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Tracial oscillation zero and stable rank one

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Abstract. Let A be a separable (not necessarily unital) simple C^*-algebra with strict comparison. We show that if A has tracial approximate oscillation zero, then A has stable rank one and the canonical map Γ from the Cuntz semigroup of A to the corresponding lower-semicontinuous affine function space is surjective. The converse also holds. As a by-product, we find that a separable simple C^*-algebra which has almost stable rank one must have stable rank one, provided it has strict comparison and the canonical map Γ is surjective.

1 Introduction

Let X be a compact metric space and T be a set of probability Borel measures on X. For each open subset O of X, we consider its measure $\mu(O)$. This gives a function $\widehat{O}(\mu) = \mu(O)$ $(\mu \in T)$ on T. This function is lower-semicontinuous on T if we endow T with the weak*-topology. Let $\alpha : X \to X$ be a homeomorphism on X and T be the set of α-invariant probability Borel measures. One considers the case that there are sufficiently many open sets O for which $\widehat{O}$ is continuous on T. This is certainly the case when the action is uniquely ergodic. The small boundary condition, or the condition of mean dimension zero, requires that in any neighborhood $N(x)$ of each point $x \in X$, there is a neighborhood $O(x) \subset N(x)$ such that $\widehat{O(x)}$ is continuous. Let $\omega(\widehat{O})$ be the oscillation of the function $\widehat{O}$. If $\widehat{O}$ is continuous, then $\omega(\widehat{O}) = 0$.

Let A be a C^*-algebra with tracial state space $T(A)$. For each $a \in A_+$, one defines the rank function of a by $\widehat{[a]}(\tau) = \lim_{n\to\infty} \tau(a^{1/n})$ for $\tau \in T(A)$. When $A = M_n$, i.e., A is the $n \times n$ matrix algebra, $\widehat{[a]}$ is just the normalized rank of a. We study the oscillation of the function $\widehat{[a]}$. It is called tracial oscillation of the element a. This notion of tracial oscillation has been studied in [11, 26] in connection with the augmented Cuntz semigroups. We introduce the notion of tracial approximate oscillation zero for C^*-algebras. Roughly speaking, a unital C^*-algebra A has tracial approximate oscillation zero, if each positive element a is approximated (tracially) by

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elements in the hereditary C^*-subalgebra generated by a with small tracial oscillation (see Definition 5.1). If α is a minimal homeomorphism on X and (X, α) has mean dimension zero, it is shown in [12] that the crossed product C^*-algebra $C(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable. As a consequence, $C(X) \rtimes_\alpha \mathbb{Z}$ has tracial approximate oscillation zero (see Theorem 1.1 below).

The notion of stable rank was introduced to C^*-algebra theory by Rieffel in [36]. A unital C^*-algebra has stable rank one if its invertible elements are dense in A. The notion plays an important role in the study of simple C^*-algebras (see some earlier work, for example, [9, 35]). It is proved by Rørdam [41] that if A is a unital finite separable simple $\mathcal{Z}$-stable C^*-algebra, then A has stable rank one. Robert in [38] introduced the notion of almost stable rank one, which is also a very useful notion, and showed that every stably projectionless $\mathcal{Z}$-stable simple C^*-algebra has almost stable rank one. A question remains open, however, whether a separable simple C^*-algebra with almost stable rank one actually has stable rank one.

There is a canonical map Γ from the Cuntz semigroup of A, denoted by $\mathrm{Cu}(A)$, to $\mathrm{LAff}_+(\widetilde{QT}(A))$, the set of strictly positive lower semi-continuous affine functions (vanishing at zero) on the cone of 2-quasitraces on A, defined by $\Gamma([a])(\tau) = d_\tau(a)$ (for $\tau \in \widetilde{QT}(A)$). A question posed by N. Brown (see the remark after Question 1.1 of [42]) asked whether this map is surjective, i.e., whether every strictly positive lower semi-continuous affine function on $\widetilde{QT}(A)$ is a rank function for some positive element in $A \otimes \mathcal{K}$. It is of course an important question. In fact, the strict comparison and surjectivity of Γ are perhaps equally important when one studies Cuntz semigroups. If we denote by $\mathrm{Cu}(A)_+$ the set of purely non-compact elements in the Cuntz semigroup of a separable stably finite simple C^*-algebra A, then strict comparison is the condition that Γ restricted on $\mathrm{Cu}(A)_+$ is injective. If Γ is also surjective, then the map Γ gives an isomorphism from $\mathrm{Cu}(A)_+$ onto $\mathrm{LAff}_+(\widetilde{QT}(A))$. In [13], it is shown that if A is $\mathcal{Z}$-stable, then the map Γ is indeed surjective, which extends an earlier result of [7]. More recently, it is proved in [1, 42] that when A has stable rank one, Γ is surjective. We show that if A is a σ-unital simple C^*-algebra which has strict comparison and tracial approximate oscillation zero, then the map Γ is surjective. On the other hand, if A is a σ-unital stably finite simple C^*-algebra with strict comparison which has almost stable rank one and Γ is surjective, then A has tracial approximate oscillation zero.

Let A be a σ-unital simple C^*-algebra. We also found that if A has tracial approximate oscillation zero, then A has a nice matricial structure, a property that we call (TM) (see Definition 8.1). We prove that if A has strict comparison and has property (TM), then A has stable rank one. As a by-product, we show that, if A has strict comparison and Γ is surjective, then the condition that A has almost stable rank one implies that A actually has stable rank one.

Our main result may be stated as follows:

Theorem 1.1 *Let A be a separable simple C^*-algebra which admits at least one densely defined non-trivial 2-quasitrace and has strict comparison.*

Then the following are equivalent:

(1) *A has tracial approximate oscillation zero;*

(2) Γ *is surjective (see Definition 2.13) and A has stable rank one;*

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(3) *A has stable rank one;*
(4) Γ *is surjective and A has almost stable rank one;*
(5) *A has property (TM).*

The technical terms in the statement above will be discussed in detail in the process and some examples of simple C^*-algebras which satisfy (1) will be given (e.g., Proposition 5.8 and Theorem 5.9). The condition that A has a non-trivial densely defined 2-quasitrace could be replaced by that A is stably finite (see Remark 9.10). Note that Theorem 1.1 is stated without assuming that A is nuclear or exact. Related to the Toms–Winter conjecture, Thiel in [42] shows that under the same assumption as that of Theorem 1.1, if A is unital and has stable rank one, then Γ is surjective, and, if, in addition, A has local finite nuclear dimension, then A is $\mathcal{Z}$-stable. With the same spirit, Corollary 9.8 below states that, under the same assumption as in Theorem 1.1, if (1) in the theorem above also holds and A has local finite nuclear dimension, then A is $\mathcal{Z}$-stable (see also Remark 9.9 for an even weaker hypothesis). In fact, the idea of tracial oscillation zero can also be directly used in the study of Toms–Winter conjecture (see [25]).

The paper is organized as follows. Section 2 is a preliminary that lists a number of notations and definitions that are used in the paper. It also includes some known facts which may not be stated explicitly in the literature. Section 3 discusses some preliminary cancellation properties that will be used later. In Section 4, we recall the notion of tracial oscillation and introduce the notion of tracial approximate oscillation for positive elements. In Section 5, we introduce the notion of tracial approximate oscillation zero for C^*-algebras and give some examples of separable C^*-algebras which have positive tracial approximate oscillation and examples which have tracial approximate oscillation zero. In particular, we show that, if the cone of 2-quasitraces of A has a basis S which has countably many extremal points, then A has tracial approximate oscillation zero. In Section 6, we study sequence algebras and its quotients for compact C^*-algebras A. We find that $l^\infty(A)/I_{\overline{QT(A)}^w}$, where $I_{\overline{QT(A)}^w}$ is the quasitrace kernel ideal, is a SAW*-algebra and has real rank zero and stable rank one, provided A has tracial approximate oscillation zero. Section 7 contains one of the main results: if A has strict comparison and tracial approximate oscillation zero, then Γ is surjective. In Section 8, we introduce the property (TM), a property of tracial matricial structure. We show that, under the assumption of strict comparison, the property (TM) is equivalent to the property of tracial approximate oscillation zero. The last section is devoted to the proof of Theorem 1.1 mentioned above, in particular, (1) $\Rightarrow$ (2) without assuming that A is separable (but σ-unital).

2 Preliminary

In this section, we will give a list of basic notations and a number of definitions which will be used throughout this paper. Most of them are familiar to experts. It also includes some basic facts about Cuntz semigroups and 2-quasitraces, as well as some ad hoc but more or less known facts. Readers are encouraged to skip them until they are needed.

2.1 Some basic notations and definitions

Notation 2.1 *In this paper, the set of all positive integers is denoted by* $\mathbb{N}$. *The set of all compact operators on a separable infinite-dimensional Hilbert space is denoted by* $\mathcal{K}$.

Let A *be a normed space and* $\mathcal{F} \subset A$ *a subset. Let* $\varepsilon > 0$. *For any pair* $a, b \in A$, *we write* $a \approx_\varepsilon b$ *if* $\|a - b\| < \varepsilon$. *We write* $a \in_\varepsilon \mathcal{F}$ *if there is* $x \in \mathcal{F}$ *such that* $a \approx_\varepsilon x$.

Let A *be a* C^**-algebra and* $x \in A$. *Let* $|x| := (x^*x)^{1/2}$. *If* $a, b \in A$ *and* $ab = ba = a^*b = ba^* = 0$, *we often write* $a \perp b$.

Notation 2.2 *Let* A *be a* C^**-algebra and* $S \subset A$ *a subset of* A. *Denote by* $\mathrm{Her}_A(S)$ *(or just* $\mathrm{Her}(S)$, *when* A *is clear) the hereditary* C^**-subalgebra of* A *generated by* S. *Denote by* $A^{\mathbf{1}}$ *the unit ball of* A, *and by* A_+ *the set of all positive elements in* A. *Put* $A^{\mathbf{1}}_+ := A_+ \cap A^{\mathbf{1}}$. *Denote by* $\widetilde{A}$ *the minimal unitization of* A. *When* A *is unital, denote by* $GL(A)$ *the group of invertible elements of* A, *and by* $U(A)$ *the unitary group of* A. *Let* $\mathrm{Ped}(A)$ *denote the Pedersen ideal of* A, $\mathrm{Ped}(A)_+ = \mathrm{Ped}(A) \cap A_+$, $\mathrm{Ped}(A)^{\mathbf{1}} = A^{\mathbf{1}} \cap \mathrm{Ped}(A)$, *and* $\mathrm{Ped}(A)^{\mathbf{1}}_+ = \mathrm{Ped}(A)_+ \cap \mathrm{Ped}(A)^{\mathbf{1}}$. *Denote by* $T(A)$ *the tracial state space of* A. *Except the Pedersen ideal, all other ideals mentioned in this paper are* closed two-sided *ideals.*

Definition 2.3 Let A and B be C^*-algebras and $\phi : A \to B$ a linear map. The map ϕ is said to be positive if $\phi(A_+) \subset B_+$. The map ϕ is said to be completely positive contractive, abbreviated to c.p.c., if $\|\phi\| \le 1$ and $\phi \otimes \mathrm{id} : A \otimes M_n \to B \otimes M_n$ are positive for all $n \in \mathbb{N}$. A c.p.c. map $\phi : A \to B$ is called order zero, if for any $x, y \in A_+$, $xy = 0$ implies $\phi(x)\phi(y) = 0$ (see Definition 2.3 of [45]).

In what follows, $\{e_{i,j}\}_{i,j=1}^n$ (or just $\{e_{i,j}\}$, if there is no confusion) stands for a system of matrix units for M_n, 1_n for the identity of M_n, $\iota \in C_0((0,1])$ for the identity function on $(0,1]$, i.e., $\iota(t) = t$ for all $t \in (0,1]$. We also write $\{e_{i,j}\}$ for a system of matrix units for $\mathcal{K}$.

Definition 2.4 A C^*-algebra A is said to have stable rank one [36] if $\widetilde{A} = \overline{GL(\widetilde{A})}$, i.e., $GL(\widetilde{A})$ is dense in $\widetilde{A}$. A C^*-algebra A is said to have almost stable rank one [38] if, for any hereditary C^*-subalgebra $B \subset A$, $B \subset \overline{GL(\widetilde{B})}$.

Notation 2.5 *Let* $\varepsilon, \delta > 0$. *Define continuous functions* $f_\varepsilon, g_\delta : [0, +\infty) \to [0,1]$ *by*

$$f_\varepsilon(t) = \begin{cases} 0, & t \in [0, \varepsilon/2], \\ 1, & t \in [\varepsilon, \infty), \\ \text{linear}, & t \in [\varepsilon/2, \varepsilon], \end{cases} \quad \text{and} \quad g_\delta(t) = \begin{cases} 0, & t \in \{0\} \cup [\delta, \infty), \\ 1, & t \in [\delta/8, \delta/2], \\ \text{linear}, & t \in [0, \delta/8] \cup [\delta/2, \delta]. \end{cases}$$

(Note that $(t - \delta/2)_+$ *and* f_δ *have the same support.)*

2.2 Cuntz semigroup and quasitraces

Definition 2.6 Let A be a C^*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$. We write $a \lesssim b$ if there are $x_k \in A \otimes \mathcal{K}$ such that $\lim_{k\to\infty} \|a - x_k^* b x_k\| = 0$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ both hold [8]. The Cuntz relation $\sim$ is an equivalence relation. Set $\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+ / \sim$. Denote by $V(A)$ the subset of those elements in $\mathrm{Cu}(A)$ which are represented by projections.

Definition 2.7 Let A be a C^*-algebra. A densely defined 2-quasitrace is a 2-quasitrace defined on $\mathrm{Ped}(A \otimes \mathcal{K})$ (see Definition II.1.1 of [2]). Denote by $\widetilde{QT}(A)$ the set of

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densely defined 2-quasitraces on $A \otimes \mathcal{K}$. In what follows, we will identify A with $A \otimes e_{1,1}$ whenever it is convenient. Note that we require that a 2-quasitrace has finite value on $\mathrm{Ped}(A \otimes \mathcal{K})$. In particular, we exclude the function on $\mathrm{Ped}(A \otimes \mathcal{K})$ with only ∞ value from the consideration.

We endow $\widetilde{QT}(A)$ with the topology in which a net $\{\tau_i\}$ converges to τ if $\{\tau_i(a)\}$ converges to $\tau(a)$ for all $a \in \mathrm{Ped}(A \otimes \mathcal{K})$ (see also (4.1) on page 985 of [13]).

Note that, for each $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$, $f_\varepsilon(a) \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. Define, for each $\tau \in \widetilde{QT}(A)$,

$$\text{(e2.1)} \qquad \widehat{a}(\tau) := \tau(a) := \lim_{\varepsilon \to 0} \tau(a f_\varepsilon(a)) \text{ and } \widehat{[a]}(\tau) := d_\tau(a) := \lim_{\varepsilon \to 0} \tau(f_\varepsilon(a)).$$

We will use properties of 2-quasitraces discussed in [2, 13] (see, in particular, Section 4.1 and Theorem 4.4 of [13]). Denote by $\widetilde{T}(A)$ the subset of $\widetilde{QT}(A)$ consisting of traces.

Definition 2.8 Recall (Theorem 4.7 of [10]) that a σ-unital C^*-algebra A is compact if and only if $A = \mathrm{Ped}(A)$. Every unital C^*-algebra is compact. Let A be a compact C^*-algebra. Since $A = \mathrm{Ped}(A)$, every (densely defined) 2-quasitrace is actually defined on A. By II 2.3 of [2], every 2-quasitrace on A is bounded. Put $QT_{[0,1]}(A) = \{\tau \in \widetilde{QT}(A) : \|\tau|_A\| \le 1\}$. Then $QT_{[0,1]}(A)$ is a compact convex subset of $\widetilde{QT}(A)$ (see [13, Theorem 4.4]). Denote by $QT(A)$ the set of 2-quasitraces τ with $\|\tau|_A\| = 1$. It is a convex subset of $\widetilde{QT}(A)$. Denote by $\overline{QT(A)}^w$ the (weak*) closure of $QT(A)$. Then, in the case that A is compact and $\widetilde{QT}(A)\setminus\{0\} \ne \emptyset$, $\mathbb{R}_+ \cdot \overline{QT(A)}^w = \widetilde{QT}(A)$ (if $QT(A) = \emptyset$, then $\overline{QT(A)}^w = \emptyset$).

Let $I \subset A$ be an ideal and $\{e_\lambda\}$ be a quasi-central approximate identity for I. Suppose that $\tau \in \widetilde{QT}(I)$. Then $\tau(a) = \lim_\lambda \tau(a e_\lambda)$ (for $a \in A$) defines a (densely defined) 2-quasitrace of A. Note that $\|\tau|_A\| = \|\tau|_I\|$. If $\tau \in \widetilde{QT}(A)$, then $\tau_I(a) = \lim_\lambda \tau(a e_\lambda)$ also densely defines a 2-quasitrace of A with $\|\tau_I|_A\| \le \|\tau\|$ (see Definition 2.5 of [23]). Let $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$ and I_a be the ideal generated by a. By [2, II.4.2], every τ in $\widetilde{QT}(\mathrm{Her}(a))$ can be uniquely extended to a 2-quasitrace τ in $\widetilde{QT}(I_a)$. In what follows, we will identify $\widetilde{QT}(\mathrm{Her}(a))$ with $\{\tau_{I_a} : \tau \in \widetilde{QT}(\mathrm{Her}(a))\}$.

The following is a quasitrace version of Lemma 4.5 of [10].

Proposition 2.9 (Lemma 4.5 of [10]) *Let A be a σ-unital compact C^*-algebra. Then $0 \notin \overline{QT(A)}^w$ and $\overline{QT(A)}^w$ is compact.*

Proof We may assume that $QT(A) \ne \emptyset$. By Lemma 4.4 of [10], there is $e_1 \in M_n(A)$ with $0 \le e_1 \le 1$ and $x \in M_n(A)$ such that $e_1 x^* x = x^* x e_1 = x^* x$ and $a_0 = x x^*$ is a strictly positive element of A (for some $n \in \mathbb{N}$). Note that $\tau(e_1) \ge d_\tau(a_0) = 1$ for all $\tau \in QT(A)$. Note also that

$$\text{(e2.2)} \qquad QT(A) = \{\tau \in \widetilde{QT}(A) : d_\tau(a_0) = 1\}.$$

Put $S = \{\tau \in QT_{[0,1]}(A) : \tau(e_1) \ge 1\}$. Then S is compact and $0 \notin S$. Since $\tau(e_1) \ge d_\tau(a_0) = 1$, $QT(A) \subset S$. So $\overline{QT(A)}^w \subset S$ and $0 \notin \overline{QT(A)}^w$. This also implies that $\overline{QT(A)}^w$ is compact. ∎

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Proposition 2.10 *Let A be a σ-unital C^*-algebra and $S_1, S_2 \subset \widetilde{QT}(A)$ nonempty compact subsets. Then one has the following (with $\|\tau\| = \|\tau|_A\|$):*

(1) *If $\mathbb{R}_+ \cdot S_1 = \widetilde{QT}(A)$ and $0 \notin S_1$, then there exists $L_1 \in \mathbb{R}_+$ such that*

$$S_2 \subset \{r \cdot s : s \in S_1 \text{ and } r \in [0, L_1]\}.$$

(2) *If $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$, then $d = \sup\{\|\tau|_{\mathrm{Her}(a)}\| : \tau \in S_1\} < \infty$.*

(3) *If A is compact, then $M_1 = \sup\{\|\tau\| : \tau \in S_1\} < \infty$.*

(4) *If a is as in (2), and S_1 is as in (1), then $\overline{QT(\mathrm{Her}(a))}^w \subset \{r \cdot \tau : \tau \in S_1, r \in [0, L]\}$ for some $L \in \mathbb{R}_+$ (see the last paragraph of Definition 2.8).*

Proof To see (1) holds, let us assume otherwise. Then there exist sequences $r_n \in \mathbb{R}_+$, $s_n \in S_1$ and $t_n \in S_2$ such that $r_n s_n = t_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = \infty$. Since both S_1, S_2 are compact, without loss of generality, we may assume that $s_n \to s \in S_1$ and $t_n \to t \in S_2$.

