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A NOTE ON STONE LATTICES

BY
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Stone lattices can be considered as forming a category of abstract algebras and thus there is a forgetful functor from this category to the category of distributive lattices with zero and unit. In this note we consider Stone lattices in this light (cf. [3], [4]) and describe an adjoint to the forgetful functor. The Stone extension of a distributive lattice with zero unit which we obtain differs markedly from the one given in [1].

For our notation we follow [6] while basic facts about minimal prime ideals are given in [5]. We assume familiarity with these works.

1. **Preliminaries.** Let $\mathscr{L}=\langle L; \vee, \wedge, 0, 1\rangle$ be an arbitrary distributive lattice with zero and unit. Then the congruence R on $\mathscr{L}$ is defined by $\langle x, y\rangle \in R$ iff $(x)^*=(y)^*$. A lattice morphism $\phi: \mathscr{L} \to \mathscr{L}'$ is called *R-compatible* if $(x)^*=(y)^*$ in $\mathscr{L}$ implies $(x\phi)^*=(y\phi)^*$ in $\mathscr{L}'$.

For any minimal prime ideal M of $\mathscr{L}$ we define a relation Φ_M by writing $\langle x, y\rangle \in \Phi_M$ iff both x and y belong to M, or both x and y belong to $L\backslash M$.

LEMMA 1. *The relation Φ_M is a congruence on $\mathscr{L}$ and the canonical epimorphism $\phi_M: \mathscr{L} \to \mathscr{L}/\Phi_M \simeq 2$ is R-compatible.*

Proof. It is easy to see that Φ_M is a congruence. Suppose $x, y \in L$ with $(x)^*=(y)^*$. Then if $x \notin M$, $(x)^*=(y)^* \subseteq M$ and so $y \notin M$. Conversely if $x \in M$, $(x)^* \nsubseteq M$ in which case $(y)^* \nsubseteq M$ whence $y \in M$ and ϕ_M is R-compatible.

Suppose now that P is a nonminimal prime ideal and M is a minimal prime ideal contained (strictly) in P. We define a relation $\Phi_{P,M}$ as follows: $\langle x, y\rangle \in \Phi_{P,M}$ iff both x and y belong to M, or both x and y belong to $P\backslash M$, or both x and y belong to $L\backslash P$.

LEMMA 2. *The relation $\Phi_{P,M}$ is a congruence on $\mathscr{L}$ and the canonical epimorphism $\phi_{P,M}: \mathscr{L} \to \mathscr{L}/\Phi_{P,M} \simeq 3$ is R-compatible.*

Proof. As for Lemma 1.

LEMMA 3. *Let x and y be distinct elements of L. If $(x)^* \neq (y)^*$ there is a minimal prime ideal M with $x \not\equiv y(\Phi_M)$. If $(x)^*=(y)^*$ there is a nonminimal prime ideal P and a minimal prime $M \subset P$ with $x \not\equiv y(\Phi_{P,M})$.*

Proof. Suppose $(x)^* \neq (y)^*$. Then by known results (see, e.g., [5]) there is a

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minimal prime ideal M of $\mathscr{L}$ containing only one of x and y. Clearly for such an M, $x \not\equiv y(\Phi_M)$.

If $(x)^* = (y)^*$ then any prime ideal which distinguishes x and y must be non-minimal. One exists by Stone's theorem, say P, and we may choose any minimal prime ideal M contained in P. Then $x \not\equiv y(\Phi_{P,M})$.

2. **The extension theorem.** In this section we prove our main result. It can be best understood if one considers any distributive lattice $\mathscr{L}$ with zero and unit as a *partial Stone lattice* in the obvious manner, i.e. by taking x^* to be defined whenever $(x)^*$ is a direct summand of L; the associated morphisms are what we have called R-compatible, and these become pseudo-complement preserving when $\mathscr{L}$ is a Stone lattice.

THEOREM 1. *Let $\mathscr{L}$ be a distributive lattice with zero and unit. Then there is a Stone lattice $S(\mathscr{L})$ and an R-compatible lattice monomorphism $\sigma: \mathscr{L} \to S(\mathscr{L})$ with the following property: for any R-compatible lattice morphism $\theta: \mathscr{L} \to \mathscr{S}$ of $\mathscr{L}$ into a Stone lattice $\mathscr{S}$, there is a unique Stone lattice morphism $\bar{\theta}: S(\mathscr{L}) \to \mathscr{S}$ such that $\sigma \circ \bar{\theta} = \theta$. The pair $(\sigma, S(\mathscr{L}))$ is defined uniquely up to isomorphism.*

Proof. For every prime ideal P of $\mathscr{L}$ we define

$$\Psi_P = \begin{cases} \Phi_P & \text{if } P \in \mathscr{M}_{\mathscr{L}} \\ \Phi_{P,M} & \text{if } P \in \mathscr{P}_{\mathscr{L}} \backslash \mathscr{M}_{\mathscr{L}} \end{cases}$$

