad = Fu_t + Gu'_t + f(t,x), & (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = 0, & (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi, x), & x \in \Omega, \\ \dfrac{\partial u}{\partial t}(0,x) = \dfrac{\partial u}{\partial t}(2\pi, x), & x \in \Omega, \end{cases}$$

where B is a bounded linear operator on X, $u_t(s,x) := u(t+s,x)$, $u'_t(s,x) := 3u'(t+s,x)$ when $s \in [-2\pi, 0]$ and $x \in \Omega$, the delay operators $F, G\colon L^p([-2\pi,0]; X) \to X$ are bounded linear operators for some fixed $1 < p < \infty$.

Let M be the operator of multiplication by m on $H^{-1}(\Omega)$ with domain $D(M)$. Then it follows from *[8, Section 3.7]* that if we consider the Laplacian Δ on X with Dirichlet boundary condition, then there exists a constant $C > 0$ such that

$$\left\| M(zM - \Delta)^{-1} \right\| \le \frac{C}{1 + |z|},$$

when $\mathrm{Re}(z) \ge -\beta(1 + |\mathrm{Im}(z)|)$ for some positive constant β depending only on m, which implies that

$$\left\| M(k^2 M - \Delta)^{-1} \right\| \le \frac{C}{1 + |k|^2} \tag{4.1}$$

when $k \in \mathbb{Z}$. If we assume that m is regular enough so that the operator of multiplication by the function m^{-1} is bounded on $H^{-1}(\Omega)$, then there exists a constant C_1 such

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that

$$\|(k^2M-\Delta)^{-1}\| \le \frac{C_1}{1+|k|^2} \tag{4.2}$$

when $k \in \mathbb{Z}$. Assume that $D(\Delta) \subset D(M)$ and the set $\{k(G_{k+1}-G_k) : k \in \mathbb{Z}\}$ is norm bounded. Furthermore, we assume that $\rho_p(P) = \mathbb{Z}$ so that for all $k \in \mathbb{Z}$, the operator $-k^2M + ikB + \Delta - F_k - ikG_k$ is a bijection from $D(\Delta)$ onto X, and $(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1} \in \mathcal{L}(X)$. We observe that

$$-k^2M + ikB + \Delta - F_k - ikG_k = \big(I - (F_k + ikG_k - ikB)(-k^2M+\Delta)^{-1}\big)(-k^2M+\Delta)$$

when $k \in \mathbb{Z}$. It follows from the estimation (4.2) that

$$\lim_{k\to\infty} \|(F_k + ikG_k - ikB)(-k^2M+\Delta)^{-1}\| = 0$$

using the norm boundedness of $(F_k)_{k\in\mathbb{Z}}$ and $(G_k)_{k\in\mathbb{Z}}$. This implies that $I - (-k^2M + \Delta)^{-1}(F_k + ikG_k - ikB)$ is invertible when $|k|$ is big enough. For such k we have

$$(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1} = \\ (-k^2M+\Delta)^{-1}\big(I - (F_k + ikG_k - ikB)(-k^2M+\Delta)^{-1}\big)^{-1}$$

when $k \in \mathbb{Z}$. It follows from (4.1) and (4.2) that

$$\sup_{k\in\mathbb{Z}} \|k(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1}\| < \infty,$$
$$\sup_{k\in\mathbb{Z}} \|k^2M(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1}\| < \infty.$$

Consequently, the sets

$$\{k(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\},$$
$$\{kB(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\},$$
$$\{k^2M(-k^2M + ikB + \Delta - F_k - ikG_k)^{-1} : k \in \mathbb{Z}\}$$

are R-bounded. Here we use the fact that if the underlying Banach space X is a Hilbert space, then each norm bounded subset of $\mathcal{L}(X)$ is R-bounded *[2, Proposition 1.13]*. We deduce from Theorem 2.6 that (P) is L^p-well-posed when $X = H^{-1}(\Omega)$.