Since $s \neq 0$, choose $c \in \mathrm{Ped}(A)_+^{\mathbf{1}}$ such that $s(c) > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that $s_n(c) > s(c)/2 > 0$ for all $n \geq n_0$. Consequently,

$$t_n(c) = r_n s_n(c) \to \infty. \tag{e2.3}$$

Hence, $t(c) = \infty$. However, $c \in \mathrm{Ped}(A)_+$. A contradiction.

For (2), since $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, there are $b_i \in (A \otimes \mathcal{K})_+$ and $f_i \in C_c((0,\infty))_+$ ($1 \leq i \leq m$), the set of continuous functions with compact supports, such that $a \leq \sum_{i=1}^m f_i(b_i)$ (see [30, Theorem 5.6.1]). It follows that $a \lesssim \mathrm{diag}(f_1(b_1), f_2(b_2), \ldots, f_m(b_m))$. One can choose $f \in C_c((0,\infty))_+$ with $0 \leq f \leq 1$ such that $f f_i = f_i, 1 \leq i \leq m$. Put $b = \mathrm{diag}(f(b_1), f(b_2), \ldots, f(b_m))$. Then

$$\tau(b) \geq d_\tau(\mathrm{diag}(f_1(b_1), f_2(b_2), \ldots, f_m(b_m))) \geq d_\tau(a) \text{ for all } \tau \in S. \tag{e2.4}$$

But $\widehat{b}$ is bounded on the compact subset S_1. Put $M = \sup\{\tau(b) : \tau \in S_1\}$. Then $M < \infty$ and

$$\sup\{\|\tau|_{\mathrm{Her}(a)}\| : \tau \in S_1\} = \sup\{d_\tau(a) : \tau \in S_1\} \leq M. \tag{e2.5}$$

To see (3), let $a \in A$ be a strictly positive element. Since $A = \mathrm{Ped}(A)$, $a \in \mathrm{Ped}(A)_+$ and $\mathrm{Her}(a) = A$. Thus, (3) follows from (2).

For (4), let I_a be the (closed) ideal of $A \otimes \mathcal{K}$ generated by a. Then $\{\tau_I : \tau \in \overline{QT(\mathrm{Her}(a))}^w\}$ is a compact subset of $\widetilde{QT}_{[0,1]}(A)$ (see the last paragraph of Definition 2.8). Hence, part (4) of the lemma then follows from (1). ∎

2.3 Comparison and canonical map Γ

Definition 2.11 A simple C^*-algebra A is said to have (Blackadar's) strict comparison, if, for any $a, b \in (A \otimes \mathcal{K})_+$, one has $a \lesssim b$, provided

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in \widetilde{QT}(A)\backslash\{0\}. \tag{e2.6}$$

2.12 *Let A be a σ-unital C^*-algebra and $e \in \mathrm{Ped}(A \otimes \mathcal{K})_+\backslash\{0\}$. If e is a full element, put $T_e = \{\tau \in \widetilde{QT}(A) : \tau(e) = 1\}$. Then T_e is a compact convex subset and is a basis for the cone $\widetilde{QT}(A)$ (see Proposition 3.4 of [43]). If, in addition, A is simple, then*

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e is always full. Put $A_1 = \mathrm{Her}(e)$. By Brown's stable isomorphism theorem (see [4]), $A \otimes \mathcal{K} \cong A_1 \otimes \mathcal{K}$. So $e \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+$. Then $A_1 = \mathrm{Ped}(A_1)$ (see, for example, Theorem 2.1(iii) of [43]); in other words, A_1 is algebraically simple. Therefore, instead of studying $A \otimes \mathcal{K}$, we will study $A_1 \otimes \mathcal{K}$. Throughout the paper, we often consider σ-unital simple C^*-algebras A with $\mathrm{Ped}(A) = A$ (in other words, algebraically simple C^*-algebras).

Definition 2.13 Let A be a C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \neq \emptyset$. Denote by $L(\widetilde{QT}(A))$ the family of continuous real-valued functions f on $\widetilde{QT}(A)$ such that $f(\alpha\tau) = \alpha f(\tau)$ for all $\alpha \in \mathbb{R}_+$ and $\tau \in \widetilde{QT}(A)$ and $f(\tau + t) = f(\tau) + f(t)$ for all $\tau, t \in \widetilde{QT}(A)$. Let $S \subset \widetilde{QT}(A)$ be a convex subset. Set

$$\mathrm{Aff}_+(S) = \{f|_S : f \in L(\widetilde{QT}(A)), f(\tau) > 0 \text{ if } \tau \in S\backslash\{0\}\} \cup \{0\}, \tag{e2.7}$$

$$\mathrm{LAff}_+(S) = \{f : S \to [0,\infty] : \exists\{f_n\}, f_n \nearrow f,\ f_n \in \mathrm{Aff}_+(S)\}. \tag{e2.8}$$

Note that if $0 \in S$, then $f(0) = 0$ for all $f \in \mathrm{LAff}_+(S)$. For a simple C^*-algebra A and $a \in (A \otimes \mathcal{K})_+$, the function $\hat{a}(\tau) = \tau(a)$ $(\tau \in S)$ is in general in $\mathrm{LAff}_+(S)$. If $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, then $\widehat{a} \in \mathrm{Aff}_+(S)$. Recall that $\widehat{[a]}(\tau) = d_\tau(a)$ for $\tau \in \widetilde{QT}(A)$. So $\widehat{[a]} \in \mathrm{LAff}_+(\widetilde{QT}(A))$. Caution: $\widehat{a}$ and $\widehat{[a]}$ are not the same in general.

We will write $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ for the canonical map defined by $\Gamma([a])(\tau) = \widehat{[a]}(\tau) = d_\tau(a)$ for all $\tau \in \widetilde{QT}(A)$.

(1) In the case that A is simple and $A = \mathrm{Ped}(A)$, Γ also induces a canonical map $\Gamma_1 : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\overline{QT(A)}^w)$. Since, in this case, $\mathbb{R}_+\overline{QT(A)}^w = \widetilde{QT}(A)$, the map Γ is surjective if and only if Γ_1 is surjective.

(2) In the case that A is stably finite and simple, denote by $\mathrm{Cu}(A)_+$ the set of purely non-compact elements (see Proposition 6.4 of [13]). Suppose that Γ is surjective. Let $p \in (A \otimes \mathcal{K})_+$ be a projection (so $p \in \mathrm{Ped}(A \otimes \mathcal{K})$). There are $a_n \in (A \otimes \mathcal{K})_+$ with $0 \leq a_n \leq 1$ such that $\widehat{[a_n]} = (1/2^n)\widehat{[p]}$, $n \in \mathbb{N}$. Define $b = \mathrm{diag}(a_1/2, a_2/2^2, \ldots, a_n/2^n, \ldots) \in A \otimes \mathcal{K}$. Then 0 is a limit point of $\mathrm{sp}(b)$. Therefore, $[b]$ cannot be represented by a projection. In other words, $[b] \in \mathrm{Cu}(A)_+$. We compute that $\widehat{[b]} = \widehat{[p]}$. It then follows that $\Gamma|_{\mathrm{Cu}(A)_+}$ is surjective.

Suppose that A is simple and a is a purely non-compact element and

$$d_\tau(a) \leq d_\tau(b) \text{ for all } \tau \in \widetilde{QT}(A). \tag{e2.9}$$

Then, for any $\varepsilon > 0$ (recall that $f_\varepsilon(a) \in \mathrm{Ped}(A)$),

$$d_\tau(f_\varepsilon(a)) < d_\tau(b) \text{ for all } \tau \in \widetilde{QT}(A). \tag{e2.10}$$

If A has strict comparison, then $f_\varepsilon(a) \lesssim b$. Since this holds for all $\varepsilon > 0$, we conclude that $a \lesssim b$.

The reader should be reminded that when A is exact, every 2-quasitrace is a trace (see [17]). These facts will be used without further explanation.

2.4 Cuntz null sequences and the ideal generated by Cuntz null sequences

Definition 2.14 Let A be a separable non-elementary simple C^*-algebra. Then A contains a sequence of nonzero elements $e_n \in \mathrm{Ped}(A)$ with $0 \leq e_n \leq 1$ such that

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$e_{n+1} \lesssim e_n$ for all $n \in \mathbb{N}$, and for any finite subset $\mathcal{F} \subset A_+ \setminus \{0\}$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$n[e_n] \le [d] \text{ for all } d \in \mathcal{F} \tag{e2.11}$$

(see Lemma 4.3 of [15]).

For general C^*-algebra A, a sequence $\{a_n\} \subset A_+$ is said to be *truly* Cuntz-null and written $a_n \xrightarrow{c.} 0$ if, for any finite subset $\mathcal{F} \subset A_+ \setminus \{0\}$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$a_n \lesssim d \text{ for all } d \in \mathcal{F}. \tag{e2.12}$$

This is equivalent to saying that, for any $d \in A_+ \setminus \{0\}$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$, $a_n \lesssim d$. We also write $a_n \overset{c.}{\searrow} 0$ if $a_{n+1} \lesssim a_n$ for all $n \in \mathbb{N}$ and $a_n \xrightarrow{c.} 0$.

A sequence $\{x_n\} \subset A \otimes \mathcal{K}$ is said to be Cuntz-null if, for any $\varepsilon > 0$, $f_\varepsilon(x_n^* x_n) \xrightarrow{c.} 0$.

Definition 2.15 Let X be a normed space, and let $l^\infty(X)$ denote the space of bounded sequences of X. When A is a C^*-algebra, $l^\infty(A)$ is also a C^*-algebra, and $c_0(A) = \{\{a_n\} \in l^\infty(A) : \lim_{n\to\infty} \|a_n\| = 0\}$ is an ideal of $l^\infty(A)$. Let $A_\infty = l^\infty(A)/c_0(A)$ and $\pi_\infty : l^\infty(A) \to A_\infty$ be the quotient map. We view A as a C^*-subalgebra of $l^\infty(A)$ via the canonical map $\iota : a \mapsto \{a, a, \dots\}$ for all $a \in A$. In what follows, we may identify a with the constant sequence $\{a, a, \dots\}$ in $l^\infty(A)$ without further warning. Let $\{C_n\}$ be a sequence of C^*-subalgebra s of A. We may also use notation $l^\infty(\{C_n\}) = \{\{c_n\} \in l^\infty(A) : c_n \in C_n\}$ for the infinite product of $\{C_n\}$.

Denote by $N_{cu}(A)$ (or just N_{cu}) the set of all Cuntz-null sequences in $l^\infty(A)$.

It follows from Proposition 3.5 of [14] that, if A has no one-dimensional hereditary C^*-subalgebra s, then $N_{cu}(A)$ is an ideal of $l^\infty(A)$. Moreover, if A is non-elementary and simple, $c_0(A) \subsetneqq N_{cu}(A)$. Denote by $\Pi_{cu} : l^\infty(A) \to l^\infty(A)/N_{cu}(A)$ the quotient map and $\Pi_{cu}(A)^\perp = \{b \in l^\infty(A)/N_{cu} : b\Pi_{cu}(a) = \Pi_{cu}(a)b = 0 \text{ for all } a \in A\}$.

Definition 2.16 Let A be a C^*-algebra with $\widetilde{QT}(A) \ne \emptyset$. Fix a compact subset $T \subset \widetilde{QT}(A)$. For each $x \in A$, define

$$\|x\|_{2,T} = \sup\{\tau(x^*x)^{1/2} : \tau \in T\}. \tag{e2.13}$$

Then $\|x^*\|_{2,T} = \|x\|_{2,T}$.

By Lemma 3.5 of [17] (one does not need to assume that A is unital),

$$\tau(a+b)^{1/2} \le \tau(a)^{1/2} + \tau(b)^{1/2} \text{ for all } a, b \in \mathrm{Ped}(A \otimes \mathcal{K})_+ \text{ and } \tau \in T, \tag{e2.14}$$

$$\|x+y\|_{2,\tau}^{2/3} \le \|x\|_{2,\tau}^{2/3} + \|y\|_{2,\tau}^{2/3} \text{ for all } x, y \in \mathrm{Ped}(A \otimes \mathcal{K}) \text{ and } \tau \in T. \tag{e2.15}$$

Then

$$\sup\{\|x+y\|_{2,\tau}^{2/3} : \tau \in T\} \le \sup\{\|x\|_{2,\tau}^{2/3} : \tau \in T\} + \sup\{\|y\|_{2,\tau}^{2/3} : \tau \in T\}. \tag{e2.16}$$

In other words,

$$\|x+y\|_{2,T}^{2/3} \le \|x\|_{2,T}^{2/3} + \|y\|_{2,T}^{2/3}. \tag{e2.17}$$

We also have

$$\|xy\|_{2,T} \le \|x\|\|y\|_{2,T} \text{ and } \|xy\|_{2,T} \le \|x\|_{2,T}\|y\|. \tag{e2.18}$$

It follows that $\{a \in A : \|a\|_{2,T} = 0\}$ is a (closed two-sided) ideal of A.

We also have the following inequality for $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$:

$$\|a\|_{2,T} \le \|a\|(\sup\{d_\tau(a) : \tau \in T\})^{1/2}. \tag{e2.19}$$

In fact, for all $n \in \mathbb{N}$, we have $\tau(a^2) = \tau(a^{1/2n}a^{2-(1/n)}a^{1/2n}) \le \|a^{2-(1/n)}\|\tau(a^{1/n})$. Let $n \to \infty$. We obtain $\tau(a^2) \le \|a\|^2 d_\tau(a)$. So (e2.19) holds.

Definition 2.17 Suppose that A is a σ-unital C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \ne \emptyset$, and $T \subset \widetilde{QT}(A)$ is a compact subset with $T \ne \{0\}$. Define

$$I_T = \{\{x_n\} \in l^\infty(A) : \lim_{n\to\infty} \sup\{\tau(x_n^* x_n) : \tau \in T\} = 0\}. \tag{e2.20}$$

Then I_T is an ideal of $l^\infty(A)$.

Suppose that A is a simple non-elementary C^*-algebra. Then it is clear that

$$N_{cu}(A) \subset I_{\overline{QT(A)}^w}. \tag{e2.21}$$

It follows from the proof of Proposition 3.8 of [14] that $I_{\overline{QT(A)}^w} = N_{cu}(A)$ if $A = \mathrm{Ped}(A)$ and A has strict comparison. Denote by $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w}$ the quotient map.

Proposition 2.18 *Let A be a σ-unital algebraically simple C^*-algebra such that $QT(A) \ne \emptyset$. Let $S \subset \widetilde{QT}(A)\setminus\{0\}$ be a compact subset such that $QT(A) \subset \mathbb{R}_+ \cdot S$. Then*

$$I_S = I_{\overline{QT(A)}^w}. \tag{e2.22}$$

Moreover, if A has strict comparison, then $I_{\overline{QT(A)}^w} = N_{cu}$.

Proof By Proposition 2.10, $0 < s_1 = \sup\{\|\tau|_A\| : \tau \in S\} < \infty$. Since $S \subset \widetilde{QT}(A)\setminus\{0\}$ and is compact,

$$s_2 := \inf\{\|\tau|_A\| : \tau \in S\} > 0. \tag{e2.23}$$

Suppose that $\{a_n\} \in (I_{\overline{QT(A)}^w})_+^1$. Then, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n \ge n_0$,

$$\tau(a_n^2) < (\varepsilon/(s_1+1))^2 \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e2.24}$$

Thus, if $n \ge n_0$, for any $t \in S$,

$$t(a_n^2) = \|t|_A\|(t/\|t|_A\|)(a_n^2) < \|t|_A\|(\varepsilon/(s_1+1))^2 \le \varepsilon^2. \tag{e2.25}$$

This implies that $\{a_n\} \in I_S$. It follows that $I_{\overline{QT(A)}^w} \subset I_S$.

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Conversely, if $\{a_n\} \in I_S$, there exists $n_1 \in \mathbb{N}$ such that, if $n \geq n_1$,

$$t(a_n^2) < s_2\varepsilon^2 \ \text{ for all } \ t \in S. \tag{e2.26}$$

Note that, for all $\tau \in QT(A)$, there are $r_\tau \in \mathbb{R}_+$ and $t_\tau \in S$ such that $\tau = r_\tau t_\tau$. Since $r_\tau \|t_\tau\| \leq 1$, we have $r_\tau \leq 1/\|t_\tau\| \leq 1/s_2$. Suppose $\tau \in \overline{QT(A)}^w$. Then there are $t_n \in S$ and $r_n > 0$ such that $r_n t_n \in QT(A)$ and $r_n t_n \to \tau$. As mentioned above, we have $r_n \leq 1/s_2$ for all $n \in \mathbb{N}$. Since S is compact (and $\{r_n\}$ is bounded), by choosing a subsequence, we may assume that $t_n \to t_\tau \in S$ and $r_n \to r_\tau$. In other words, $\tau = r_\tau t_\tau$. Note that $\|\tau\| \leq 1$. So we also have $r_\tau \leq 1/s_2$. Therefore, for any $n \geq n_1$, if $\tau \in \overline{QT(A)}^w$,

$$\tau(a_n^2) = r_\tau t_\tau(a_n^2) \leq (1/s_2)t_\tau(a_n^2) < \varepsilon^2. \tag{e2.27}$$

Thus, $\{a_n\} \in I_{\overline{QT(A)}^w}$.

To see the last part of the statement, choose $b \in \mathrm{Ped}(A)_+^{\mathbf{1}} \backslash \{0\}$. Let $S = \{\tau \in \widetilde{QT}(A) : \tau(b) = 1\}$. Then $S \subset \widetilde{QT}(A) \backslash \{0\}$ is a compact subset and, $\widetilde{QT}(A) = \mathbb{R} \cdot S$. By (the "Moreover" part of) Proposition 3.8 of [14], $N_{cu} = I_S = I_{\overline{QT(A)}^w}$. ∎

Let A be a σ-unital simple C^*-algebra and $\{e_n\}$ be an approximate identity with $e_{n+1}e_n = e_n e_{n+1}$ ($n \in \mathbb{N}$). Recall that A is said to have continuous scale, if, for any $a \in A_+ \backslash \{0\}$, there is $n_0 \in \mathbb{N}$ such that

$$e_m - e_n \lesssim a \quad \text{for all} \ \ m > n \geq n_0. \tag{e2.28}$$

This definition does not depend on the choice of $\{e_n\}$ (see [22, Definition 2.1] and [19, Definition 2.5]). With terminology of Definition 2.14, A has continuous scale if and only if, for any $m(n) > n$, $e_{m(n)} - e_n \xrightarrow{c.} 0$ for any $\{e_n\}$ for which $e_{n+1}e_n = e_n e_{n+1} = e_n$ ($n \in \mathbb{N}$).

The following is known. The proof of it is exactly the same as that of the case $T(A) = QT(A)$ (see [10, Definition 5.1, Remark 5.2, Theorem 5.3, and Proposition 5.4] for details, and also see the remark after Definition 6.3 of [14]).