In the second case we select any minimal prime ideal contained inside the non-minimal prime ideal P. Now define $\phi: \mathscr{L} \to \overline{\mathscr{L}} = \times_{P \in \mathscr{P}_{\mathscr{L}}} \mathscr{L}/\Psi_P$ by $x\phi = \langle x\psi_P \rangle_{P \in \mathscr{P}}$ and let $S(\mathscr{L})$ be the intersection of all the Stone sublattices of $\overline{\mathscr{L}}$ containing $\mathscr{L}_\phi$. Writing σ for the restriction of ϕ to codomain $S(\mathscr{L})$ we will prove that the pair $(\sigma, S(\mathscr{L}))$ has the desired properties. Firstly, σ is an R-compatible monomorphism by Lemmas 1, 2, 3. To prove the universal mapping property we need to obtain a representation for the elements of $S(\mathscr{L})$. It is as follows:

For any $s \in S(\mathscr{L})$ there are elements $x_1, x_2, \ldots, x_m$ in L and $z_1, z_2, \ldots, z_m$ in $S(\mathscr{L})$ with $z_i \wedge z_j = 0$ $(i \neq j)$ and $\bigvee_{i=1}^m z_i = 1$ such that $s = \bigvee_{i=1}^m (x_i\sigma) \wedge z_i$. Further, each such central element z_i in the representation of s can be written in the form $\bigvee_{j=1}^n [(a_j\sigma)^{**} \wedge (b_j\sigma)^*]$ for $\{a_j, b_j : j = 1, 2, \ldots, n\} \subseteq L$.

These facts are easily seen. For $x\sigma = (x\sigma \wedge 1) \vee (x\sigma \wedge 0)$, $(x\sigma)^* = (1\sigma \wedge (x\sigma)^*) \vee (0\sigma \wedge (x\sigma)^{**})$, and if two elements $s, t \in S(\mathscr{L})$ have representations

$$s = \bigvee_{i=1}^m (x_i\sigma) \wedge z_i, \qquad t = \bigvee_{j=1}^n (y_j\sigma) \wedge \tilde{z}_j,$$

for suitable $\{z_i\}$, $\{\tilde{z}_j\}$, then

$$s \wedge t = \bigvee_{i,j} (x_i \wedge y_j)\sigma \wedge (z_i \wedge \tilde{z}_j)$$

$$s \vee t = \bigvee_{i,j} (x_i \wedge y_j)\sigma \wedge (z_i \wedge \tilde{z}_j)$$

where the joins are taken over pairs (i, j) such that $z_i \wedge \tilde{z}_j \neq 0$. Also $s^* = (1\sigma \wedge s^*) \vee (0\sigma \wedge s^{**})$, where $s^* = \bigwedge_{i=1}^m ((x_i\sigma)^{**} \wedge z_i)^*$. Thus it can be seen that a typical central element z in such a representation is of the form $\bigvee_{k=1}^p [(a_k\sigma)^{**} \wedge (b_k\sigma)^*]$ as required.

The remaining details are now easy; let $\theta: \mathscr{L} \to \mathscr{S}$ be an R-compatible lattice morphism of $\mathscr{L}$ into a Stone lattice $\mathscr{S}$, and define $\bar{\theta}$ on $S(\mathscr{L})$ by

$$\left[\bigvee_{i=1}^m (x_i\sigma) \wedge z_i\right]\bar{\theta} = \bigvee_{i=1}^m (x_i\theta) \wedge (z_i\bar{\theta})$$

where, if,

$$z_i = \bigvee_{k=1}^p [(a_k\sigma)^{**} \wedge (b_k\sigma)^*],$$

we put

$$z_i\bar{\theta} = \bigvee_{k=1}^p [(a_k\theta)^{**} \wedge (b_k\theta)^*].$$

Because θ is R-compatible this definition makes sense, $\bar{\theta}$ is clearly a Stone lattice morphism, and the unique one for which $\sigma \circ \bar{\theta} = \theta$.

Finally, the uniqueness of the pair $(\sigma, S(\mathscr{L}))$ can be proved by routine methods; it is in fact (up to natural equivalence) the adjoint functor to the forgetful functor from Stone lattices (with Stone lattice morphisms) to distributive lattices with zero and unit (with zero, unit preserving lattice morphisms).

COROLLARY 1. (GRÄTZER [4]) *Let $\mathscr{L}$ be a Stone lattice. Then there is a Stone lattice monomorphism ϕ from $\mathscr{L}$ onto a *-sublattice of a direct product of 2- or 3-element chains.*

Proof. In Stone lattices the concepts of R-compatible lattice morphism and Stone lattice morphism coincide. We then apply Theorem 1, since in this case $\mathscr{L}\phi$ will already be a *-sublattice of $\bar{\mathscr{L}}$.