If we consider $F, G \in \mathcal{L}(B^s_{p,q}([-2\pi,0];X),X)$, we can also apply Theorem 3.7 to obtain the $B^s_{p,q}$-well-posedness of (P) under suitable assumptions on F and G.

Example 4.2 Let H be a complex Hilbert space, $1 < p < \infty$ and let

$$F, G \in \mathcal{L}\big(L^p([-2\pi,0],H),H\big)$$

be delay operators. Let P be a densely defined positive selfadjoint operator on H with $P \ge \delta > 0$. Let $M = P - \epsilon$ with $\epsilon < \delta$, and let $A = \sum_{i=0}^k a_i P^i$ with $a_i \ge 0$, $a_k > 0$. Then there exists a constant $C > 0$, such that

$$\|M(zM+A)^{-1}\| \le \frac{C}{1+|z|}$$

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whenever $\operatorname{Re} z \geq -\beta(1+|\operatorname{Im} z|)$ for some positive constant β depending only on A and M by [8, p. 73]. This implies in particular that

$$\sup_{k\in\mathbb{Z}} \|k^2 M(k^2 M + A)^{-1}\| < \infty.$$

If we assume $0 \in \rho(M)$, then

$$\sup_{k\in\mathbb{Z}} \|k^2 (k^2 M + A)^{-1}\| < \infty.$$

Furthermore, we assume that the set $\{k(G_{k+1} - G_k : k \in \mathbb{Z})\}$ is norm bounded. Then the argument used in Example 4.1 shows that the degenerate differential system with finite delay

$$(\mathrm{P}') \quad \begin{aligned} (Mu)''(t) + Bu'(t) &= Au(t) + Gu'_t + Fu_t + f(t), \quad (t \in \mathbb{T}),\\ (Mu)(0) &= (Mu)(2\pi), \qquad (Mu)'(0) = (Mu)'(2\pi), \end{aligned}$$

is L^p-well-posed when $\rho_p(P') = \mathbb{Z}$, where B is a bounded linear operator on H. Under suitable assumptions on F and G, we can also apply Theorem 3.7 to obtain the $B^s_{p,q}$-well-posedness of (P') for all $1 \leq p, q \leq \infty$, $s > 0$.

Now we give a concrete example of (P'). Consider the problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(t,x) + B\frac{\partial}{\partial t}u(t,x) = \frac{\partial^4}{\partial x^4}u(t,x)\\ &\qquad + Fu_t(\cdot,x) + G(\frac{\partial u}{\partial t})_t(\cdot,x) + f(t,x), \quad (t,x) \in (0,2\pi)\times\Omega,\\ &u(t,0) = u(t,1) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,1) = 0, \quad t \in [0,2\pi],\\ &u(0,x) = u(2\pi,x), \quad \Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(0,x) = \Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(2\pi,x), \quad x \in \Omega,\\ &\frac{\partial}{\partial t}\Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(0,x) = \frac{\partial}{\partial t}\Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(2\pi,x), \quad x \in \Omega, \end{aligned}$$

where $\Omega = (0,1)$, $F, G \in \mathcal{L}(L^p([-2\pi,0]; L^2(\Omega)), L^2(\Omega))$ and $u_t(s,x) := u(t+s,x)$ when $t \in [0,2\pi]$, $x \in \Omega$ and $s \in [-2\pi, 0]$. Let $X = L^2(\Omega)$, let $P = -\frac{\partial^2}{\partial x^2}$ with domain $D(P) = H^2(\Omega) \cap H^1_0(\Omega)$, *i.e.,* P is the Laplacian on $L^2(\Omega)$ with Dirichlet boundary conditions, B is a bounded linear operator on X. Then P is positive self adjoint on X. Let $M = P + I_X$ and $A = P^2$. It is clear that $-P$ generates an contraction semigroup on $L^2(\Omega)$ [1, Example 3.4.7]; hence, $1 \in \rho(-P)$, or equivalently $M = I_X + P$ has a bounded inverse, *i.e.*, $0 \in \rho(M)$. Then the abstract results obtained above for the problem (P') can be applied.

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