Theorem 2.19 (cf. Theorem 5.3 and Proposition 5.4 of [10], also [19]) *Let A be a σ-unital simple C^*-algebra with a strict positive element e_A, continuous scale and $QT(A) \neq \emptyset$. Then $QT(A)$ is compact and $\widehat{[e_A]}$ is continuous on $\widetilde{QT}(A)$. Assuming A has strict comparison, then A has continuous scale if and only if $\widehat{[e_A]}$ is continuous on $\widetilde{QT}(A)$.*

Proposition 2.20 *Let A be a separable non-elementary simple C^*-algebra. Then A has continuous scale if and only if $\Pi_{cu}(A)^\perp = \{0\}$.*

Proof Suppose that A has continuous scale. Let $\{b_n\} \in l^\infty(A)_+^{\mathbf{1}}$ be such that $b = \Pi_{cu}(\{b_n\}) \in \Pi_{cu}(A)^\perp$. Fix a truly Cuntz-null sequence of $\{a_n\}$ in the unit ball of A_+ such that $a_n \neq 0$ for all $n \in \mathbb{N}$ (see Definition 2.14). Let $e \in A_+^{\mathbf{1}}$ be a strictly positive element. Note that, for each $k \in \mathbb{N}$, $\{f_{1/2k}(e)b_n\}_{n \in \mathbb{N}}$ is a Cuntz-null sequence. For each $k \in \mathbb{N}$, there are $l(k), n(k) \in \mathbb{N}$ such that

$$[f_{1/2k}(b_n^* f_{1/2k}(e)^2 b_n)] \leq [a_i] \ (1 \leq i \leq k) \ \text{ for all } \ n \geq l(k) \tag{e2.29}$$

$$\text{and } \ \|(1 - f_{1/n}(e))b_k\| < 1/k \ \text{ for all } \ n \geq n(k). \tag{e2.30}$$

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We may assume that $l(k+1) > l(k) \geq k$ and $n(k) > 2k$ for all $k \in \mathbb{N}$. Define $c_{0,i} = c_{1,i} = 0$ if $1 \leq i < l(1)$, and, if $l(k) \leq i < l(k+1)$, define

$$c_{0,i} = f_{1/2k}(e)b_i \text{ and } c_{1,i} = (1 - f_{1/n(i)}(e))b_i. \tag{e2.31}$$

Put $\bar{b}_i = b_i - c_{0,i} - c_{1,i}$, $i \in \mathbb{N}$. Note that, if $l(k) \leq i < l(k+1)$, $\bar{b}_i = (f_{1/n(i)}(e) - f_{1/2k}(e))b_i$ $(k > 1)$. By (e2.29), one verifies that $\{c_{0,n}\} \in N_{cu}(A)$. In fact, for a fixed $1/2 > \varepsilon > 0$, choose k_0 such that $1/k_0 < \varepsilon$. For any finite subset $\mathcal{F} \subset A_+ \backslash \{0\}$, choose $J \in \mathbb{N}$ such that $a_i \lesssim b$ for all $b \in \mathcal{F}$ and for all $i \geq J$. It follows that, if $k_1 \geq J$, by (e2.29), for all $l(k) \leq i < l(k+1)$ and $k \geq \max\{k_0, k_1\}$,

$$f_\varepsilon(c_{0,i}^* c_{0,i}) \leq f_{1/2k}(c_{0,i}^* c_{0,i}) \lesssim a_J \lesssim b$$

for all $b \in \mathcal{F}$. Hence, $\{c_{0,n}\} \in N_{cu}(A)$. Also, by (e2.30), $\{c_{1,i}\} \in c_0(A)$. It follows that

$$\Pi_{cu}(\{b_n\}) = \Pi_{cu}(\{\bar{b}_n\}). \tag{e2.32}$$

It suffices to show that $\{\bar{b}_n\} \in N_{cu}$. In fact, for all $l(k) \leq i < l(k+1)$,

$$\bar{b}_i^* \bar{b}_i \lesssim f_{1/2n(l(k+1))}(e) - f_{1/2k}(e), \ k \in \mathbb{N}. \tag{e2.33}$$

Since A has continuous scale, then $f_{1/2n(l(k+1))}(e) - f_{1/2k}(e) \xrightarrow{c.} 0$ (see [22, Definition 2.1] and [19, Definition 2.5], for example). It follows that $\{\bar{b}_n\} \in N_{cu}(A)$. This implies that $\{b_n\} \in N_{cu}(A)$. Consequently, $\Pi_{cu}(A)^\perp = \{0\}$.

Conversely, suppose that $\Pi_{cu}(A)^\perp = \{0\}$. Let $e_n = f_{1/2n}(e)$, $n \in \mathbb{N}$. Choose any $m(n) > n$. Define $d_n = e_{4m(n)} - f_{1/n}(e)$. Then, for any $a \in A$, $\lim_{n\to\infty} a d_n = 0$. It follows that $\Pi_{cu}(\{d_n\}) \in \Pi_{cu}(A)^\perp = \{0\}$. In other words, $\{d_n\} \in N_{cu}(A)$. Therefore, for any $0 < \delta < 1/4$,

$$f_\delta(d_n) \xrightarrow{c.} 0. \tag{e2.34}$$

Note that, for all $n \in \mathbb{N}$,

$$f_{1/4m(n)} - f_{1/2n} \leq f_\delta(f_{1/8m(n)} - f_{1/n}) \text{ in } C_0((0,1]).$$

Thus,

$$e_{2m(n)} - e_n \lesssim f_\delta(d_n), \ n \in \mathbb{N}. \tag{e2.35}$$

It follows that $(e_{2m(n)} - e_n) \xrightarrow{c.} 0$. Hence, A has continuous scale. ■

3 Comparison and cancellation of projections

Lemma 3.1 *Let A be a C^*-algebra and $\tau \in \widetilde{QT}(A)\backslash\{0\}$. Let $e \in A_+$ and $a \in A$ such that $ea = a = ae$. Suppose that $\tau(e) < \infty$. Then, for any $f \in C_0(\mathbb{R})$, it holds that $\tau(f(e - a^*a)) = \tau(f(e - aa^*))$. In particular, $\|e - a^*a\|_{2,\tau} = \|e - aa^*\|_{2,\tau}$. Moreover, $d_\tau(g(e - a^*a)) = d_\tau(g(e - aa^*))$ for any $g \in C_0(\mathbb{R})$, assuming $g(e - a^*a)$ and $g(e - aa^*)$ are positive.*

Proof Note that since $C^*(e, a^*a)$ and $C^*(e, aa^*)$ are commutative, the restrictions of τ on them are linear. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
\tau((e-a^*a)^n) &= \tau\left(e^n + \sum_{m=1}^{n} \frac{n!}{m!(n-m)!}(-a^*a)^m\right)\\
&= \tau(e^n) + \sum_{m=1}^{n} \frac{n!}{m!(n-m)!}\tau((-a^*a)^m)\\
&= \tau(e^n) + \sum_{m=1}^{n} \frac{n!}{m!(n-m)!}\tau((-aa^*)^m)\\
&= \tau\left(e^n + \sum_{m=1}^{n} \frac{n!}{m!(n-m)!}(-aa^*)^m\right) = \tau((e-aa^*)^n). \qquad \text{(e3.1)}
\end{aligned}$$

Thus, for any polynomial $P, \tau(P(e-a^*a)) = \tau(P(e-aa^*))$. In particular, $\|e-a^*a\|_{2,\tau} = \|e-aa^*\|_{2,\tau}$. Therefore, by the continuity of 2-quasitraces (see [2, Corollary II.2.5]), and the Stone–Weierstrass theorem, $\tau(f(e-a^*a)) = \tau(f(e-aa^*))$ for all $f \in C_0(\mathbb{R})$. Moreover, for any $g \in C_0(\mathbb{R})$, assuming $g(e-a^*a)$ and $g(e-aa^*)$ are positive,

(e3.2)
$$d_\tau(g(e-a^*a)) = \sup_{\varepsilon>0} \tau(f_\varepsilon(g(e-a^*a))) = \sup_{\varepsilon>0} \tau(f_\varepsilon(g(e-aa^*))) = d_\tau(g(e-aa^*)).$$

∎

Lemma 3.2 *Let A, B be C^*-algebras and $\pi : A \to B$ be a surjective homomorphism. Assume $p, q \in B$ are projections, and $x \in B$ satisfies $px = x = xq$. Then there are $\tilde{p}, \tilde{q} \in A_+^{\mathbf{1}}$ and $\tilde{x} \in A$, such that $\pi(\tilde{p}) = p, \pi(\tilde{q}) = q, \pi(\tilde{x}) = x$, and $\tilde{p}\tilde{x} = \tilde{x} = \tilde{x}\tilde{q}$. Moreover, if $p = q$, we can take $\tilde{p} = \tilde{q}$.*

Proof We may assume that $\|x\| \le 1$. Let $p_1, q_1 \in A_+^{\mathbf{1}}$ such that $\pi(p_1) = p, \pi(q_1) = q$. Since p, q are projections, we also have $\pi(f_{1/2}(p_1)) = f_{1/2}(\pi(p_1)) = p$, and $\pi(f_{1/2}(q_1)) = f_{1/2}(\pi(q_1)) = q$. Note $x^*x \le q$. By [30, Proposition 1.5.10], there exists $y \in A^{\mathbf{1}}$ such that $\pi(y) = x$ and $y^*y \le f_{1/2}(q_1)$. Put $\tilde{x} = f_{1/2}(p_1)y$. Then

$$\pi(\tilde{x}) = px = x, \tilde{x}\tilde{x}^* = f_{1/2}(p_1)yy^*f_{1/2}(p_1) \le f_{1/4}(p_1)$$

and $\tilde{x}^*\tilde{x} \le y^*y \le f_{1/2}(q_1)$. Set $\tilde{p} = f_{1/8}(p_1)$ and $\tilde{q} = f_{1/8}(q_1)$. Then $\pi(\tilde{p}) = p, \pi(\tilde{q}) = q$. The facts that $f_{1/8}(p_1)f_{1/4}(p_1) = f_{1/4}(p_1)$ and $f_{1/8}(q_1)f_{1/4}(q_1) = f_{1/4}(q_1)$ imply that $\tilde{p}\tilde{x} = \tilde{x} = \tilde{x}\tilde{q}$. Moreover, if $p = q$, we can take $p_1 = q_1$, and hence $\tilde{p} = \tilde{q}$. ∎

Proposition 3.3 *Let A be a C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \ne \emptyset$. Suppose that $T \subset \widetilde{QT}(A)\backslash\{0\}$ is a compact subset. Then every projection in $l^\infty(A)/I_T(A)$ is finite (see Definition 2.17).*

Proof Let $B = l^\infty(A)/I_T(A)$ and $\pi : l^\infty(A) \to B$ be the quotient map. Assume $p \in B$ is a projection, $u \in B$ satisfies $u^*u = p$, and $uu^* \le p$. We need to show $uu^* = p$.

By Lemma 3.2, there are $a = \{a_1, a_2, \dots\} \in l^\infty(A)_+^{\mathbf{1}}$ and $v = \{v_1, v_2, \dots\} \in l^\infty(A)$ such that $\pi(a) = p, \pi(v) = u$, and $av = v = va$. Since $\pi(a) = p = \pi(v^*v)$, we

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have $\lim_{n\to\infty}\|a_n - v_n^* v_n\|_{2,T} = 0$. By Lemma 3.1, $\|a_n - v_n v_n^*\|_{2,T} = \|a_n - v_n^* v_n\|_{2,T} \to 0 (n \to \infty)$. Hence,

$$p - uu^* = \pi(\{a_1 - v_1 v_1^*, a_2 - v_2 v_2^*, \dots\}) = 0,$$

which shows p is a finite projection. ∎

Proposition 3.4 *Let A be a non-elementary simple C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$. Let $T \subset \widetilde{QT}_{[0,1]}(A)\setminus\{0\}$. Then, for any $a \in \mathrm{Ped}(A)_+^{\mathbf{1}}\setminus\{0\}$, any $\varepsilon > 0$, there is $b \in \mathrm{Her}(a)_+$ such that $b \le a$, $\|a - b\|_{2,T} < \varepsilon$, and $d_\tau(b) < d_\tau(a)$ for all $\tau \in T$.*

Proof It follows from the first paragraph of Definition 2.14 that there exists $c \in \mathrm{Her}(a)_+$ with $\|c\| = 1$ such that $d_\tau(c) < \varepsilon^2$ for all $\tau \in T$. Define $b = a^{1/2}(1 - f_{1/4}(c))a^{1/2}$. Then $0 \le b \le a$. It follows from (e2.17) that

$$\|a - b\|_{2,T} = \|a^{1/2} f_{1/4}(c) a^{1/2}\|_{2,T} \le \|f_{1/4}(c)\|_{2,T} \le (d_\tau(c))^{1/2} \le \varepsilon.$$

For all $\tau \in T$,

$$\begin{aligned} d_\tau(b) &= d_\tau(a^{1/2}(1 - f_{1/4}(c))a^{1/2}) = d_\tau((1 - f_{1/4}(c))^{1/2} a (1 - f_{1/4}(c))^{1/2}) \\ &< d_\tau((1 - f_{1/4}(c))^{1/2} a (1 - f_{1/4}(c))^{1/2}) + d_\tau(f_{1/2}(c)) \\ &\overset{\text{(orthogonality)}}{=} d_\tau((1 - f_{1/4}(c))^{1/2} a (1 - f_{1/4}(c))^{1/2} + f_{1/2}(c)) \le d_\tau(a). \end{aligned}$$

∎

Theorem 3.5 *Let A be a non-elementary algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. Assume that A has strict comparison. Then $l^\infty(A)/I_{\overline{QT(A)}^w}$ has cancellation of projections, i.e., for any projections $p, q, r \in l^\infty(A)/I_{\overline{QT(A)}^w}$, if $p, q \le r$ and $p \sim q$, then $r - p \sim r - q$.*

Proof Set $B := l^\infty(A)/I_{\overline{QT(A)}^w}$ and let $\Pi : l^\infty(A) \to B$ denote the quotient map.

Let $p, q, r \in B$ be projections with $p, q \le r$, and assume that there is a partial isometry $v \in B$ such that $v^* v = p$, $vv^* = q$. By Lemma 3.2, there are $e = \{e_1, e_2, \dots\} \in l^\infty(A)_+^{\mathbf{1}}$ and $w = \{w_1, w_2, \dots\} \in l^\infty(A)$ such that $\pi(e) = r$, $\pi(w) = v$, and $ew = w = we$. Then, by Lemma 3.1, we have $d_\tau(f_{1/4}(e_n - w_n^* w_n)) = d_\tau(f_{1/4}(e_n - w_n w_n^*))$ for all $\tau \in \overline{QT(A)}^w$ and $n \in \mathbb{N}$. By Proposition 3.4, for each $n \in \mathbb{N}$, there is $b_n \in A_+^{\mathbf{1}}$ such that

$$\|f_{1/4}(e_n - w_n^* w_n) - b_n\|_{2,\overline{QT(A)}^w} < 1/n, \text{ and} \tag{e3.3}$$

$$d_\tau(b_n) < d_\tau(f_{1/4}(e_n - w_n^* w_n)) = d_\tau(f_{1/4}(e_n - w_n w_n^*)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e3.4}$$

Since A has strict comparison, we have $b_n \lesssim f_{1/4}(e_n - w_n w_n^*)$. By [40, Proposition 2.4(iv)], for each $n \in \mathbb{N}$, there is $x_n' \in A$ such that

$$(x_n')^*(x_n') = f_{1/n}(b_n) \quad\text{and}\quad x_n'(x_n')^* \in \mathrm{Her}(f_{1/4}(e_n - w_n w_n^*)). \tag{e3.5}$$

Note that $f_{1/n}(b)(b - 1/n)_+ = (b - 1/n)_+$. Choose $x_n = x_n'(b - 1/n)_+^{1/2}$. Then

$$x_n^* x_n = (b_n - 1/n)_+ \quad\text{and}\quad x_n x_n^* \in \mathrm{Her}(f_{1/4}(e_n - w_n w_n^*)). \tag{e3.6}$$

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Note $\|x_n\|^2 = \|x_n^* x_n\| = \|(b_n - 1/n)_+\| \le 1$. The second part of (e3.6) implies

$$x_n x_n^* \le f_{1/8}(e_n - w_n w_n^*). \tag{e3.7}$$

Let $c_n = e_n - w_n^* w_n$ and $d_n = e_n - w_n w_n^*$ $(n \in \mathbb{N})$. Let $x = \{x_1, x_2, \ldots\}$, $b = \{b_1, b_2, \ldots\}$, $c = \{c_1, c_2, \ldots\}$, and $d = \{d_1, d_2, \ldots\} \in l^\infty(A)$. Then

$$\Pi(x)^*\Pi(x) \overset{(3.3)}{=} \Pi(b) \overset{(3.3)}{=} \Pi(f_{1/4}(c)) = f_{1/4}(\Pi(c)) = f_{1/4}(r-p) = r - p, \tag{e3.8}$$

and

$$\Pi(x)\Pi(x)^* \overset{(3.7)}{\le} \Pi(f_{1/8}(d)) = f_{1/8}(\Pi(d)) = f_{1/8}(r-q) = r - q. \tag{e3.9}$$

Let $y = v + \Pi(x)$. Note that $v\Pi(x)^* = vp(r-p)\Pi(x)^* = 0$ and $\Pi(x)v^* = (v\Pi(x)^*)^* = 0$. Also, note that $v^*\Pi(x) = v^*q(r-q)\Pi(x) = 0$ and $\Pi(x)^*v = (v^*\Pi(x))^* = 0$. Then we compute (using also (e3.8) and (e3.9)) that

$$r = y^*y \sim yy^* = \Pi(x)\Pi(x)^* + q \le r.$$

By Proposition 3.3, r is a finite projection. Hence, $\Pi(x)\Pi(x)^* + q = r$. Consequently, $\Pi(x)\Pi(x)^* = r - q$. Together with (e3.8), we obtain $r - p \sim r - q$. The theorem then follows. ∎

4 Tracial oscillations

In this section, we will introduce the notion of tracial approximate oscillation for positive elements in a C^*-algebra and present some basics around the notion.

Definition 4.1 Let A be a C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \ne \emptyset$. Let $S \subset \widetilde{QT}(A)$ be a compact subset. Define, for each $a \in (A \otimes \mathcal{K})_+$,

$$\omega(a)|_S = \inf\{\sup\{d_\tau(a) - \tau(c) : \tau \in S\} : c \in \mathrm{Ped}(\overline{a(A\otimes\mathcal{K})a}),\ 0 \le c \le 1\} \tag{e4.1}$$

(see A1 of [11]). The number $\omega(a)|_S$ is called the (tracial) oscillation of a on S.

If $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, then $\omega(a)|_S < \infty$ (see (2) of Proposition 2.10). Since $\tau(f_{1/n}(a)) \nearrow d_\tau(a)$ (point-wisely) and $\widehat{c}$ is continuous on compact set S for each $c \in \mathrm{Ped}(\overline{a(A\otimes\mathcal{K})a})_+$, one has

$$\omega(a)|_S = \lim_{n\to\infty} \sup\{d_\tau(a) - \tau(f_{1/n}(a)) : \tau \in S\}. \tag{e4.2}$$

Note that, exactly as in A1 of [11], if $a, b \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ and $a \sim b$, then $\omega(a)|_S = \omega(b)|_S$ (cf. Proposition 4.2 below). For each $h \in \mathrm{LAff}_+(\widetilde{QT}(A))$, define

$$\omega(h)|_S = \inf\{\sup\{h(\tau) - f(\tau) : \tau \in S\} : 0 < f < h,\ f \in \mathrm{Aff}_+(\widetilde{QT}(A))\}. \tag{e4.3}$$

Recall that, in general, for any real function f defined on S, the oscillation of f at $s \in S$ is defined as

$$\omega(f)(s) = \inf\{\sup\{|f(s') - f(s'')| : s', s'' \in O(s)\} : O(s)\}, \tag{e4.4}$$

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where $O(s)$ is an open neighborhood of s and the infimum above is taken among all such $O(s)$. Denote by $\omega(f)|_S = \sup\{\omega(f)(s) : s \in S\}$. Then (recall that S is compact)

$$\omega(a)|_S = \omega(\widehat{[a]})|_S. \tag{e4.5}$$

Let $a \in (A \otimes \mathcal{K})_+$. For each $\tau \in S$, and its neighborhood $O(\tau)$, define

$$\omega_{O(\tau)}(a)|_S = \lim_{n\to\infty} \sup\{d_t(a) - t(f_{1/n}(a)) : t \in O(\tau) \cap S\}. \tag{e4.6}$$

One may note that, if $O_1(\tau) \subset O_2(\tau)$, then $\omega_{O_1(\tau)}(a)|_S \le \omega_{O_2(\tau)}(a)|_S$. Define

$$\omega(a)(\tau)|_S = \inf\{\omega_{O(\tau)}(a) : \tau \in O(\tau) \cap S\} \tag{e4.7}$$

(the infimum is taken among all neighborhood $O(\tau)$ of τ in S). In other words, when S is fixed, $\omega(a)(\tau)|_S$ is the oscillation of the lower-semicontinuous function $\widehat{[a]}$ at τ. In particular, $\widehat{[a]}$ is continuous on S if and only if $\omega(a)(\tau)|_S = 0$ for all $\tau \in S$.