To formulate a second corollary we need some notation. For $x \in L$ we write $\mathscr{M}_{\mathscr{L}}(x) = \{M \in \mathscr{M}_{\mathscr{L}}: x \notin M\}$ and put $\mu_{\mathscr{L}} = \{\mathscr{M}_{\mathscr{L}}(x): x \in L\}$. As shown in [5] $\mu_{\mathscr{L}}$ is a disjunctive lattice of subsets of $\mathscr{M}_{\mathscr{L}}$ and we denote by $\bar{\mu}_{\mathscr{L}}$ the Boolean lattice of subsets of $\mathscr{M}_{\mathscr{L}}$ generated by $\mu_{\mathscr{L}}$. A little computation shows that a typical element of $\bar{\mu}_{\mathscr{L}}$ has the form

$$\bigcup_{i=1}^m [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c] \quad \text{for} \quad \{x_i, y_i: i = 1, \ldots, m\} \subseteq L.$$

COROLLARY 2. *The pair $(\sigma, S(\mathscr{L}))$ satisfies the following properties:*

(1) *$\sigma: \mathscr{L} \to S(\mathscr{L})$ is an R-compatible lattice monomorphism from $\mathscr{L}$ into a Stone lattice $S(\mathscr{L})$;*

(2) *The induced map $\sigma^*: \bar{\mu}_{\mathscr{L}} \to \mu_{S(\mathscr{L})}$ defined by*

$$\left\{\bigcup_{i=1}^m [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c]\right\}\sigma^* = \bigcup_{i=1}^m [\mathscr{M}_{S(\mathscr{L})}(x_i\sigma) \cap \mathscr{M}_{S(\mathscr{L})}(y_i\sigma)^c]$$

is a Boolean isomorphism.

(3) *For any element* $s \in S(\mathscr{L})$ *there are elements* $x_1, \ldots, x_m$ *of* L *and* $z_1, \ldots, z_m$ *of* $S(\mathscr{L})$ *with* $z_i \wedge z_j = 0$, $i \neq j$, *and* $\bigvee_{i=1}^m z_i = 1$ *such that* $s = \bigvee_{i=1}^m (x_i\sigma) \wedge z_i$.

Proof. (1) and (3) have already been noted in the course of the proof of Theorem 1. It remains to show that σ^* is a bijection since it is clearly a Boolean morphism. Suppose σ^* maps $\bigcup_{i=1}^m [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c]$ to the zero $\square$ of $\mu_{S(\mathscr{L})}$. Then we must have

$$\mathscr{M}_{S(\mathscr{L})}(x_i\sigma) \cap \mathscr{M}_{S(\mathscr{L})}(y_i\sigma)^c = \square, \quad i = 1, 2, \ldots, m.$$

But this is just the relation

$$\mathscr{M}_{S(\mathscr{L})}(x_i\sigma) \subseteq \mathscr{M}_{S(\mathscr{L})}(y_i\sigma), \quad i = 1, 2, \ldots, m,$$

from which we deduce that $(x_i\sigma)^* \supseteq (y_i\sigma)^*$, $i = 1, 2, \ldots, m$. Now the fact that σ is a lattice monomorphism implies that $(x_i)^* \supseteq (y_i)^*$ in L, $i = 1, 2, \ldots, n$, and so $\mathscr{M}_{\mathscr{L}}(x_i) \subseteq \mathscr{M}_{\mathscr{L}}(y_i)$, $i = 1, 2, \ldots, m$, whence

$$\bigcup_{i=1}^m [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c] = \square.$$

This result, which shows that σ^* is a Boolean monomorphism, can also be obtained directly by identifying the minimal prime ideals of $S(\mathscr{L})$.

Finally, σ^* is an epimorphism, since every element in $\mu_{S(\mathscr{L})}$ must be of the form $\mathscr{M}_{S(\mathscr{L})}(z)$ (see [6]), and if

$$z = \bigvee_{i=1}^m [(x_i\sigma)^{**} \wedge (y_i\sigma)^*]$$

we have

$$\begin{aligned}\left\{\bigcup_{i=1}^m [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c]\right\}\sigma^* &= \bigcup_{i=1}^m [\mathscr{M}_{S(\mathscr{L})}(x_i\sigma) \cap \mathscr{M}_{S(\mathscr{L})}(y_i\sigma)^c] \\ &= \bigcup_{i=1}^m [\mathscr{M}_{S(\mathscr{L})}((x_i\sigma)^{**}) \cap \mathscr{M}_{S(\mathscr{L})}((y_i\sigma)^*)] \\ &= \bigcup_{i=1}^m \mathscr{M}_{S(\mathscr{L})}\left(\bigvee_{i=1}^m [(x_i\sigma)^{**} \wedge (y_i\sigma)^*]\right) \\ &= \mathscr{M}_{S(\mathscr{L})}(z).\end{aligned}$$

This completes the proof of the corollary.

