DERIVATIONS ON COMMUTATIVE
OPERATOR ALGEBRAS

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algebra is implemented by a bounded operator. In this note we
present a simple alternative proof, which generalises the result
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algebra is implemented by a bounded operator. This result can be extended
to certain non-self-adjoint commutative algebras by similarity. In this
note we present a simple alternative proof, which generalises the result
further within Hilbert space, and to reflexive Banach spaces.

Let us call a set of Banach space operators a *projection algebra* if
it is the algebra, or norm-closed algebra, generated by its projections.
We will say that a projection algebra is *bounded*, with *bound* k , if the
real number k is a uniform bound for the projections in the algebra.
If E is a Banach space, then $B(E)$ denotes the set of bounded operators
on E and a *derivation* on a subalgebra A is a linear map δ from A
into $B(E)$ such that $\delta(ab) = a\delta(b) + \delta(a)b$ for a,b in A . A
derivation is said to be *implemented* if it is of the form $d_x(a) = xa - ax$
for some fixed operator x on E and all a in A . All Banach spaces
mentioned are assumed to be complex, and all algebras to be unital. Note
that any "complete" Boolean algebra of projections is bounded. (See [2]).

Every derivation on a commutative algebra generated by a finite set
of Banach space projections is known to be implemented by an operator

which can be described explicitly. We obtain a bound for this operator
which is independent of the number of generating projections. It follows
immediately that continuous derivations on commutative bounded projection
algebras on reflexive Banach spaces are implemented.

We abuse notation a little by saying that projections p and q are
orthogonal if $pq = qp = 0$. Denote $1 - p$ by $p^{\perp}$.

Then, given any finite commutative set $\{q_i\}, i = 1,\ldots,n$ of
projections, there is a set $\{p_i\}, i = 1,\ldots,2^n$, of orthogonal projections
(some of which may be zero) generating the same algebra as $\{q_i\}$.

Before proving the crucial result we mention the fact that a Banach
space E is reflexive if and only if the unit ball of $B(E)$ is compact
in the weak operator topology (Chapter VI, Dunford and Schwartz [2]).
Note also that a von Neumann algebra on a Hilbert space is generated by its
self-adjoint projections [1].

We first state as a lemma the known finite case.

LEMMA 1. *Let A be a finitely generated commutative projection
algebra on any Banach space E and let δ be any derivation on A . If
$\{p_i\}$, $i = 1,\ldots,n$, is an orthogonal generating set, then the operator*

$$x = \sum_{i=1}^{n} \delta(p_i)p_i \quad \text{implements} \quad \delta \quad \text{on} \quad A.$$

The proof follows easily from the relations

$$p\delta(p)p = 0 \qquad \text{and} \qquad p\delta(q) = -\delta(p)q$$

for orthogonal projections p , q .

THEOREM 2. $\|x\| \leqslant 4k^2 \|\delta\|$ *where k is a bound for A . If E is a
Hilbert space and the projections $\{p_i\}$ are self-adjoint, then $\|x\| \leqslant \|\delta\|$*

Proof. We claim that for some integer m we can write

$x = \dfrac{1}{m} \sum_{r=1}^{m} \delta(u_r)u_r$, where for each r , $\|u_r\| \leqslant 2k$. In fact we will put

$u_r = \sum_{i=1}^{n} \lambda_{i,r} p_i$ where each $\lambda_{i,r}$ is 1 or -1 .

Put $m = 2^{n-1}$, and let $\{\lambda_{i,r}\}$ form the (finite) matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & . & . & . \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & & & \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & & & \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & & & \\
. & & & & & & & & & & \\
. & & & & & & & & & & \\
. & & & & & & & & & &
\end{pmatrix}
$$

One can easily convince oneself, by checking with $n = 2$ or 3, that

$$
\sum_{r=1}^{m} \lambda_{i,r} \lambda_{j,r} = 0 \quad \text{for} \quad i \neq j .
$$

and $\sum_{r=1}^{m} (\lambda_{i,r})^2 = m$ for each i. Consequently,

$$
\sum_{r=1}^{m} \delta(u_r) u_r = \sum_{r=1}^{m} \sum_{i,j=1}^{n} \lambda_{i,r} \lambda_{j,r} \delta(p_i) p_j
$$

$$
= \sum_{r=1}^{m} \sum_{i=1}^{n} \delta(p_i) p_i = mx , \text{ as required.}
$$

Finally, since each $\lambda_{i,r}$ is ± 1, u_r is the difference of two projections in A and $u_r^2 = 1$. Hence $\|u_r\| \leqslant 2k$, and $\|x\| \leqslant (2k)^2 \|\delta\|$. If $\{p_i\}$ is a set of self-adjoint Hilbert space projections then each u_r is a self-adjoint unitary operator, and so $\|u_r\| = 1$. Then $\|x\| \leqslant \|\delta\|$ as claimed.

We can immediately state the corollary.

COROLLARY 3. *Let A be any commutative bounded projection algebra on a reflexive Banach space E, and let δ be a continuous derivation on A. Then δ is implemented by an operator x such that $\|x\| \leqslant 4k^2 \|\delta\|$, where k is a bound for A. If, in particular, E is a Hilbert space and A a von Neumann algebra then we can choose x so that $\|x\| \leqslant \|\delta\|$.*

Proof. The restriction of δ to any finitely generated projection subalgebra B is implemented by such an operator, by Theorem 2. Let S_B denote the set of operators which implement δ on B and whose norms are bounded by $4k^2 \|\delta\|$. Then S_B is compact in the weak operator topology,

since E is reflexive, and any finite intersection of such sets is non-empty. Hence the intersection of all such sets is non-empty, and the result follows.

REMARK 1. The condition on E of reflexivity applies of course to Hilbert space, where the self-adjoint projections are exactly those of norm 1. Any projection on a Hilbert space which commutes with its adjoint must be self-adjoint. So a commutative projection algebra on a Hilbert space is self-adjoint if and only if it has bound 1. Note that the result holds for general commutative projection C^*-algebras.

REMARK 2. We mention finally that this proof is closely related to an existing one for the case of a von Neumann algebra. I am indebted to Professor Erik Christensen for describing that one to me. We sketch it here for comparison: Let δ be a derivation on a commutative von Neumann algebra A. Note that by Ringrose [3] δ is continuous. Let C be the weakly closed convex set generated by elements of the form $u^*\delta(u)$ where u is unitary in A. For each unitary u define a map $T_u : C \rightarrow C$ by $T_u(a) = uau^* + u\delta(u^*)$. Then C and the maps T_u satisfy the conditions of the Markov-Kakutani fixed-point theorem [2], and so for some $x \in C$, $T_u(x) = x$ for all u. A simple manipulation shows that x implements δ. Our proof is different even in this case in that it substitutes a partially constructive method for the use of the fixed-point theorem.

References

[1]   J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, 2nd edition (Gauthier-Villars, 1969).

[2]   N. Dunford and J.T. Schwartz, *Linear operators*, (Interscience, 1958).

[3]   J.R. Ringrose, "Automatic continuity of derivations on operator algebras", *J. London Maths. Soc.* 5 (1972), 432-438.

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