(1) If $c_n \in \mathrm{Her}(a)_+^{\mathbf{1}}$ and $\tau(c_n) \nearrow d_\tau(a)$ for all $\tau \in S$, then

$$\omega_{O(\tau)}(a)|_S = \lim_{n\to\infty} \sup\{d_t(a) - t(c_n) : t \in O(\tau) \cap S\}. \tag{e4.8}$$

In general, one checks that

$$\sup\{\omega(a)(\tau)|_S : \tau \in S\} = \omega(a)|_S. \tag{e4.9}$$

(2) For most of the time, we will assume that A is simple and S is a compact subset of $\widetilde{QT}(A)\backslash\{0\}$ such that $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$, for example, $S = T_b$ for some $b \in \mathrm{Ped}(A)_+\backslash\{0\}$. Or, in the case that $A = \mathrm{Ped}(A)$, $S = \overline{QT(A)}^w$. When S is understood, we may omit S in the notation. In fact, when A is compact, we may write $\omega(a)$ instead of $\omega(a)|_{\overline{QT(A)}^w}$.

(3) Let $S_1, S_2 \subset \widetilde{QT}(A)\backslash\{0\}$ be compact subsets such that $\mathbb{R}_+ \cdot S_i = \widetilde{QT}(A)$, $i = 1, 2$. If $\omega(a)|_{S_1} = 0$, then $\omega(a)|_{S_2} = 0$ (see also Proposition 2.10). In what follows, we write $\omega(a) = 0$ if $\omega(a)|_S = 0$ for one compact subset of $\widetilde{QT}(A)$ such that $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$.

Proposition 4.2 *[11, A1] Let $a, b \in (A \otimes \mathcal{K})_+$. Let $S \subset \widetilde{QT}(A)$ be a compact subset. If $a \sim b$, then $\omega(a)(\tau)|_S = \omega(b)(\tau)|_S$ for all $\tau \in S$, and $\omega(a)|_S = \omega(b)|_S$.*

Proof Let $\tau \in S$. Let $O(\tau)$ be any open neighborhood of τ. For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sup\{d_t(a) - t(f_\delta(a)) : t \in O(\tau) \cap S\} < \omega_{O(\tau)}(a)|_S + \varepsilon. \tag{e4.10}$$

Since $a \sim b$, there exists a sequence $x_n \in A \otimes \mathcal{K}$ such that $x_n x_n^* \to a$ and $x_n^* x_n \in \mathrm{Her}(b) = \overline{b(A \otimes \mathcal{K})b}$. Since $a^{1/m} a a^{1/m} \to a$ as $m \to \infty$, replacing x_n by $a^{1/m(n)} x_n$ for some subsequence $\{m(n)\}$, we may assume that $x_n x_n^* \in \mathrm{Her}(a)$. Note that, for any $\delta > 0$,

$$\lim_{n\to\infty} \|f_\delta(x_n x_n^*) - f_\delta(a)\| = 0.$$

Since S is compact, by (2) of Proposition 2.10, $\sup\{\|t|_{\mathrm{Her}(a)}\| : t \in S\} < \infty$. It follows that there is $m \in \mathbb{N}$ such that

$$\sup\{|t(f_\delta(a)) - t(f_\delta(x_m x_m^*))| : t \in S\} < \varepsilon. \tag{e4.11}$$

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Note that $d_t(a) = d_t(b)$ for all $t \in S$ because of $a \sim b$. Also, note that $t(f_\delta(x_m x_m^*)) = t(f_\delta(x_m^* x_m))$ for all $t \in S$. Then

$$\omega_{O(\tau)}(b)|_S \le \sup\{d_t(b) - t(f_\delta(x_m^* x_m)) : t \in O(\tau) \cap S\} \tag{e4.12}$$
$$= \sup\{d_t(a) - t(f_\delta(x_m x_m^*)) : t \in O(\tau) \cap S\} \tag{e4.13}$$
$$\le \sup\{d_t(a) - t(f_\delta(a)) : t \in O(\tau) \cap S\} \tag{e4.14}$$
$$+ \sup\{|t(f_\delta(a)) - t(f_\delta(x_m x_m^*))| : t \in O(\tau) \cap S\} \tag{e4.15}$$
$$\le \omega_{O(\tau)}(a)|_S + \varepsilon + \varepsilon. \tag{e4.16}$$

Since ε is arbitrary, we have $\omega_{O(\tau)}(b)|_S \le \omega_{O(\tau)}(a)|_S$. Exactly the same argument shows that $\omega_{O(\tau)}(a)|_S \le \omega_{O(\tau)}(b)|_S$. Hence, $\omega_{O(\tau)}(a)|_S = \omega_{O(\tau)}(b)|_S$. Since $O(\tau)$ is an arbitrary open neighborhood of τ, we have

$$\omega(b)(\tau)|_S = \inf\{\omega_{O(\tau)}(b)|_S : \tau \in O(\tau)\cap S\} = \inf\{\omega_{O(\tau)}(a)|_S : \tau \in O(\tau)\cap S\} = \omega(a)(\tau)|_S.$$

For the last identity in the proposition, we note that, by (e4.9),

$$\omega(a)|_S = \sup\{\omega(a)(\tau)|_S : \tau \in S\} = \sup\{\omega(b)(\tau)|_S : \tau \in S\} = \omega(b)|_S. \qquad \blacksquare$$

Definition 4.3 In the case that A does not have strict comparison, we may still want to consider elements with zero tracial oscillation. We write $\omega^c(a) = 0$ if $g_{1/n}(a) \stackrel{c.}{\to} 0$ (recall Definition 2.5 for g_δ, and also see Definition 2.14). Let $\{a_n\} \in l^\infty(A)_+$. We write $\lim_{n\to\infty} \omega^c(a_n) = 0$, if there exists $\delta_n \in (0, 1/2)$ such that $g_{\delta_n}(a_n) \stackrel{c.}{\to} 0$.

Note that, by Proposition 2.10, if A is compact, then the number s in part (1) of the next proposition is always finite. Let $\tau \in S$. In the next lemma, we write $O(\tau)$ for an open neighborhood of τ in S.

Proposition 4.4 *Let A be a σ-unital C^*-algebra. Let $S \subset \widetilde{QT}(A)\backslash\{0\} \neq \emptyset$ be a compact subset.*

(1) *Suppose that $s := \{\|\tau|_A\| : \tau \in S\} < \infty$. If $a, b \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$, then*

$$\omega(a)(\tau)|_S - \overline{d_\tau}(b)|_S \le \omega(a+b)(\tau)|_S \le \omega(a)(\tau)|_S + \overline{d_\tau}(b)|_S \ \textit{for all}\ \tau \in S, \tag{e4.17}$$

where $\overline{d_\tau}(b)|_S := \inf\{\sup\{d_t(b) : t \in O(\tau)\} : O(\tau)$ open neighborhoods of τ in $S\}$.

(2) *If $a \perp b$, then*

$$\max\{\omega(a)|_S, \omega(b)|_S\} \le \omega(a+b)|_S \le \omega(a)|_S + \omega(b)|_S. \tag{e4.18}$$

Moreover,

$$\max\{\omega(a)(\tau)|_S, \omega(b)(\tau)|_S\} \le \omega(a+b)(\tau)|_S \le \omega(a)(\tau)|_S + \omega(b)(\tau)|_S \ \textit{for all}\ \tau \in S. \tag{e4.19}$$

(3) *For σ-unital simple C^*-algebra A, $\omega^c(a) = 0$ if and only if* $\mathrm{Her}(a)$ *has continuous scale.*

Proof (1) For the inequality on the left, let $\varepsilon > 0$. Fix $\tau \in S$. Choose an open neighborhood $O(\tau)$ of τ in S such that

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$$\omega_{O(\tau)}(a+b)|_S \le \omega(a+b)(\tau)|_S + \varepsilon, \text{ and} \tag{e4.20}$$

$$\sup\{d_t(b) : t \in O(\tau)\} \le \overline{d_\tau}(b)|_S + \varepsilon. \tag{e4.21}$$

Note that there is $\delta > 0$ such that

$$\sup\{d_t(a+b) - t(f_\delta(a+b)) : t \in O(\tau)\} \le \omega_{O(\tau)}(a+b)|_S + \varepsilon. \tag{e4.22}$$

Note that $a+b \approx_{\delta/2} (a-\delta/2)_+ + b$. By [40, Proposition 2.2], we have $f_\delta(a+b) \lesssim (a-\delta/2)_+ + b$. Then, for any $t \in O(\tau)$, we have

$$t(f_\delta(a+b)) \le d_t(f_\delta(a+b)) \le d_t((a-\delta/2)_+ + b) \le d_t((a-\delta/2)_+) + d_t(b) \tag{e4.23}$$

$$\stackrel{(4.21)}{\le} t(f_{\delta/2}(a)) + \overline{d_\tau}(b)|_S + \varepsilon. \tag{e4.24}$$

Then, for $t \in O(\tau)$, $d_t(a) + t(f_\delta(a+b)) \le d_t(a+b) + t(f_{\delta/2}(a)) + \overline{d_\tau}(b)|_S + \varepsilon$. It follows that

$$d_t(a) - t(f_{\delta/2}(a)) \le d_t(a+b) - t(f_\delta(a+b)) + \overline{d_\tau}(b)|_S + \varepsilon \tag{e4.25}$$

$$\stackrel{(4.22)}{\le} \omega_{O(\tau)}(a+b)(\tau)|_S + \overline{d_\tau}(b)|_S + 2\varepsilon \tag{e4.26}$$

$$\stackrel{(4.20)}{\le} \omega(a+b)(\tau)|_S + \overline{d_\tau}(b)|_S + 3\varepsilon \quad \text{for all } t \in O(\tau). \tag{e4.27}$$

Hence,

$$\omega(a)(\tau)|_S \le \omega_{O(\tau)}(a)(\tau) \le \sup\{d_t(a) - t(f_{\delta/2}(a)) : t \in O(\tau)\} \tag{e4.28}$$

$$\le \omega(a+b)(\tau)|_S + \overline{d_\tau}(b)|_S + 3\varepsilon. \tag{e4.29}$$

Let $\varepsilon \to 0$, then we have the desired inequality.

Now we turn to the inequality on the right. By definition, for any $\varepsilon > 0$, there are open neighborhood $O(\tau)$ of τ in S, and $\delta > 0$ such that

$$\sup\{d_t(b) : t \in O(\tau)\} \le \overline{d_\tau}(b) + \varepsilon, \text{ and} \tag{e4.30}$$

$$\sup\{d_t(a) - t(f_\delta(a)) : t \in O(\tau)\} \le \omega_{O(\tau)}(a)|_S + \varepsilon/2 \le \omega(a)(\tau)|_S + \varepsilon. \tag{e4.31}$$

Note that $a \in \mathrm{Her}(a+b)$, then there is $\eta > 0$ such that $f_\delta(a) \approx_{\varepsilon/(s+1)} f_\eta(a+b)f_\delta(a)f_\eta(a+b)$. Hence, for any $t \in O(\tau)$, by [2, Corollary II.2.5(iii)],

$$d_t(a+b) - t(f_\eta(a+b)) \le d_t(b) + d_t(a) - t(f_\eta(a+b)f_\delta(a)f_\eta(a+b))$$

$$\le d_t(b) + d_t(a) - t(f_\delta(a)) + \varepsilon \tag{e4.32}$$

$$\stackrel{(4.30),(4.31)}{\le} \overline{d_\tau}(b) + \omega(a)(\tau)|_S + 3\varepsilon. \tag{e4.33}$$

Hence,

$$\omega(a+b)(\tau)|_S \le \sup\{d_t(a+b) - t(f_\eta(a+b)) : t \in O(\tau)\}$$

$$\le \overline{d_\tau}(b) + \omega(a)(\tau)|_S + 3\varepsilon. \tag{e4.34}$$

Let $\varepsilon \to 0$, (1) then follows.

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For (2), we have, for any $1/2 > \varepsilon > 0$, since $a \perp b$,

$$d_\tau(a+b) - \tau(f_\varepsilon(a+b)) = (d_\tau(a) - \tau(f_\varepsilon(a)) + (d_\tau(b) - \tau(f_\varepsilon(b)) \tag{e4.35}$$

for all $\tau \in S$. Thus,

$$\omega(a+b)|_S \le \omega(a)|_S + \omega(b)|_S \text{ and} \tag{e4.36}$$

$$\omega(a+b)(\tau)|_S \le \omega(a)(\tau)|_S + \omega(b)(\tau)|_S \text{ for all } \tau \in S. \tag{e4.37}$$

Hence, the inequality on the right in (e4.18) holds.

Now we turn to the inequality on the left of (e4.18). Since $a \perp b$, for all $\tau \in S$ and all $\eta > 0$,

$$d_\tau(a) - \tau(f_\eta(a)) \le (d_\tau(a) - \tau(f_\eta(a))) + (d_\tau(b) - \tau(f_\eta(b))) = d_\tau(a+b) - \tau(f_\eta(a+b)). \tag{e4.38}$$

Thus, $\omega(a)|_S \le \sup\{d_\tau(a) - f_\eta(a) : \tau \in S\} \le \sup\{d_\tau(a+b) - f_\eta(a+b) : \tau \in S\}$. Since η can be arbitrary small, we have

$$\omega(a)|_S \le \inf_{\eta>0} \sup\{d_\tau(a+b) - f_\eta(a+b) : \tau \in S\} = \omega(a+b)|_S. \tag{e4.39}$$

Similarly, $\omega(b)|_S \le \omega(a+b)|_S$. Thus, the inequality on the left of (e4.18) holds. The estimates (e4.19) can be checked similarly.

For (3), recall that $\mathrm{Her}(a)$ has continuous scale if and only if $e_{m(n)} - e_n \overset{c.}{\to} 0$ for any $m(n) > n$, where $e_n = f_{1/2^n}(a), n \in \mathbb{N}$. Suppose that $\omega^c(a) = 0$. Then, for each $n \in \mathbb{N}$ and any $m(n) \ge n$,

$$e_{m(n)} - e_n \lesssim g_{1/n}(a) \overset{c.}{\to} 0. \tag{e4.40}$$

It follows that $\mathrm{Her}(a)$ has continuous scale.

Conversely, suppose that $\mathrm{Her}(a)$ has continuous scale. For any $d \in A_+ \setminus \{0\}$, choose $n_0 \in \mathbb{N}$ such that, for any $m(n) > n \ge n_0$,

$$e_{m(n)} - e_n \lesssim d. \tag{e4.41}$$

Suppose that $k_0 > 2^{n_0}$. Fix $k \ge k_0$. For any $\varepsilon \in (0, 1/4)$, there is $m(n) > n_0$ such that

$$f_\varepsilon(g_{1/k}(a)) \lesssim e_{m(n)} - e_{n_0}. \tag{e4.42}$$

In other words, for any $\varepsilon \in (0, 1/4)$, $f_\varepsilon(g_{1/k}(a)) \lesssim d$. It follows that $g_{1/k}(a) \lesssim d$ (for any $k \ge k_0$). This proves (3). ∎

Lemma 4.5 *Let A be a C^*-algebra with a nonempty compact subset $S \subset \widetilde{QT}(A)$, and let $a \in (A \otimes \mathcal{K})_+$. Then, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that*

$$\omega(f_\delta(a))|_S < \omega(a)|_S + \varepsilon \textit{ for all } 0 < \delta < \delta_0. \tag{e4.43}$$

Proof We may assume that $\omega(a)|_S < \infty$. There exists $\delta_0 > 0$ such that, for all $0 < \eta \le 2\delta_0$,

$$d_\tau(a) - \tau(f_\eta(a)) < \omega(a) + \varepsilon/2 \text{ for all } \tau \in S. \tag{e4.44}$$

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Then, there exists $\sigma_0 > 0$ such that, if $0 < \delta < \delta_0$,

$$d_\tau(f_\delta(a)) - \tau(f_{\sigma_0}(f_\delta(a))) \le d_\tau(a) - \tau(f_{2\delta_0}(a)) < \omega(a) + \varepsilon/2 \tag{e4.45}$$

for all $\tau \in S$. Note that $d_\tau(f_\delta(a)) \le \tau(f_{\delta/2}(a))$ for all $\tau \in \widetilde{QT}(A)$. It follows that (see (2) of Proposition 2.10)

$$\omega(f_\delta(a))|_S < \omega(a)|_S + \varepsilon. \qquad \blacksquare$$

Proposition 4.6 *Let A be a C^*-algebra with a nonempty compact subset $S \subset \widetilde{QT}(A)$ and $a \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ with $\omega(a) < \infty$, and $0 < \delta < 1/2$. Then, for any $\varepsilon > 0$, there is $0 < \eta < \delta/2$ and $n_0 \in \mathbb{N}$ such that*

$$\sup\{\tau(f_\eta(a)) - d_\tau(f_\delta(a)) : \tau \in S\} \ge \omega(a)|_S - \varepsilon \ \textit{ and} \tag{e4.46}$$

$$\sup\{\tau(a^{1/n_0}) - d_\tau(f_\delta(a)) : \tau \in S\} \ge \omega(a)|_S - \varepsilon. \tag{e4.47}$$

Proof Fix $0 < \delta < 1/2$. For any $\varepsilon > 0$, there exists $\tau_0 \in S$ such that

$$d_{\tau_0}(a) - \tau_0(f_{\delta/2}(a)) > \omega(a)|_S - \varepsilon/4. \tag{e4.48}$$

For this τ_0, choose $0 < \eta < \delta/2$ such that $d_{\tau_0}(a) - \tau_0(f_\eta(a)) < \varepsilon/4$. Then

$$\tau_0(f_\eta(a)) - d_{\tau_0}(f_\delta(a)) > d_{\tau_0}(a) - \tau_0(f_{\delta/2}(a)) - (d_{\tau_0}(a) - \tau_0(f_\eta(a))) \tag{e4.49}$$

$$> \omega(a)|_S - \varepsilon/4 - \varepsilon/4. \tag{e4.50}$$

Hence,

$$\sup\{\tau(f_\eta(a)) - d_\tau(f_\delta(a)) : \tau \in S\} \ge \omega(a)|_S - \varepsilon/2. \tag{e4.51}$$

To see the second inequality, choose $n_0 \in \mathbb{N}$ such that

$$\|a^{1/n_0} f_\eta(a) - f_\eta(a)\| < \varepsilon/4. \tag{e4.52}$$

It follows that, for all $\tau \in S$,

$$\tau(a^{1/n_0}) - d_\tau(f_\delta(a)) \ge \tau(a^{1/n_0} f_\eta(a)) - d_\tau(f_\delta(a)) \tag{e4.53}$$

$$\ge \tau(f_\eta(a)) - d_\tau(f_\delta(a)) - \varepsilon/4. \tag{e4.54}$$

Therefore, by (e4.51), $\sup\{\tau(a^{1/n_0}) - d_\tau(f_\delta(a)) : \tau \in S\} \ge \omega(a)|_S - \varepsilon.$ $\blacksquare$

Definition 4.7 Let A be a C^*-algebra, let $S \subset \widetilde{QT}(A)$ be a compact subset, and let $a \in (A \otimes \mathcal{K})_+$. Put $B = \mathrm{Her}(a)$ and $I_{_{S,B}} = \{\{b_n\} \in l^\infty(B) : \lim_{n\to\infty} \|b_n\|_{_{2,S}} = 0\}$. Denote by $\Pi_S : l^\infty(B) \to l^\infty(B)/I_{_{S,B}}$ and $\Pi_{cu} : l^\infty(B) \to l^\infty(B)/N_{cu}(B)$ (in the case that B has no one-dimensional hereditary C^*-subalgebra) the quotient maps, respectively.