Our final result states that the three conditions in Corollary 2 describe the extension $(\sigma, S(\mathscr{L}))$ up to Stone lattice isomorphism over $\mathscr{L}$.

THEOREM 2. *Let* $(\tau, T(\mathscr{L}))$ *be an extension of the distributive lattice with zero and unit* $\mathscr{L}$ *satisfying*:

(1) $\tau: \mathscr{L} \to T(\mathscr{L})$ *is an R-compatible lattice monomorphism of* $\mathscr{L}$ *into a Stone lattice* $T(\mathscr{L})$;

(2) *The induced map* $\tau^*: \bar{\mu}_{\mathscr{L}} \to \mu_{T(\mathscr{L})}$ *is a Boolean isomorphism*;

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(3) *For any element t of* $T(\mathscr{L})$ *there are elements* $x_1, \ldots, x_m$ *of L and* $z_1, \ldots, z_m$ *of* $T(\mathscr{L})$ *with* $z_i \wedge z_j = 0$, $i \neq j$, $\bigvee_{i=1}^{m} z_i = 1$ *such that* $t = \bigvee_{i=1}^{m} (x_i\tau) \wedge z_i$. *Then there are Stone lattice isomorphisms* $\bar{\tau}: S(\mathscr{L}) \to T(\mathscr{L})$, $\bar{\sigma}: T(\mathscr{L}) \to S(\mathscr{L})$ *such that* $\bar{\tau} \circ \bar{\sigma} = 1_{S(\mathscr{L})}$, $\bar{\sigma} \circ \bar{\tau} = 1_{T(\mathscr{L})}$.

Proof. First we need to locate the central elements of $T(\mathscr{L})$. By (2) for a central z there is $\{x_i, y_i : i = 1, \ldots, m\} \subseteq L$ such that

$$\mathscr{M}_{T(\mathscr{L})}(z) = \left\{\bigcup_{i=1}^{m} [\mathscr{M}_{\mathscr{L}}(x_i) \cap \mathscr{M}_{\mathscr{L}}(y_i)^c]\right\}\tau^*$$

$$= \bigcup_{i=1}^{m} [\mathscr{M}_{T(\mathscr{L})}(x_i\tau) \cap \mathscr{M}_{T(\mathscr{L})}(y_i\tau)^c].$$

Using the fact that in a Stone lattice $z \leftrightarrow \mathscr{M}(z)$ is 1:1 when z is central, and other results from [5], [6] we conclude

$$z = \bigvee_{i=1}^{m} [(x_i\tau)^{**} \wedge (y_i\tau)^*].$$

For such a central $z \in T(\mathscr{L})$ we define $z\bar{\sigma}$ by

$$z\bar{\sigma} = \bigvee_{i=1}^{m} [(x_i\sigma)^{**} \wedge (y_i\sigma)^*].$$

Using the representation of (3) and putting $(x\tau)\bar{\sigma} = x\sigma$ we define the map $\bar{\sigma}: T(\mathscr{L}) \to S(\mathscr{L})$. Exactly similarly (by Corollary 2) we define $\tau: S(\mathscr{L}) \to T(\mathscr{L})$ and it is easy to check that $\bar{\sigma} \circ \bar{\tau} = 1_{T(\mathscr{L})}$, $\bar{\tau} \circ \bar{\sigma} = 1_{S(\mathscr{L})}$. This completes the proof of Theorem 2.

3. **Final remarks.** The relation between the construction of a Stone extension described above and the one given in C. C. Chen [1] is not clear; indeed if the centre of the Stone extension given in [1] coincides with the Boolean extension of a distributive lattice with zero and unit, then in this case a Stone extension of a Stone lattice is strictly bigger than the Stone lattice in general. The problem seems to be universal mapping problem; it is hoped that the above solution is clear on this point.

Also it is natural to ask for a triple description [2] $(B_{S(\mathscr{L})}, D_{S(\mathscr{L})}, \psi^{S(\mathscr{L})})$ of the Stone extension. Clearly $B_{S(\mathscr{L})} = \bar{\mu}_{\mathscr{L}}$ but the exact details of $D_{S(\mathscr{L})}$ are rather intricate and we decided to omit this aspect. In fact $D_{S(\mathscr{L})}$ is constructed in a very similar manner to the construction given in [1]; the map $\phi^{S(\mathscr{L})}$ can also be (somewhat clumsily) defined. If a simple construction for this triple exists, the present author is unaware of it.

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