Let A be a σ-unital C^*-algebra and $a \in (A \otimes \mathcal{K})_+$ with $\|a\|_{_{2,S}} < \infty$. Define (here we assume that $b_n \in \mathrm{Ped}(A \otimes \mathcal{K})_+$ and $\mathrm{Her}(a) = \overline{a(A \otimes \mathcal{K})a}$)

$$\Omega^T(a)|_S = \inf\{\|\Pi_S(\iota(a) - \{b_n\})\| : \{b_n\} \in l^\infty(\mathrm{Her}(a))_+, \|b_n\| \le \|a\|, \lim_{n\to\infty} \omega(b_n)|_S = 0\},$$

$$\Omega^T_T(a)|_S = \inf\{\limsup_n \|a - b_n\|_{_{2,S}} : b_n \in \mathrm{Her}(a)_+, \|b_n\| \le \|a\|, \lim_{n\to\infty} \omega(b_n)|_S = 0\},$$

$$\Omega^T_C(a)|_S = \inf\{\|\Pi_{cu}(\iota(a) - \{b_n\})\| : \{b_n\} \in l^\infty(\mathrm{Her}(a))_+, \|b_n\| \le \|a\|, \lim_{n\to\infty} \omega(b_n)|_S = 0\},$$

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$$\Omega^C(a) = \inf\{\|\Pi_{cu}(\iota(a) - \{b_n\})\| : \{b_n\} \in l^\infty(\mathrm{Her}(a))_+,\ \|b_n\| \le \|a\|,\ \lim_{n\to\infty} \omega^c(b_n) = 0\} \text{ and}$$
$$\Omega^N(a)|_S = \inf\{\|\pi_\infty(\iota(a) - \{b_n\})\| : b_n \in l^\infty(\mathrm{Her}(a))_+,\ \lim_{n\to\infty} \omega(b_n)|_S = 0\}.$$
(e4.55)

We will focus on $\Omega^T(a)$.

(1) Note, for the convenience, in the definition above, we always assume that $\|a\|_{2,S} < \infty$.

(2) Note also that $\lim_{n\to\infty} \|af_{1/n}(a) - a\| = 0$ and

$$af_{1/n}(a) = a^{1/2} f_{1/n}(a) a^{1/2} \sim f_{1/n}(a),\ \ n \in \mathbb{N}.$$

Hence, if $\omega(a)|_S = 0$, by Lemma 4.5, then $\Omega^N(a)|_S = \Omega^T(a)|_S = \Omega^T_T(a)|_S = 0$.

One may call $\Omega^T(a)|_S$ the tracial approximate oscillation of a (on S). If $\Omega^T(a)|_S = 0$, we say a has approximately tracial oscillation zero (on S). Often, when S is understood, we may omit S in notation above. In particular, when A is algebraically simple, we write $\Omega^T(a) := \Omega^T(a)|_{\overline{QT(A)}^w}$.

(3) It is, perhaps, convenient to use (1) and (2) of Proposition 4.8 for the definition of $\Omega^T(a)|_S = 0$. We would like to mention that, for the definition of $\Omega^T_C(a)$ and $\Omega^C(a)$, we also require that C^*-algebra A has no one-dimensional hereditary C^*-subalgebra s (see Definition 2.15).

(4) Moreover, since $\omega(0)|_S = 0$, we have

$$\Omega^T(a)|_S \le \|\Pi_S(\iota(a))\| \le \|a\|, \text{ and } \Omega^T_C(a)|_S,\ \Omega^C(a),\ \Omega^N(a) \le \|a\|. \tag{e4.56}$$

When A is unital, $\Omega^T(a)|_S = 0$ for any $a \in GL(A) \cap A_+$, since $\omega(1_A) = 0$.

(5) In the case that A is a σ-unital algebraically simple C^*-algebra with strict comparison, if $S = \overline{QT(A)}^w$ and $\Omega^T_T(a)|_S = 0$, then $\Omega^T(a)|_S = \Omega^N(a)|_S = \Omega^T_C(a)|_S = \Omega^C(a) = 0$ (see (2) of Proposition 4.8, (2) after Definition 5.1, and Proposition 5.7).

Proposition 4.8 *Let A be a C^*-algebra, $a \in (A \otimes \mathcal{K})^{\mathbf{1}}_+$, and $S \subset \widetilde{QT}(A)$ a compact subset such that $\|a\|_{2,S} < \infty$.*

(1) If $\Omega^T(a)|_S = 0$, then there exists a sequence $\{b_n\} \subset \mathrm{Ped}(\mathrm{Her}(a))^{\mathbf{1}}_+$ such that

$$\lim_{n\to\infty} \omega(b_n)|_S = 0 \ \textit{and}\ \|\Pi_S(\iota(a) - \{b_n\})\| = 0, \tag{e4.57}$$

and, if $\Omega^T_T(a)|_S = 0$, there exists $b_n \in \mathrm{Ped}(\mathrm{Her}(a))^{\mathbf{1}}_+$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \omega(b_n)|_S = 0 \ \textit{and}\ \lim_{n\to\infty} \|a - b_n\|_{2,S} = 0. \tag{e4.58}$$

(2) $\Omega^T(a)|_S = 0$ if and only if $\Omega^T_T(a)|_S = 0$.

(3) If there exists $M \ge 1$ such that

$$\inf\{\|\Pi_S(\iota(a) - \{b_n\})\| : b_n \in \mathrm{Ped}(\mathrm{Her}(a))_+,\ \|b_n\| \le M,\ \lim_{n\to\infty} \omega(b_n)|_S = 0\} = 0,$$

then $\Omega^T(a)|_S = 0$.

(4) If $\{a_n\} \in (I_s)^{\mathbf{1}}_+$, then there is $\{b_n\} \in (I_s)^{\mathbf{1}}_+$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup\{d_\tau(b_n) : \tau \in S\} = 0 \text{ and } \lim_{n\to\infty} \|a_n - b_n\| = 0. \tag{e4.59}$$

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Proof Recall that $B = \mathrm{Her}(a)$ and $\Pi_S : l^\infty(B) \to l^\infty(B)/I_{S,B}$ is the quotient map.

For (1), there is, for each $k \in \mathbb{N}$, a sequence $\{c_n^{(k)}\} \in l^\infty(\mathrm{Ped}(\mathrm{Her}(a)))$ with $0 \le c_n^{(k)} \le 1$ and $\lim_{n\to\infty} \omega(c_n^{(k)})|_S = 0$ such that

$$\|\Pi_S(\iota(a) - \{c_n^{(k)}\})\| < 1/k. \tag{e4.60}$$

Therefore, for each $k \in \mathbb{N}$, there is $n(k) \in \mathbb{N}$ such that

$$\|a - c_{n(k)}^{(k)}\|_{_{2,S}} < 2/k \text{ and } \omega(c_{n(k)}^{(k)}) < 1/k. \tag{e4.61}$$

Define $b_k = c_{n(k)}^{(k)}$, $k \in \mathbb{N}$. Then $0 \le b_k \le 1, b_k \in \mathrm{Ped}(\mathrm{Her}(a))$ and $\lim_{n\to\infty} \omega(b_n)|_S = 0$. Moreover,

$$\|a - b_n\|_{_{2,S}} < 2/n \text{ for all } n \in \mathbb{N}. \tag{e4.62}$$

It follows that $\|\Pi_S(\iota(a) - \{b_n\})\| = 0$, and (e4.58) holds. A similar proof above shows that, if $\Omega_T^T(a) = 0$ implies that there is $b_n \in \mathrm{Ped}(\mathrm{Her}(a))_+^{\mathbf{1}}$ such that (e4.58) holds.

For (2), we note that (e4.58) implies that $\iota(a) - \{b_n\} \in I_{_S}$. So, if $\Omega_T^T(a)|_S = 0$, then $\|\Pi_S(\iota(a) - \{b_n\})\| = 0$. Hence, $\Omega^T(a)|_S = 0$. The converse also holds.

To show (3) holds, suppose that there are $c_n^{(k)} \in \mathrm{Ped}(\mathrm{Her}(a))$ such that $0 \le c_n^{(k)} \le M$, $\lim_{n\to\infty} \omega(c_n^{(k)})|_S = 0$ and

$$\|\Pi_S(\iota(a) - \{c_n^{(k)}\})\| < 1/k \text{ for all } k \in \mathbb{N}. \tag{e4.63}$$

By the proof of (1), one obtains $b_n \in \mathrm{Ped}(\mathrm{Her}(a))_+$ with $0 \le b_n \le M$ such that

$$\lim_{n\to\infty} \omega(b_n)|_S = 0 \text{ and } \Pi_S(\iota(a)) = \Pi_S(\{b_n\}). \tag{e4.64}$$

Define $g \in C_0((0, \|a\| + M])$ by $g(t) = \|a\|$ if $t \in [\|a\|, \|a\| + M]$ and $g(t) = t$ if $t \in [0, \|a\|]$. Since M is fixed and $g(a) = a$, we have

$$\Pi_S(a) = \Pi_S(\{g(b_n)\}). \tag{e4.65}$$

Put $c_n = g(b_n)$. Then $c_n \in \mathrm{Ped}(\mathrm{Her}(a))_+$ and $\|c_n\| \le \|a\|$. Since $c_n = g(b_n) \sim b_n$, then $\lim_{n\to\infty} \omega(c_n)|_S = 0$. Therefore, by (e4.65), $\Omega^T(a)|_S = 0$.

(4) Since $\{a_n\} \in (I_{_S})_+^{\mathbf{1}}$, $\lim_{n\to\infty} \|a_n\|_{_{2,S}} = 0$. Choose $\delta_n := \sqrt{\|a_n\|_{2,S} + 1/n}$ and $b_n = (a_n - \delta_n)_+, n \in \mathbb{N}$. Then $\|b_n\| \le 1, n \in \mathbb{N}, \|b_n\|_{_{2,S}} \le \|a_n\|_{_{2,S}} \to 0$ and $\lim_{n\to\infty} \|a_n - b_n\| = 0$.

Note that

$$f_\eta((x - \delta_n)_+^2) \le \chi_{[\delta_n,+\infty)}(x) \le (1/\delta_n^2)x^2 \text{ for all } x \in \mathbb{R}_+, \tag{e4.66}$$

where $\eta \in (0,1)$ and $\chi_{[\delta_n,+\infty)}$ is the characteristic function of $[\delta_n, +\infty)$. Then, for all $\tau \in S$,

(e4.67)
$$d_\tau(b_n) = d_\tau(b_n^2) = \sup_{\eta>0} \tau(f_\eta((a_n - \delta_n)_+^2)) \le (1/\delta_n^2)\tau(a_n^2) \le (1/\delta_n^2)\|a_n\|_{_{2,S}}^2 = \|a_n\|_{_{2,S}}.$$

It follows that $\lim_{n\to\infty} \sup\{d_\tau(b_n) : \tau \in S\} = 0$. ∎

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The next proposition also justifies that we often write $\Omega^T(a) = 0$ instead of $\Omega^T(a)|_S$ for some compact set $S \subset \widetilde{QT}(A)\setminus\{0\}$ such that $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$.

Proposition 4.9 *Let A be a C^*-algebra and $S_1, S_2 \subset \widetilde{QT}(A)\setminus\{0\}$ be nonempty compact subsets such that $\mathbb{R}_+ \cdot S_1 = \mathbb{R}_+ \cdot S_2 = \widetilde{QT}(A)$. Suppose that $a \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$. Then $\Omega^T(a)|_{S_1} = 0$ ($\Omega^T_C(a)|_{S_1} = 0$, or $\Omega^N(a)|_{S_1} = 0$) if and only if $\Omega^T(a)|_{S_2} = 0$ ($\Omega^T_C(a)|_{S_2} = 0$, or $\Omega^N(a)|_{S_2} = 0$). Moreover, if $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$ and $\Omega^T(a)|_{S_a} = 0$, where $S_a = \overline{\{\tau \in \widetilde{QT}(A) : \|\tau|_{\mathrm{Her}(a)}\| = 1\}}^w$, then $\Omega^T(a)|_{S_1} = 0$.*

Proof It follows from (1) of Proposition 2.10 that there is $L \in \mathbb{R}_+$ such that

$$S_2 \subset \{rs : s \in S_1 \text{ and } r \in [0, L]\}.$$

If $\Omega^T(a)|_{S_1} = 0$, then, by Proposition 4.8, there exists a sequence $b_n \in \mathrm{Ped}(\mathrm{Her}(a))$ with $\|b_n\| \le \|a\|$ such that $\lim_{n\to\infty} \omega(b_n)|_{S_1} = 0$ and $\lim_{n\to\infty} \|a - b_n\|_{2,S_1} = 0$. It follows that

$$\lim_{n\to\infty} \omega(b_n)|_{S_2} \le \lim_{n\to\infty} L \cdot \omega(b_n)|_{S_1} = 0 \text{ and } \lim_{n\to\infty} \|a - b_n\|_{2,S_2} \le \lim_{n\to\infty} L\|a - b_n\|_{2,S_1} = 0.$$

Then $\Omega^T(a)|_{S_2} = 0$. Exactly the same argument shows that if $\Omega^T_C(a)|_{S_1} = 0$ (or $\Omega^N(a)|_{S_1} = 0$), then $\Omega^T_C(a)|_{S_2} = 0$ (or $\Omega^N(a)|_{S_2} = 0$).

To see the last statement, we note that $(S_1)|_{\mathrm{Her}(a)}$ is bounded (see (2) of Proposition 2.10). In other words, there is $L > 0$ such that $(S_1)|_{\mathrm{Her}(a)} \subset \{r \cdot \tau : \tau \in S_a, r \in [0, L]\}$. ∎

Lemma 4.10 *Let A be a C^*-algebra, $a \in (A \otimes \mathcal{K})_+$, and $S \subset \widetilde{QT}(A)$ a compact subset. Suppose $e \in \mathrm{Her}(a)_+$. Then $e \sim a^{1/2}ea^{1/2} \sim e^{1/2}ae^{1/2}$, and*

$$\omega(e)(\tau)|_S = \omega(a^{1/2}ea^{1/2})(\tau)|_S = \omega(e^{1/2}ae^{1/2})(\tau)|_S \text{ for all } \tau \in S, \text{ and} \tag{e4.68}$$

$$\omega(e)|_S = \omega(a^{1/2}ea^{1/2})|_S = \omega(e^{1/2}ae^{1/2})|_S. \tag{e4.69}$$

Proof Since $e \in \mathrm{Her}(a)_+^{\mathbf{1}}$, we compute that

$$\lim_{n\to\infty} \|(a + 1/n)^{-1/2}a^{1/2}ea^{1/2}(a + 1/n)^{-1/2} - e\| = 0. \tag{e4.70}$$

It follows that $e \sim a^{1/2}ea^{1/2} \sim e^{1/2}ae^{1/2} \le e$. Therefore, by Proposition 4.2, (e4.68) and (e4.69) hold. ∎

Let us end this section with the following fact. The proof could be simplified when $QT(A) = T(A)$. Recall that when $A = \mathrm{Ped}(A)$, we write $\omega(a) = \omega(a)|_{\overline{QT(A)}^w}$ and $\Omega^T_T(a) = \Omega^T_T(a)|_{\overline{QT(A)}^w}$.

Proposition 4.11 *Let A be an algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. Suppose that A has strict comparison and Γ is surjective (see Definition 2.13). Then, for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$,*

$$\Omega^T_T(a) \le \|a\|\sqrt{\omega(a)}. \tag{e4.71}$$

Proof Fix $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+\setminus\{0\}$. Let $\varepsilon > 0$. If there is a subsequence $\{n_k\} \subset \mathbb{N}$ such that $f_{1/4n_k}(a) - f_{1/n_k}(a) = 0$, then $f_{1/4n_k}(a)$ is a projection. Consequently,

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$\omega(f_{1/4n_k}(a)) = 0$ for all $k \in \mathbb{N}$. Note that $af_{1/4n_k}(a) \sim f_{1/4n_k}(a)$. It follows that $\omega(af_{1/4n_k}(a)) = 0, k \in \mathbb{N}$. Since $\lim_{k\to\infty} \|a - af_{1/4n_k}(a)\| = 0, \Omega^T(a) = 0$. Hence, (e4.71) holds.

Next, we assume that for some $n_0 \in \mathbb{N}, f_{1/4n}(a) - f_{1/n}(a) \neq 0$ for all $n \geq n_0$. Moreover,

$$\sup\{d_\tau(a) - \tau(f_{1/n}(a)) : \tau \in \overline{QT(A)}^w\} < \omega(a) + \varepsilon/2 \text{ for all } n \geq n_0. \tag{e4.72}$$

For the rest of this proof, we will assume $n \geq n_0$. Since the map Γ is surjective, there is $c_n \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $d_\tau(c_n) = \tau(f_{1/n}(a))$ for all $\tau \in \overline{QT(A)}^w$. Since $\widehat{f_{1/n}(a)}$ is continuous on $\overline{QT(A)}^w$, $\omega(c_n) = 0$. Choose $0 < \delta_n < 1$ such that (by also Lemma 4.5),

$$d_\tau(c_n) - \tau(f_{\delta_n}(c_n)) < 1/2^n \text{ for all } \tau \in \overline{QT(A)}^w \text{ and } \omega(f_{\delta_n}(c_n)) < 1/2^n. \tag{e4.73}$$

Note that we have

$$d_\tau(c_n) = \tau(f_{1/n}(a)) < d_\tau(f_{1/4n}(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e4.74}$$

Since A has strict comparison, $c_n \lesssim f_{1/4n}(a)$. By Proposition 2.4 of [40], there is $x_n \in A \otimes \mathcal{K}$ such that

$$x_n^* x_n = f_{\delta_n}(c_n) \text{ and } x_n x_n^* \in \mathrm{Her}(f_{1/4n}(a)). \tag{e4.75}$$

Put $b_n = a^{1/2} x_n x_n^* a^{1/2}$. Then $\|b_n\| \leq \|a\|$ and $b_n \in \mathrm{Her}(f_{1/4n}(a))$. By Lemma 4.10 and (e4.73), $\omega(b_n) = \omega(x_n x_n^*) = \omega(x_n^* x_n) = \omega(f_{\delta_n}(c_n)) \leq 1/2^n \to 0$. Note that

$$f_{1/8n}(a)(x_n x_n^*) = x_n x_n^* = x_n x_n^* f_{1/8n}(a). \tag{e4.76}$$

Put $a_n = a^{1/2} f_{1/8n}(a) a^{1/2}$ and $d_n = f_{1/8n}(a) - x_n x_n^*$. Then $0 \leq d_n \leq 1$ and $0 \leq a_n - b_n, n \in \mathbb{N}$. For all $\tau \in \overline{QT(A)}^w$, we compute that

(e4.77)

$$\begin{aligned}
\tau((a_n - b_n)^2) &= \tau(a^{1/2} d_n a d_n a^{1/2}) \leq \|a\| \tau(a^{1/2} d_n^2 a^{1/2}) \\
&\leq \|a\| \tau(a^{1/2} d_n a^{1/2}) \leq \|a\|^2 \tau(d_n) && \text{(e4.78)} \\
&\overset{(4.76)}{=} \|a\|^2 (\tau(f_{1/8n}(a)) - \tau(x_n x_n^*)) \leq \|a\|^2 (d_\tau(a) - \tau(x_n^* x_n)) && \text{(e4.79)} \\
&= \|a\|^2 (d_\tau(a) - \tau(f_{1/n}(a)) + \tau(f_{1/n}(a)) - \tau(f_{\delta_n}(c_n))) && \text{(e4.80)} \\
&= \|a\|^2 (d_\tau(a) - \tau(f_{1/n}(a)) + d_\tau(c_n) - \tau(f_{\delta_n}(c_n))) && \text{(e4.81)} \\
&\overset{(4.72),(4.73)}{\leq} \|a\|^2 ((\omega(a) + \varepsilon/2) + 1/2^n). && \text{(e4.82)}
\end{aligned}$$

Note that $\|a - a_n\| < 1/n$. It follows that, by (e2.17), (e2.19), and (e4.82),

$$\begin{aligned}
\|a - b_n\|_{2,\overline{QT(A)}^w}^{2/3} &\leq \|a - a_n\|_{2,\overline{QT(A)}^w}^{2/3} + \|a_n - b_n\|_{2,\overline{QT(A)}^w}^{2/3} && \text{(e4.83)} \\
&< (1/n)^{2/3} \sup\{d_\tau(a)^{1/3} : \tau \in \overline{QT(A)}^w\} + (\omega(a) + \varepsilon/2 + 1/2^n)^{1/3}. && \text{(e4.84)}
\end{aligned}$$

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Hence (by also (2) of Proposition 2.10),

$$\limsup_{n\to\infty} \|a-b_n\|_{2,\overline{QT(A)}^w} \le \|a\|\sqrt{\omega(a)+\varepsilon/2}. \tag{e4.85}$$

Let $\varepsilon \to 0$. We then obtain (e4.71) (recall that $\lim_{n\to\infty}\omega(b_n)=0$). ∎

5 C^*-algebras with tracial approximate oscillation zero

In this section, we will introduce the notion of T-tracial approximate oscillation zero for C^*-algebras with non-trivial 2-quasitraces. We also present some examples of simple C^*-algebras with tracial approximate oscillation zero (see Proposition 5.8 and Theorems 5.9 and 5.10).

Definition 5.1 Let A be a C^*-algebra with $\widetilde{QT}(A)\backslash\{0\}\neq\emptyset$ and $S\subset\widetilde{QT}(A)\backslash\{0\}$ a compact convex subset of A such that $\mathbb{R}_+\cdot S=\widetilde{QT}(A)$.$C^*$-algebra A is said to have norm approximate oscillation zero (relative to S) if for any $a\in \mathrm{Ped}(A\otimes\mathcal{K})_+$, $\Omega^N(a)|_S=0$. It is said to have tracial approximate oscillation zero (relative to S), if for any $a\in \mathrm{Ped}(A\otimes\mathcal{K})_+$, $\Omega_C^T(a)|_S=0$. We say that A has T-tracial approximate oscillation zero (relative to S) if $\Omega^T(a)|_S=0$ for all $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$. We say that A has C-tracial approximate oscillation zero if $\Omega^C(a)=0$ for all $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$.

Note that, by Proposition 4.9, these definitions do not depend on the choices of S. Therefore, we often omit S in the notation.

(1) If A is a σ-unital simple C^*-algebra and $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+^{\mathbf{1}}$, then $QT(\mathrm{Her}(a))$ may be viewed as a convex subset of $\widetilde{QT}(A)$. Put $S_1=\overline{QT(\mathrm{Her}(a))}^w$. Therefore (see also Proposition 2.9), $\Omega^T(a)|_S=0$ if and only if $\Omega^T(a)|_{S_1}=0$.

(2) By Proposition 4.8, $\Omega_T^T(a)|_S=0$ if and only if $\Omega^T(a)|_S=0$ for all $a\in(A\otimes\mathcal{K})_+$ with $\|a\|_{2,S}<\infty$. By Proposition 5.7 below, that A has norm approximate oscillation zero is the same as that A has T-tracial approximate oscillation zero.

If A is σ-unital, non-elementary, and simple and has strict comparison, then, by Proposition 2.18, for any nonzero $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$, one has $I_{\overline{QT(A_a)}^w}=N_{cu}(A_a)$, where $A_a=\mathrm{Her}(a)$. It follows that, in this case, the notion of tracial approximate oscillation zero, the notion of T-tracial approximate oscillation zero, the notion of C-tracial approximate oscillation zero, and that of norm approximate oscillation zero all coincide (see also Proposition 5.7).

(3) Note also that, if A has (T-) tracial approximate oscillation zero, then $M_n(A)$ also has (T-) tracial approximate oscillation zero.

If we view $\|\cdot\|_{2,\overline{QT(A)}^w}$ as an L^2-norm, then that A has T-tracial approximate oscillation zero has an analogue to that "almost" continuous functions are dense in the L^2-norm. It is worth mentioning that a σ-unital simple C^*-algebra has (T-) tracial approximate oscillation zero, if, for some $e\in\mathrm{Ped}(A)_+\backslash\{0\}$, $\mathrm{Her}(e)$ has (T-) tracial approximate oscillation zero.

Definition 5.2 Let A be a σ-unital C^*-algebra with $\widetilde{QT}\backslash\{0\}\neq\emptyset$. Define

$$\mathbb{O}(A)=\sup\{\Omega^T(a)|_{S_a}: a\in\mathrm{Ped}(A\otimes\mathcal{K})_+^{\mathbf{1}}\}, \tag{e5.1}$$

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where $S_a = \overline{QT(\mathrm{Her}(a))}^w$ (see the last paragraph of Definition 2.8). Note that, since $\Omega^T(a)|_{S_a} \le \|a\|$ for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$ and for any C^*-algebra A, one has that $0 \le \mathbb{O}(A) \le 1$. The number $\mathbb{O}(A)$ is called T-tracial approximate oscillation of A.

Suppose that $S \subset \widetilde{QT}(A)\backslash\{0\}$ is a compact convex set such that $\mathbb{R}_+ \cdot S = \widetilde{QT}(A)$. By (the "Moreover" part of) Proposition 4.9, if $\mathbb{O}(A) = 0$, then A has T-tracial approximate oscillation zero. Conversely, if A has T-tracial approximate oscillation zero, then $\mathbb{O}(A) = 0$ (see (4) of Proposition 2.10).

The next example shows that there are (commutative) C^*-algebras A of stable rank one such that $\mathbb{O}(A) > 0$. By Theorem 1.1, if A is a separable simple C^*-algebra which has strict comparison but does not have stable rank one, then $\mathbb{O}(A) > 0$. However, we do not have any such examples.

Example 5.3 Let $A = C([0,1])$. Then $T(A)$ is compact and $\widetilde{QT}(A) = \widetilde{T}(A)$. Let $a \in A_+\backslash\{0\}$ be such that $0 \le a \le 1$ that is not invertible. Let $G = \{t \in [0,1] : a(t) > 0\}$. Then G is an open subset. Note $0 \in \mathrm{sp}(a)$. Since A has no nontrivial projection, there are $t_n \in \mathrm{sp}(a)$ with $\lim_{n\to\infty} t_n = 0$. For any $0 < \delta < 1/2$, let $b = f_\delta(a)$. Let $s_n \in G$ such that $a(s_n) = t_n, n \in \mathbb{N}$. Then, for some $0 < \eta < \delta/2$, there is s_n in the support of $c = f_\eta(a) - f_{\delta/2}(a)$. There is a Borel probability measure μ_n on $[0,1]$ such that $\mu(\{s_n\}) = 1$. Let τ_{μ_n} be the tracial state induced by μ_n, then $\tau_{\mu_n}(f_{\eta/2}(a) - b) = 1$. This implies that $\omega(a)|_{S_a} = 1$. This also holds for any nonzero $g \in \mathrm{Her}(a)_+ \subset A_+$. In other words, for any $g \in \mathrm{Her}(a)_+, \omega(g)|_{S_a} = 1$. Therefore, $\Omega^T(a) = 1$. It follows $\mathbb{O}(A) = 1$. Recall that A has stable rank one.

Proposition 5.4 *If A is a simple C^*-algebra which has (T-) tracial approximate oscillation zero, then every hereditary C^*-subalgebra of A also has (T-) tracial approximate oscillation zero.*

Proof This follows from the definition immediately. ∎

Definition 5.5 Let S be a compact subset of $\widetilde{QT}(A)\backslash\{0\}$ such that $\widetilde{QT}(A) = \mathbb{R}_+ \cdot S$ and $B \subset A$ be a hereditary C^*-subalgebra. A sequence of elements $\{e_n\} \subset \mathrm{Ped}(B)_+^{\mathbf{1}}$ is said to be tracial approximate identity for B, if, for any $b \in B$,

$$\|\Pi_{cu}(\iota(b) - \iota(b)\{e_n\})\| = 0, \tag{e5.2}$$

and $\{e_n\}$ is said to be T-tracial approximate identity for B (relative to S), if, for any $b \in B$ with $\|b\|_{2,S} < \infty$,

$$\lim_{n\to\infty} \|b - be_n\|_{2,S} = \lim_{n\to\infty} \|b - e_n b\|_{2,S} = 0. \tag{e5.3}$$

We do not require that $\{e_n\}$ is increasing.

Proposition 5.6 *Let A be a C^*-algebra, $a \in (A \otimes \mathcal{K})_+$ with $\|a\|_{2,S} < \infty$, and $S \subset \widetilde{QT}(A)\backslash\{0\}$ be a compact subset.*

(1) *Then $\Omega^N(a)|_S = 0$ if and only if* $\mathrm{Her}(a)$ *has a (not necessarily increasing) approximate identity $\{e_n\}$ such that $\lim_{n\to\infty} \omega_n(e_n)|_S = 0$.*

(2) *Moreover, $\Omega^T(a)|_S = 0$ (or $\Omega_C^T(a)|_S = 0$) if and only if* $\mathrm{Her}(a)$ *admits a T-tracial (or tracial) approximate identity (relative to S) $\{e_n\}$ with $\lim_{n\to\infty} \omega(e_n)|_S = 0$.*

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Proof For (1), let us assume that $\{e_n\}$ is a (not necessarily increasing) approximate identity for $\mathrm{Her}(a)$ such that $\lim_{n\to\infty}\omega(e_n)=0$. Then

$$\lim_{n\to\infty}\|a-e_n^{1/2}ae_n^{1/2}\|=0. \tag{e5.4}$$

By Lemma 4.10, $\omega(e_n^{1/2}ae_n^{1/2})|_S\le\omega(e_n)|_S\to 0$. Thus, $\Omega^N(a)|_S=0$.

Conversely, suppose that $a\in(A\otimes\mathcal{K})_+^{\mathbf{1}}$ and $\{b_n\}\in l^\infty(\mathrm{Her}(a))_+$ such that

$$\lim_{n\to\infty}\|a-b_n\|=0 \ \text{ and } \ \lim_{n\to\infty}\omega(b_n)|_S=0. \tag{e5.5}$$

Note that $\lim_{n\to\infty}\|b_n\|=\|a\|\le 1$. Let $g\in C([0,\infty))_+^{\mathbf{1}}$ such that $g(t)=t$ if $t\in[0,1]$ and $g(t)=1$ if $t>1$. Then, for any $n\in\mathbb{N}$, $g(b_n)\sim b_n$. We also have $\lim_{n\to\infty}\|g(a)-g(b_n)\|=0$. But $g(a)=a$. Then, replacing b_n by $g(b_n)$, we may assume that $\|b_n\|\le 1$. For each $k\in\mathbb{N}$, choose n_k such that

$$\|b_{n_k}^{1/k}-a^{1/k}\|<1/k,\ \ k\in\mathbb{N}. \tag{e5.6}$$

Put $e_k=b_{n_k}^{1/k}$, $k\in\mathbb{N}$. Then, for any $x\in\mathrm{Her}(a)$,

$$\|x-xe_k\|\le\|x-xa^{1/k}\|+\|x\|\|a^{1/k}-e_k\|\le\|x-xa^{1/k}\|+\frac{\|x\|}{k}. \tag{e5.7}$$

This shows that $\{e_k\}$ is a (not necessarily increasing) approximate identity for $\mathrm{Her}(a)$. Since $e_k=b_{n_k}^{1/k}$, we have that $\omega(e_k)=\omega(b_{n_k})$ and

$$\lim_{k\to\infty}\omega(e_k)|_S=\lim_{k\to\infty}\omega(b_{n_k})|_S=0.$$

For (2), let us prove one case.

Fix $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$. Without loss of generality, we may assume that $0\le a\le 1$. Suppose that $\{e_n\}$ is a T-tracial approximate identity of $\mathrm{Her}(a)$ relative to S. Then

$$\lim_{n\to\infty}\|a-e_na\|_{_{2,S}}=\lim_{n\to\infty}\|a-ae_n\|_{_{2,S}}=0 \ \text{ and}$$
$$\begin{aligned}\lim_{n\to\infty}\|a-e_nae_n\|_{_{2,S}}^{2/3}&\le\lim_{n\to\infty}(\|a-e_na\|_{_{2,S}}^{2/3}+\|e_na-e_nae_n\|_{_{2,S}}^{2/3})\\&\le\lim_{n\to\infty}\|e_n(a-ae_n)\|_{_{2,S}}^{2/3}\le\lim_{n\to\infty}(\|e_n^2\|^{1/3}\|a-ae_n\|_{_{2,S}}^{2/3})=0.\end{aligned}$$

By Lemma 4.10, $\omega(e_nae_n)|_S\le\omega(e_n^2)|_S=\omega(e_n)|_S\to 0$. Thus, $\Omega^T(a)|_S=0$.

Conversely, suppose that $\Omega^T(a)|_S=0$. Then, by Proposition 4.8, there exists $\{b_n\}\subset\mathrm{Ped}(\mathrm{Her}(a))_+^{\mathbf{1}}$ such that

$$\lim_{n\to\infty}\|a-b_n\|_{_{2,S}}=0 \ \text{ and } \ \lim_{n\to\infty}\omega(b_n)|_S=0. \tag{e5.8}$$

Let $B=\mathrm{Her}(a)$ and $\Pi_S:l^\infty(B)\to l^\infty(B)/I_{_{S,B}}$ be the quotient map (see Definition 4.7 for $I_{_{S,B}}$). Then

$$\Pi_S(\iota(a))=\Pi_S(\{b_n\}). \tag{e5.9}$$

For each $k\in\mathbb{N}$, we have

$$\Pi_S(\iota(a)^{1/k})=\Pi_S(\{b_n\}^{1/k}). \tag{e5.10}$$

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It follows that, for each $k \in \mathbb{N}$, there exists $n(k) \in \mathbb{N}$ such that

$$\|a^{1/k} - b_{n(k)}^{1/k}\|_{2,s} < 1/2k. \tag{e5.11}$$

Choose $e_k = b_{n(k)}^{1/k}$, $k \in \mathbb{N}$. Then, for any $c \in \mathrm{Her}(a)$,

$$\|c - ce:k\|_{2,s}^{2/3} \le \|c - ca^{1/k}\|_{2,s}^{2/3} + \|c(a^{1/k} - b_{n(k)}^{1/k})\|_{2,s}^{2/3} \to 0, \text{ as } k \to \infty. \tag{e5.12}$$

Since $b_{n(k)} \sim e_k$,

$$\lim_{k\to\infty} \omega(e_k)|_S = 0. \tag{e5.13}$$

The proposition then follows. ∎

Proposition 5.7 *Let A be a σ-unital C^*-algebra, let $S \subset \widetilde{QT}(A)$ be a compact subset, and let $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. Then $\Omega^T(a)|_S = 0$ if and only if $\Omega^N(a)|_S = 0$.*

Proof For the "if" part, let us assume $\Omega^N(a)|_S = 0$. By the definition there is $\{b_n\} \in l^\infty(\mathrm{Her}(a))_+$ such that $\omega(b_n)|_S < 1/n$ and $\|a - b_n\| < 1/n$. Let $b_n' = \frac{\|a\| b_n}{\|b_n\| + 1/n}$. Then $\|b_n'\| \le \|a\|$,

$$\lim_{n\to\infty} \omega(b_n')|_S = \lim_{n\to\infty} \omega(b_n)|_S = 0, \lim_{n\to\infty} \|b_n - b_n'\| = 0 \text{ and} \tag{e5.14}$$

$$0 \le \Omega^T(a)|_S \le \|\Pi_S(\iota(a) - \{b_n'\})\| \le \limsup_{n\to\infty} \|a - b_n'\| \le \limsup_{n\to\infty} \|a - b_n\| = 0. \tag{e5.15}$$

For the "only if" part, let us assume that $\Omega^T(a)|_S = 0$. Then, by Proposition 5.6, there are $e_n \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that

$$\lim_{n\to\infty} \omega(e_n)|_S = 0 \text{ and } \lim_{n\to\infty} \|a - a^{1/2} e_n a^{1/2}\|_{2,s} = 0. \tag{e5.16}$$

It follows that $\{b_n\} = \{a - a^{1/2} e_n a^{1/2}\} \in (I_{_S})_+$ (see Definition 2.16 for the definition of $I_{_S}$). By (4) of Proposition 4.8, there exists $\{c_n\} \in (I_{_S})_+$ such that

$$\lim_{n\to\infty} \sup\{d_\tau(c_n) : \tau \in S\} = 0 \text{ and } \lim_{n\to\infty} \|b_n - c_n\| = 0. \tag{e5.17}$$

Put $d_n = a^{1/2} e_n a^{1/2} + c_n, n \in \mathbb{N}$. Then $d_n \ge 0$. Put $\bar{d}_n = \frac{\|a\| d_n}{\|d_n\| + 1/n}$, $n \in \mathbb{N}$. Then $\|\bar{d}_n\| \le \|a\|$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \|a - d_n\| = 0$, we have $\lim_{n\to\infty} \|d_n\| = \|a\|$. It follows that

$$\lim_{n\to\infty} \|a - \bar{d}_n\| = 0. \tag{e5.18}$$

On the other hand, by Proposition 4.4(1) (see also (2) of Proposition 2.10) for all $\tau \in S$,

$$\omega(d_n)(\tau)|_S \le \omega(a^{1/2} e_n a^{1/2})(\tau)|_S + \overline{d_\tau}(c_n)|_S \le \omega(a^{1/2} e_n a^{1/2})|_S + \sup\{d_\tau(c_n) : \tau \in S\} \tag{e5.19}$$

for all $n \in \mathbb{N}$. By Lemma 4.10, $\lim_{n\to\infty} \omega(a^{1/2} e_n a^{1/2}) = \lim_{n\to\infty} \omega(e_n) = 0$. By the fact that $\bar{d}_n \sim d_n$ and Proposition 4.2, we have

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$$\lim_{n\to\infty}\omega(\bar{d}_n) = \lim_{n\to\infty}\omega(d_n) \overset{(4.9)}{=} \lim_{n\to\infty}\sup_{\tau\in S}\{\omega(d_n)(\tau)|_S\}$$

$$\overset{(5.19)}{\leq} \lim_{n\to\infty}\omega(a^{1/2}e_na^{1/2})|_S + \lim_{n\to\infty}\sup\{d_\tau(c_n) : \tau\in S\} = 0. \tag{e5.20}$$

Then (e5.18) and (e5.20) show that $\Omega^N(a)|_S = 0$. ∎

Let us present some examples of C^*-algebras which have norm approximate oscillation zero.

Proposition 5.8 *Let A be a C^*-algebra of real rank zero. Then A has norm approximate oscillation zero.*

Proof Let $a\in \mathrm{Ped}(A)$ with $0\leq a\leq 1$. Put $B = \mathrm{Her}(a)$. Then B has an approximate identity $\{e_n\}$ consisting of projections. Note that, since e_n is a projection, $\widehat{[e_n]} = \widehat{e_n}$ is continuous on S. The proposition follows from Proposition 5.6. ∎

Let $T_b = \{s\in \widetilde{QT}(A) : s(b) = 1\}$ for some nonzero $b\in(\mathrm{Ped}(A\otimes\mathcal{K}))_+$. It is a compact convex subset and T_b is a basis for the cone $\widetilde{QT}(A)$ if A is simple. The proof of the following is taken from Lemma 4.8 of [20] (see also Remark 4.7 of [20]).

Theorem 5.9 *Let A be a C^*-algebra with countable $\partial_e(T_b)$ (for some $b\in \mathrm{Ped}(A)_+\backslash\{0\}$), where $\partial_e(T_b)$ is the set of extremal points of T_b. Then*

$$\Omega^N(a) = 0 \ \textit{for all}\ \ a\in \mathrm{Ped}(A\otimes\mathcal{K})_+. \tag{e5.21}$$

In particular, A has norm approximate oscillation zero.

Proof We may assume that $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+\backslash\{0\}$ and $0\leq a\leq 1$. If $0\in\overline{\mathbb{R}_+\backslash \mathrm{sp}(a)}$, then $\Omega^N(a) = 0$. To see this, let $s_n\in\mathbb{R}_+\backslash\mathrm{sp}(a)$ such that $s_n\searrow 0$. Then the characteristic function $\chi_{[s_n,1]}$ is continuous on $\mathrm{sp}(a)$. Therefore, $p_n = \chi_{[s_n,1]}(a)\in C^*(a)\subset \mathrm{Her}(a)$ is a projection. Note that $p_n\leq p_{n+1}$ for all $n\in\mathbb{N}$. Then

$$\|a - p_na\|\leq s_n\to 0. \tag{e5.22}$$

In other words, $\{p_n\}$ is an approximate identity for $\mathrm{Her}(a)$. Moreover, $\omega(p_n) = 0$. So this case follows from Proposition 5.6.

Therefore, without loss of generality, we may assume that $[0,r]\subset\mathrm{sp}(a)$ for some $0<r<1$. Let $r/2>\eta>0$.

Note that, since $a\in\mathrm{Ped}(A\otimes\mathcal{K})_+$, $\sup\{d_\tau(a) : \tau\in T_b\} = M<\infty$ (see Proposition 2.10). For each $\tau\in\partial_e(T_b)$, τ induces a Borel measure μ_τ on $\mathrm{sp}(a)$ which is bounded by M.

We claim that there is $s\in(r-\eta, r]$ such that

$$\sup\{\mu_\tau(\{s\}) : \tau\in T_b\} = 0. \tag{e5.23}$$

To see this, write $\partial_e(T_b) = \{\tau_n\}_{n\in\mathbb{N}'}$, where $\mathbb{N}'$ is a subset of $\mathbb{N}$. For each $k\in\mathbb{N}'$, and $n\in\mathbb{N}$, define

$$S_{k,n} = \{s\in(r-\eta, r] : \mu_{\tau_k}(\{s\})\geq 1/n\}. \tag{e5.24}$$

Since $\mu_{\tau_k}(\mathrm{sp}(a))\leq M$, $S_{k,n}$ must be finite. It follows that $S_k = \cup_{n=1}^\infty S_{k,n}$ is countable. Thus, $S = \cup_{k\in\mathbb{N}'}S_k$ is countable. Therefore, there must be $s\in(r-\eta, r]$ such that $s\notin S$.

In other words, there must be $s \in (r - \eta, r]$ such that

$$\mu_\tau(\{s\}) = 0 \text{ for all } \tau \in \partial_e(T_b). \tag{e5.25}$$

Since T_b is compact, by the Krein–Milman theorem, this implies that

$$\mu_\tau(\{s\}) = 0 \text{ for all } \tau \in T_b. \tag{e5.26}$$

This proves the claim.

By the claim, for each integer $n \in \mathbb{N}$, there is $\delta > 0$ such that $1/2^{n+1} < \delta < 1/2^n$ and

$$\mu_\tau(\{\delta\}) = 0 \text{ for all } \tau \in T_b. \tag{e5.27}$$

Note that

$$d_\tau((a - \delta)_+) = \mu_\tau((\delta, 1] \cap \mathrm{sp}(a)) \text{ for all } \tau \in T_b \tag{e5.28}$$

is a lower semi-continuous function. By Portmanteau Theorem, the function $h : T_b \to \mathbb{R}_+$ given by

$$h(\tau) := \mu_\tau([\delta, 1] \cap \mathrm{sp}(a)) \tag{e5.29}$$

is an upper semi-continuous function.[1]

By (e5.27),

$$h(\tau) = d_\tau((a - \delta)_+) \text{ for all } \tau \in T_b. \tag{e5.30}$$

It follows that $d_\tau((a - \delta)_+)$ is continuous.

To prove the lemma, let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $1/2^n < \varepsilon$. Choose $1/2^{n+1} < \delta_n < 1/2^n$ such that $\mu_\tau(\{\delta_n\}) = 0$ for all $\tau \in S$ as above. Put $d_n = (a - \delta_n)_+$. Then

$$\|a - d_n\| < \varepsilon. \tag{e5.31}$$

Since $d_\tau(d_n) = d_\tau((a - \delta_n)_+)$, we have that $\omega(d_n)(\tau)|_{T_b} = 0$ for all $\tau \in T_b$. The lemma follows. ∎

The following is a restatement of Lemma 7.2 of [10] with the same proof (and with some necessary modification and correcting a typo).

Theorem 5.10 *Let A be a σ-unital simple C^*-algebra which has strict comparison and almost stable rank one. Suppose that the canonical map $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\widetilde{QT}(A))$ is surjective (see Definition 2.13). Then A has norm approximate oscillation zero.*

Proof Let $e_A \in \mathrm{Ped}(A)_+ \setminus \{0\}$ and $A_1 = \mathrm{Her}(e_A)$. Then $\mathrm{Ped}(A_1) = A_1$. Since A is σ-unital, $A \otimes \mathcal{K} \cong A_1 \otimes \mathcal{K}$ by Brown's stable isomorphism theorem [4]. Therefore, it suffices to show that A_1 has norm approximate oscillation zero. To simplify notation, we may assume that $A = A_1$ (and $A = \mathrm{Ped}(A)$).

Let $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. It follows from the proof of Lemma 7.2 of [10] that, since $\mathrm{Her}(a)$ also has strict comparison, almost stable rank one, and Γ is surjective, $\mathrm{Her}(a)$ has an approximate identity consisting of elements $\{e_n\}$ such that $\widehat{e_n}$ is continuous on $\overline{QT(A)}^w$. Then the theorem follows from Proposition 5.6. Since we do not assume

[1] This can be directly obtained as follows: Let $f^{(n)} \in C_0((0, \|a\|])_+^1$ be such that $f^{(n)}(t) = 1$ for $t \in [\delta, \|a\|]$, $f^{(n)} = 0$ if $t \in [0, \delta - \delta/2^n]$ and linear in $[\delta - \delta/2^n, \delta]$. Then $\tau(f^{(n)}(a)) \searrow h(\tau)$.

that $QT(A) = T(A)$, to be complete, let us repeat some of the argument in the proof of Lemma 7.2 of [10] which will be, again, used in the proof of Lemma 8.9.

By Proposition 5.6, it suffices to show that, for any $\varepsilon \in (0, 1/2)$, there is $e \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that $f_\varepsilon(a) \le e$ and $\widehat{[e]}$ is continuous. By the first part of the proof of Theorem 5.9, we may assume that $[0, \varepsilon_0) \subset \mathrm{sp}(a)$ for some $\varepsilon_0 \in (0, \varepsilon)$.

Note that, without loss of generality, we may assume that

(e5.32)

$$d_\tau(f_{\varepsilon/2}(a)) < \tau(f_{\delta_1}(a)) < d_\tau(f_{\eta_1}(a)) < \tau(f_{\delta_2}(a)) < d_\tau(f_\eta(a)) \text{ for all } \tau \in \overline{QT(A)}^w,$$

where $\varepsilon/4 > \delta_1, \delta_1/2 > \eta_1, \eta_1/2 > \delta_2$, and $\delta_2/2 > \eta > 0$. Put $h_i(\tau) = \tau(f_{\delta_i}(a))$ for all $\tau \in \overline{QT(A)}^w$, $i = 1, 2$. Then $h_i \in \mathrm{Aff}_+(\overline{QT(A)}^w)$. Since Γ is surjective, there is $c \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $d_\tau(c) = h_2(\tau)$ for all $\tau \in \overline{QT(A)}^w$. Since A has strict comparison, (e5.32) and the choice of c implies $c \lesssim f_\eta(a)$. Since A has almost stable rank one, by Lemma 3.2 of [10] and (e5.32), there is $x \in A \otimes \mathcal{K}$ such that

$$x^*x = c \text{ and } xx^* \in \mathrm{Her}(f_\eta(a)). \tag{e5.33}$$

Put $c_0 = xx^*$. Then $0 \le c_0 \le 1$. Note that $d_\tau(c_0) = d_\tau(c)$ for all $\tau \in \overline{QT(A)}^w$. Since h_1 is continuous, $h_1(\tau) < h_2(\tau) = d_\tau(c) = d_\tau(c_0) = \lim_{n\to\infty} \tau(f_{1/n}(c_0))$ for all $\tau \in \overline{QT(A)}^w$, and $\overline{QT(A)}^w$ is compact, there is an integer $m > 2$ such that

$$h_1(\tau) < \tau(f_{1/m}(c_0)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e5.34}$$

By (e5.32) and Lemma 3.2 of [10], there is a unitary u in the unitization of $\mathrm{Her}(f_\eta(a))$ such that

$$u^* f_{\varepsilon/8}(f_{\varepsilon/2}(a))u \in \mathrm{Her}(f_{1/m}(c_0)). \tag{e5.35}$$

Set $c_1 = uc_0u^*$. Then

$$f_{\varepsilon/8}(f_{\varepsilon/2}(a)) \in \mathrm{Her}(f_{1/m}(c_1)) \subset \mathrm{Her}(c_1). \tag{e5.36}$$

There is $g \in C_0((0,1])$ such that $0 \le g \le 1$, $g(t) > 0$ for all $t \in (0,1]$, and

$$g f_{1/2m} = f_{1/2m}. \tag{e5.37}$$

Put $e = g(c_1)$. Then $[e] = [c_1] = [c_0] = [c]$. In particular, $d_\tau(e)$ is continuous on $\overline{QT(A)}^w$. But we also have, by (e5.36),

$$f_\varepsilon(a) \le f_{\varepsilon/8}(f_{\varepsilon/2}(a)) \le e. \tag{e5.38}$$

This completes the proof. ∎

6 C^*-algebra $l^\infty(A)/I_{\overline{QT(A)}^w}$

Definition 6.1 Let A be a compact C^*-algebra, and let $p \in l^\infty(A)/I_{\overline{QT(A)}^w}$ (or in $l^\infty(A)/N_{cu}(A)$) be a projection and $\{e_n\} \in l^\infty(A)_+^{\mathbf{1}}$ such that $\Pi(\{e_n\}) = p$ (recall

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that $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w}$ is the quotient map). The sequence $\{e_n\}$ is said to be a permanent projection lifting, if for any sequence of positive integers $\{m(n)\}$,

$$\Pi(\{e_n^{1/m(n)}\}) = p \ \ (or\ \Pi_{cu}(\{e_n^{1/m(n)}\}) = p). \tag{e6.1}$$

Proposition 6.2 *Let A be a compact C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$ and $\{e_n\} \in l^\infty(A)^{\mathbf{1}}_+$.*

(1) Let $p = \Pi(\{e_n\})$ (or $p = \Pi_{cu}(\{e_n\})$) be a projection. Then $\{f_\delta(e_n)\}$ is a permanent projection lifting of p for any $0 < \delta < 1/2$ (for both cases) and

(e6.2)
$$\lim_{n\to\infty} \sup\{\tau(e_n - f_\delta(e_n)e_n) : \tau \in \overline{QT(A)}^w\} = 0 \ \ (\text{or } \{e_n - e_n^{1/2} f_\delta(e_n)e_n^{1/2}\} \in N_{cu}).$$

(2) If $\{e_n\}$ is a permanent projection lifting, then $\lim_{n\to\infty}\omega(e_n) = 0$. Moreover, an element $\{e_n\}$ is a permanent projection lifting (from $l^\infty(A)/I_{\overline{QT(A)}^w}$) if and only if

$$\lim_{n\to\infty} \sup\{d_\tau(e_n) - \tau(e_n^2) : \tau \in \overline{QT(A)}^w\} = 0.$$

(3) If $\{e_n\} \in l^\infty(A)^{\mathbf{1}}_+$ and $\lim_{n\to\infty}\omega(e_n) = 0$, then for some $l(k) \in \mathbb{N}$, $p = \Pi(\{e_k^{1/l(k)}\})$ is a projection, and $\{e_k^{1/l(k)}\}$ is a permanent projection lifting of p.

(4) Suppose that $p = \Pi_{cu}(\{e_n\})$ is a projection for some $\{e_n\} \in l^\infty(A)^{\mathbf{1}}_+$. Then $\{e_n\}$ is a permanent projection lifting (from $l^\infty(A)/N_{cu}$) if $g_\delta(e_n) \stackrel{c.}{\to} 0$ for some $\delta \in (0, 1/4)$.

(5) If $\{e_n\}$ is a permanent projection lifting of $p \in l^\infty(A)/I_{\overline{QT(A)}^w}$, then

$$l^\infty(\{\mathrm{Her}(e_n)\})/I_{\overline{QT(A)}^w} = p(l^\infty(A)/I_{\overline{QT(A)}^w})p$$

(see Definition 2.15 for $l^\infty(\{\mathrm{Her}(e_n)\})$).

(6) If A is algebraically simple and $QT(A) \neq \emptyset$ and $e \in A^{\mathbf{1}}_+$ is a strictly positive element such that $\widehat{[e]}$ is continuous on $\overline{QT(A)}^w$, then $l^\infty(A)/I_{\overline{QT(A)}^w}$ is unital.

(7) A σ-unital simple C^*-algebra A has continuous scale if and only if $l^\infty(A)/N_{cu}$ is unital.

Proof (1) Note that $\Pi(f_\delta(\{e_n\})) = f_\delta(\Pi(\{e_n\})) = p$ for any $0 < \delta < 1/2$. Therefore, $\Pi(f_{\delta/2}(\{e_n\})) = p$. Put $b_n = f_\delta(e_n)$, $n \in \mathbb{N}$. For any integers $\{m(n)\}$, we have

$$b_n^{1/m(n)} \le f_{\delta/2}(e_n), \ \ n \in \mathbb{N}. \tag{e6.3}$$

It follows that

$$p = \Pi(\{f_\delta(e_n)\}) \le \Pi(\{b_n^{1/m(n)}\}) \le \Pi(\{f_{\delta/2}(e_n)\}) = p. \tag{e6.4}$$

This proves the first part of (1) (the proof for $p = \Pi_{cu}(\{f_\delta(e_n)\})$ is similar).

Since $\Pi(\{f_\delta(e_n)\}) = p$ (or $\Pi_{cu}(\{f_\delta(e_n)\}) = p$), we have

$$e_n - f_\delta(e_n)e_n \in I_{\overline{QT(A)}^w}, \ \ \text{hence } (e_n - f_\delta(e_n)e_n)^{1/2} \in I_{\overline{QT(A)}^w} \tag{e6.5}$$

$$(\text{or } e_n - f_\delta(e_n)e_n,\ (e_n - f_\delta(e_n)e_n)^{1/2} \in N_{cu}). \tag{e6.6}$$

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Part (1) follows.

(2) If

$$\lim_{n\to\infty} \sup\{d_\tau(e_n) - \tau(e_n^2) : \tau \in \overline{QT(A)}^w\} = 0,$$

then, for any $\{m(n)\} \subset \mathbb{N}$,

(e6.7)
$$\sup\{\tau(e_n^{1/m(n)}) - \tau(e_n^2) : \tau \in \overline{QT(A)}^w\} \le \sup\{d_\tau(e_n) - \tau(e_n^2) : \tau \in \overline{QT(A)}^w\} \to 0.$$

It follows that $\{e_n^{1/m(n)} - e_n^2\} \in I_{\overline{QT(A)}^w}$. Since $e_n^2 \le e_n$ for all $n \in \mathbb{N}$, this also implies that $\{e_n - e_n^2\} \in I_{\overline{QT(A)}^w}$. Hence, $\Pi(\{e_n\})$ is a projection and $\{e_n\}$ is a permanent projection lifting.

Now suppose that $\{e_n\}$ is a permanent projection lifting of $p = \Pi(\{e_n\})$. Let us show first that $\lim_{n\to\infty} \omega(e_n) = 0$. Otherwise, there exists a subsequence $\{l(k)\}$ such that $\omega(e_{l(k)}) > \sigma$ for some $\sigma > 0$. Fix any $\delta \in (0, 1/4)$. By Proposition 4.6, for each of these $l(k)$, there are $m(l(k))$ such that

$$\sup\{\tau(e_{l(k)}^{1/m(l(k))}) - \tau(f_\delta(e_{l(k)})) : \tau \in \overline{QT(A)}^w\} > \omega(e_{l(k)}) - \sigma/4 > \sigma/2. \tag{e6.8}$$

Choose a sequence $m(n)$ of integers which extends $m(l(k))$. Then

$$\limsup_n \|(e_n^{1/m(n)} - f_\delta(e_n))^{1/2}\|_{2,\overline{QT(A)}^w} \ge \sigma/2. \tag{e6.9}$$

Therefore, $\Pi(\{e_n^{1/m(n)}\}) \ne \Pi(f_\delta(\{e_n\})) = p$. A contradiction. Hence, $\lim_{n\to\infty} \omega(e_n) = 0$.

Therefore, there exists a sequence $\{m(n)\}$ such that

$$\sup\{d_\tau(e_n) - \tau(e_n^{1/m(n)}) : \tau \in \overline{QT(A)}^w\} < 1/n, \ \ n \in \mathbb{N}. \tag{e6.10}$$

Then, since $\{(e_n^{1/m(n)} - e_n^2)^{1/2}\} \in I_{\overline{QT(A)}^w}$ (for any $\{m(n)\}$), we also have that

$$\begin{aligned}
&\sup\{d_\tau(e_n) - \tau(e_n^2) : \tau \in \overline{QT(A)}^w\} \\
&\le \sup\{d_\tau(e_n) - \tau(e_n^{1/m(n)}) : \tau \in \overline{QT(A)}^w\} + \sup\{\tau(e_n^{1/m(n)} - e_n^2) : \tau \in \overline{QT(A)}^w\} \\
&< 1/n + \|(e_n^{1/m(n)} - e_n^2)^{1/2}\|_{2,\overline{QT(A)}^w} \ \to 0 \ \text{ as n} \to \infty.
\end{aligned} \tag{e6.11}$$

(3) In this case, since $\lim_{n\to\infty} \omega(e_n) = 0$, there are $l(n) \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \sup\{d_\tau(e_n) - \tau(e_n^{1/l(n)}) : \tau \in \overline{QT(A)}^w\} = 0. \tag{e6.12}$$

It follows that

$$\{e_n^{1/m(n)} - e_n^{1/l(n)}\} \in I_{\overline{QT(A)}^w} \tag{e6.13}$$

for any integers $m(n) \ge l(n)$. Since

(e6.14)
$$\|e_n^{1/2l(n)} - (e_n^{1/2l(n)})^2\|^2_{2,\overline{QT(A)}^w} \le \sup\{d_\tau(e_n) - \tau(e_n^{l(n)}) : \tau \in \overline{QT(A)}^w\} \to 0,$$

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as $n \to \infty$, $\Pi(\{e_n^{1/2l(n)}\}) = p$ is a projection. By (e6.13), for any integers $m(n) \geq l(n)$,

$$\Pi(\{e_n^{1/m(n)}\}) = \Pi(\{e_n^{1/l(n)}\}) = (\Pi(\{e_n^{1/2l(n)}\}))^2 = p. \tag{e6.15}$$

It follows that $\{e_n^{1/l(n)}\}$ is a permanent projection lifting of p.

(4) Suppose that $g_\delta(e_n) \stackrel{c.}{\to} 0$ for some $0 < \delta < 1/4$. We have, for any $m(n) \in \mathbb{N}$,

$$e_n^{1/m(n)} - f_{\delta/2}(e_n)e_n^{1/m(n)} \lesssim g_\delta(e_n) \ \text{ for all } \ n \in \mathbb{N}. \tag{e6.16}$$

It follows that $\{e_n^{1/m(n)} - f_{\delta/2}(e_n)e_n^{1/m(n)}\} \in N_{cu}$. One then checks that

(e6.17)
$$p \leq \Pi_{cu}(\{e_n^{1/m(n)}\}) = \Pi_{cu}(\{f_{\delta/2}(e_n)e_n^{1/m(n)}\}) \leq \Pi_{cu}(\{f_{\delta/2}(e_n)\}) = f_{\delta/2}(\Pi_{cu}(\{e_n\})) = p.$$

Thus, (4) follows.

For (5), let $B = l^\infty(A)/I_{\overline{QT(A)}^w}$. It is clear that $pBp \subset \Pi(l^\infty(\{\mathrm{Her}(e_n)\}))$. Suppose that $g \in \Pi(l^\infty(\{\mathrm{Her}(e_n)\}))_+^{\mathbf{1}}$. Then we may write $g = \Pi(\{g_n\})$ such that $g_n \in (\mathrm{Her}(e_n))_+^{\mathbf{1}}$, $n \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $m(n) \in \mathbb{N}$ such that

$$\|e_n^{1/m(n)} g_n e_n^{1/m(n)} - g_n\| < \varepsilon \ \text{ for all } \ n \in \mathbb{N}. \tag{e6.18}$$

Thus,

$$\|\Pi(\{e_n^{1/m(n)}\}) g \Pi(\{e_n^{1/m(n)}\}) - g\| < \varepsilon. \tag{e6.19}$$

However, since $\{e_n\}$ is a permanent projection lifting of p, $\Pi(\{e_n^{1/m(n)}\}) = p$. Thus,

$$\|pgp - g\| < \varepsilon. \tag{e6.20}$$

It follows $g \in pBp$. This shows that $C = pBp = \Pi(l^\infty(\{\mathrm{Her}(e_n)\}))$.

(6) In this case, $\omega(e) = 0$. Therefore, by (3), $\{e^{1/l(n)}\}$ is a permanent projection lifting for $p = \Pi(\{e^{1/l(n)}\})$ (for some $l(n) \in \mathbb{N}$).

For any $\{x_n\} \in l^\infty(A)$, there is a sequence $\{m(n)\}$ of integers such that

$$\|x_n e^{1/m(n)} - x_n\| < 1/n \ \text{ and } \ \|e^{1/m(n)} x_n - x_n\| < 1/n, \ \ n \in \mathbb{N}. \tag{e6.21}$$

Hence $p\{x_n\} = \{x_n\}p = \{x_n\}$. So p is the unit of $l^\infty(A)/I_{\overline{QT(A)}^w}$.

For (7), suppose that A has continuous scale. Let $e \in A_+^{\mathbf{1}}$ be a strictly positive element. By (3) of Proposition 4.4, $\omega^c(e) = 0$. Then, for any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists an integer $l(n) \in \mathbb{N}$ such that

$$f_\varepsilon(e^{1/(m(n)} - e^{1/l(n)}) \lesssim g_{1/n}(e) \ \text{ for any } \ m(n) > l(n). \tag{e6.22}$$

Since $g_{1/n}(e) \stackrel{c.}{\to} 0$ (see Definition 4.3), we conclude that $\{e^{1/(m(n)} - e^{1/l(n)}\}$, $\{e^{1/2l(n)} - e^{1/l(n)}\} \in N_{cu}$. It follows that $p = \Pi(\{e^{1/l(n)}\})$ is a projection and $\{e^{1/l(n)}\}$ is a permanent projection lifting.

Let $\{b_n\} \in l^\infty(A)$. Then, for each $n \in \mathbb{N}$, there is $m(n) \in \mathbb{N}$ with $m(n) \geq l(n)$ such that $\|e^{1/m(n)} b_n - b_n\| < 1/n$. Recall that $p = \Pi(\{e^{1/m(n)}\})$. Thus, $p\Pi(\{b_n\}) = \Pi(\{b_n\})$. It follows $l^\infty(A)/N_{cu}$ is unital.

Conversely, let $p \in l^\infty(A)/N_{cu}$ be the unit. Let $\{e_n\} \in l^\infty(A)_+^{\mathbf{1}}$ such that $\Pi(\{e_n\}) = p$. We claim that $\Pi(A)^\perp = \{0\}$. Otherwise, for each n, there exists $b_n \in A_+$

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with $\|b_n\| = 1$ such that $\|e_n b_n\| < 1/n$. Then $p\Pi(\{b_n\}) = 0$. Impossible. Thus, $\Pi(A)^\perp = \{0\}$. By Proposition 2.20, A has continuous scale. ∎

Lemma 6.3 *Let A be a compact C^*-algebra with $QT(A) \neq \emptyset$. If A has T-tracial approximate oscillation zero, then, for any $x \in (l^\infty(A)/I_{\overline{QT(A)}^w})_+$, there is a projection $p \in l^\infty(A)/I_{\overline{QT(A)}^w}$ such that $px = x = xp$.*

Proof Let $B = l^\infty(A)/I_{\overline{QT(A)}^w}$ and $\Pi : l^\infty(A) \to B$ be the quotient map. Let $x \in B_+$. Without loss of generality, we may assume that $0 \le x \le 1$.

Let $y = \{y_n\} \in l^\infty(A)$ with $0 \le y \le 1$ such that $\Pi(y) = x$. Since A has T-tracial approximate oscillation zero, there are $d_n \in \mathrm{Her}(y_n)_+^{\mathbf{1}}$ and $\delta_n \in (0, 1/4n)$ such that

$$\|y_n - d_n\|_{2,\overline{QT(A)}^w} < 1/4n, \text{ and} \tag{e6.23}$$

$$d_\tau(d_n) - \tau(f_{\delta_n}(d_n)) < 1/4n \text{ for all } \tau \in \overline{QT(A)}^w \text{ and } n \in \mathbb{N}. \tag{e6.24}$$

Define $d = \{d_n\} \in l^\infty(A)$. Then $\Pi(d) = \Pi(y) = x$. Put $e_n = f_{\delta_n/2}(d_n)$, $n \in \mathbb{N}$ and $e = \{e_n\}$. Then $e \in l^\infty(A)_+^{\mathbf{1}}$. Moreover,

$$\lim_{n\to\infty} \|e_n d_n - d_n\| = 0. \tag{e6.25}$$

It follows that

$$\Pi(e)x = \Pi(e)\Pi(d) = \Pi(d) = x. \tag{e6.26}$$

It remains to show that $p := \Pi(e)$ is a projection. To do this, we compute that

$$\tau(e_n - e_n^2) \le \tau(e_n - f_{\delta_n}(d_n)) < d_\tau(d_n) - \tau(f_{\delta_n}(d_n)) < 1/n \text{ for all } n \in \mathbb{N}. \tag{e6.27}$$

It follows that

$$\|e_n - e_n^2\|_{2,\overline{QT(A)}^w} < 1/\sqrt{n} \to 0. \tag{e6.28}$$

Thus, $p = \Pi(e) = \Pi(e)^2$, or $p \in B$ is a projection. ∎

Theorem 6.4 *Let A be a compact C^*-algebra with non-empty $QT(A)$. If A has T-tracial approximate oscillation zero, then $l^\infty(A)/I_{\overline{QT(A)}^w}(A)$ has real rank zero.*

Proof Let $B = l^\infty(A)/I_{\overline{QT(A)}^w}(A)$ and $\Pi : l^\infty(A) \to B$ be the quotient map.

We claim that, if $p \in B$ is a nonzero projection, then $C := p\widetilde{B}p = pBp$ has real rank zero.

Let $\{e_n\} \in l^\infty(A)_+^{\mathbf{1}}$ such that $\Pi(\{e_n\}) = p$. Upon replacing e_n by $f_{1/4}(e_n)$, by Lemma 6.2, we may assume that $\{e_n\}$ is a permanent projection lifting of p. By (5) of Proposition 6.2, $C = pBp = \Pi(l^\infty(\{\mathrm{Her}(e_n)\}))$.

Let $a, b \in C_+$ be such that $ab = 0$. We may assume that $\|a\|, \|b\| \le 1$.

Then, by Proposition 10.1.10 of [28], for example, we may assume that $a = \Pi(\{a_n\})$ and $b = \Pi(\{b_n\})$, where $a_n, b_n \in \mathrm{Her}(e_n)_+$ and $\{a_n\}, \{b_n\} \in l^\infty(\{\mathrm{Her}(e_n)\})$ such that $a_n b_n = b_n a_n = 0$ for all $n \in \mathbb{N}$. Since A has T-tracial approximate oscillation zero, there are $d_n \in \mathrm{Her}(a_n)_+^{\mathbf{1}}$ and $\delta_n \in (0, 1/4n)$ such that

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$$\|a_n - d_n\|_{2,\overline{QT(A)}^w} < 1/4n \text{ and} \tag{e6.29}$$

$$|d_\tau(d_n) - \tau(f_{\delta_n}(d_n))| < 1/4n \text{ for all } \tau \in \overline{QT(A)}^w \text{ and } n \in \mathbb{N}. \tag{e6.30}$$

Define $d = \{d_n\} \in l^\infty(\{\mathrm{Her}(a_n)\})$. Then $a = \Pi(\{a_n\}) = \Pi(d)$. Put

$$g_n = f_{\delta_n/2}(d_n), \ n \in \mathbb{N} \text{ and } g' := \{g_n\} \in l^\infty(\{\mathrm{Her}(a_n)\}) \subset l^\infty(\{\mathrm{Her}(e_n)\}) \subset l^\infty(A). \tag{e6.31}$$

Put $g = \Pi(g')$. Recall that $l^\infty(\{\mathrm{Her}(e_n)\}) = pBp$. Thus, $g \in pBp_+$. Since $d_n \in \mathrm{Her}(a_n)$, we have $g_n b_n = b_n g_n = 0$. In other words,

$$gb = 0. \tag{e6.32}$$

Note that $f_{\delta_n}(d_n) g_n = f_{\delta_n}(d_n)$ for all $n \in \mathbb{N}$. It follows that

$$g_n^2 \geq f_{\delta_n}(d_n) \text{ for all } n \in \mathbb{N}. \tag{e6.33}$$

We compute that

$$\tau(g_n - g_n^2) \leq \tau(g_n - f_{\delta_n}(d_n)) < d_\tau(d_n) - \tau(f_{\delta_n}(d_n)) < 1/n \text{ for all } n \in \mathbb{N}. \tag{e6.34}$$

It follows that

$$\|g_n - g_n^2\|_{2,\overline{QT(A)}^w} < 1/\sqrt{n} \to 0. \tag{e6.35}$$

Thus, $g = g^2$, or $g \in pBp$ is a projection. Recall that

$$\lim_{n\to\infty} \|g_n d_n - d_n\| = \lim_{n\to\infty} \|f_{\delta_n/2}(d_n) d_n - d_n\| = 0. \tag{e6.36}$$

In other words,

$$ga = g\Pi(d) = \Pi(d) = a. \tag{e6.37}$$

This and (e6.32) imply that pBp has real rank zero and the claim is proved.

To show that B has real rank zero, let $x \in B_{s.a.}$ and let $\varepsilon > 0$. Put $z = x^2$. Then, by Lemma 6.3, there is a projection $p \in B$ such that $pz = z = zp$. Hence, $x \in pBp$. By the claim that we have just shown, pBp has real rank zero. Then there is $y \in (pBp)_{s.a.}$ with finite spectrum such that $\|x - y\| < \varepsilon$. By Theorem 2.9 of [6], B has real rank zero. ■

If A is unital, then $l^\infty(A)$ is the multiplier algebra of $c_0(A)$. Thus, $l^\infty(A)/c_0(A)$ is a corona algebra. Therefore, $l^\infty(A)/I_{\overline{QT(A)}^w}$ is an SSAW*-algebra (see Proposition 3 of [31] and Section 3 of [33]). The above theorem also implies that, for non-unital case, we also have the next corollary. (One may also compare Corollaries 6.5 and 6.7 (at least in unital case) with those of Lemma 1.8, Theorem 1.9, and Corollary 1.10 of [21].)

Corollary 6.5 *Let A be a σ-unital compact C^*-algebra with $QT(A) \neq \emptyset$. Suppose that A has T-tracial approximate oscillation zero. Then $l^\infty(A)/I_{\overline{QT(A)}^w}$ is an SSAW*-algebra with real rank zero.*

Proof This is contained in the proof of Lemma 6.4. Since $M_n(A)$ also has T-tracial approximate oscillation zero, it suffices to show that $B := l^\infty(A)/I_{\overline{QT(A)}^w}$ is an

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SAW*-algebra. Consider elements $a, b \in B_+$ such that $ab = ba = 0$. By Lemma 6.4, there is a projection $p \in B$ such that

$$p(a+b) = (a+b)p = a+b.$$

Then the first part of the proof of Theorem 6.4 provides a projection $g \in B$ such that $ga = a$ and $gb = 0$. Consequently, B is an SAW*-algebra of real rank zero. ∎

Theorem 6.6 *Let A be an algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. Suppose that A has strict comparison and T-tracial approximate oscillation zero. Then $l^\infty(A)/I_{\overline{QT(A)}^w}$ has stable rank one.*

Proof Put $B = l^\infty(A)/I_{\overline{QT(A)}^w}$. We first show that pBp has stable rank one if p is a nonzero projection. By Theorem 6.4, pBp has real rank zero. Let $q, f \in pBp$ be projections such that $q \sim f$. Then, by Theorem 3.5, $p - q \sim p - f$. By Proposition 2.4(III) of [2] and Theorem 2.6 of [6], pBp has stable rank one.

To show B has stable rank one, let $x \in \widetilde{B}$ and $\varepsilon > 0$. Write $x = \lambda + y$, where $\lambda \in \mathbb{C}$ and $y \in B$. By Lemma 6.3, there is a projection $p \in B$ such that $p(y^*y + yy^*) = (y^*y + yy^*)$. It follows that $y \in pBp$. From what has been shown, pBp has stable rank one. Choose $z_1 \in GL(pBp)$ such that

$$\|\lambda p + y - z_1\| < \varepsilon. \tag{e6.38}$$

Define $z_2 = z_1 + \lambda(1-p)$, if $\lambda \neq 0$, and $z_2 = z_1 + \varepsilon(1-p)$, if $\lambda = 0$. Then $z_2 \in GL(\widetilde{B})$ and

$$\|x - z_2\| < \varepsilon.$$ ∎

Corollary 6.7 *Let A be an algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. Suppose that A has strict comparison and T-tracial approximate oscillation zero. Then $l^\infty(A)/I_{\overline{QT(A)}^w}$ has unitary polar decomposition.*

Proof This follows from Corollary 6.5 above and Theorem 3.5 of [33]. ∎

7 Range of dimension functions

In this section, we will show that if A has T-tracial approximate oscillation zero, then the image of Γ is "dense," and if, in addition, A has strict comparison, then Γ is surjective.

Lemma 7.1 *Let $0 \le a \le 1$, $b, c \in A^{\mathbf{1}}$ be such that*

$$a \le (b+c)^*(b+c) \tag{e7.1}$$

(or $a \le b + c$, $b, c \in A^{\mathbf{1}}_+$). Then, for any $\delta \in (0, 1/2)$,

$$[f_\delta(a)] \le [f_{\delta/4}(b^*b)] + [f_{\delta/4}(c^*c)] \tag{e7.2}$$

(or $[f_\delta(a)] \le [f_{\delta/4}(b)] + [f_{\delta/4}(c)]$).

Proof Note that

$$(b+c)^*(b+c) \le 2(b^*b + c^*c). \tag{e7.3}$$

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Let $0 < \eta < 1/4$. Then, by Lemma 1.7 of [34],

(e7.4) $$((b+c)^*(b+c)-\delta/2)_+ \lesssim (2(b^*b+c^*c)-\delta/2)_+ \lesssim f_{\delta/2}(b^*b+c^*c)$$

(recall Notation 2.5). Put $z = \begin{pmatrix} b & 0 \\ c & 0 \end{pmatrix}$. Then

$$z^*z = \mathrm{diag}(b^*b+c^*c, 0) \text{ and } zz^* \le 2\mathrm{diag}(bb^*, cc^*).$$

Hence (see Lemma 1.7 of [34], for example, for the first "$\lesssim$" sign below),

(e7.5) $$f_{\delta/2}(b^*b+c^*c) \sim f_{\delta/2}(zz^*) \sim (zz^*-\delta/4)_+$$

(e7.6) $$\lesssim \mathrm{diag}(f_{\delta/4}(bb^*), f_{\delta/4}(cc^*)) \sim \mathrm{diag}(f_{\delta/4}(b^*b), f_{\delta/4}(c^*c)).$$

We then have (see also (e7.4))

(e7.7)
$$f_\delta(a) \sim (a-\delta/2)_+ \lesssim ((b+c)^*(b+c)-\delta/2)_+ \lesssim \mathrm{diag}(f_{\delta/4}(b^*b), f_{\delta/4}(c^*c)).$$

For the case that $a \le b+c$, as computed above, there is $z \in M_2(A)$ such that

(e7.8) $$z^*z = \mathrm{diag}(b+c, 0) \text{ and } zz^* \le 2\mathrm{diag}(b,c).$$

One then sees the proof of the second part is exactly the same as that of the first part. ∎

Lemma 7.2 *Let A be a non-elementary simple C^*-algebra with $\mathrm{Ped}(A) = A$ and with $QT(A) \ne \emptyset$. Let $\{e_n\}, \{b_n\} \in l^\infty(\mathrm{Ped}(A)_+^{\mathbf{1}})$. Recall that $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w}$ is the quotient map.*

(1) *Suppose that $\Pi(\{e_n\}) \le \Pi(\{b_n\})$ (or $\Pi_{cu}(\{e_n\}) \le \Pi_{cu}(\{b_n\})$). Then, any integer $m \in \mathbb{N}$ and $\varepsilon > 0$, (any $d \in \mathrm{Ped}(A)_+^{\mathbf{1}}\backslash\{0\}$), there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$,*

(e7.9) $$[f_\varepsilon(e_k)] \le [b_k] + [d_k],$$

where $d_k \in A_+$ and $\sup\{d_\tau(d_k) : \tau \in \overline{QT(A)}^w\} < 1/m$ (or $d_k \lesssim d$ and $\sup\{d_\tau(d_k) : \tau \in \overline{QT(A)}^w\} < 1/m$).

(2) *Suppose that $p = \Pi(\{e_n\})$ is a projection (or $p = \Pi_{cu}(\{e_n\})$ is a projection and $\omega^c(e_n) \to 0$), $\{e_n\}$ is a permanent projection lifting of p and*

(e7.10) $$p \le \Pi(\{b_n\}) \quad (\text{or } p \le \Pi_{cu}(\{b_n\})).$$

Then, any integer $m \in \mathbb{N}$, and $\varepsilon \in (0,1)$ (and any nonzero $d \in \mathrm{Ped}(A)_+$), there exists $k_0 \ge \mathbb{N}$ such that, for all $k \ge k_0$,

(e7.11) $$[e_k] \le [b_k] + [d_k],$$

where $0 \le d_k \le 1$, $d_k \in (A \otimes \mathcal{K})_+$ and

(e7.12) $$d_\tau(e_k) < d_\tau(b_k) + d_\tau(d_k) + \varepsilon \text{ for all } \tau \in \overline{QT(A)}^w,$$

where $\sup\{d_\tau(d_k) : \tau \in \overline{QT(A)}^w\} < 1/m$ (or, $d_k \lesssim d$ and $\sup\{d_\tau(d_k) : \tau \in \overline{QT(A)}^w\} < 1/m$).

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Proof We will use the following easy claim: If B is a C^*-algebra and $I \subset B$ is a (closed two-sided) ideal, and, if $x, y \in B_+$ such that $\pi(x) \le \pi(y)$, then there exists $j \in I_+$ such that $x \le y + j$. In fact, there is $z \in A_+$ such that $(y - x) - z \in I$. Put $c = -y + x + z \in I$. Then

$$x = y - z + c \le y + c \le y + c_+.$$

Choose $j = c_+ \in I_+$. This proves the claim.

Since A is a non-elementary simple C^*-algebra, one may choose $d_0 \in \mathrm{Her}(d)_+^{\mathbf{1}} \setminus \{0\}$ such that $2(m+1)[d_0] \le [d]$. By the easy claim above, there is $\{h_n\} \in (I_{\overline{QT(A)}^w})_+^{\mathbf{1}}$ (or $\{h_n\} \in N_{cu}(A)_+^{\mathbf{1}}$) such that, in all cases,

$$b_n + h_n \ge e_n \ \text{ for all } \ n \in \mathbb{N}. \tag{e7.13}$$

To show (1), we apply Lemma 7.1 to obtain

$$f_\varepsilon(e_n) \lesssim \mathrm{diag}(f_{\varepsilon/8}(b_n), f_{\varepsilon/8}(h_n)). \tag{e7.14}$$

Since h_n is in $(I_{\overline{QT(A)}^w})_+$ (or in $(N_{cu})_+$), there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$,

$$d_\tau(f_{\varepsilon/8}(h_k)) < 1/m \ \text{ for all } \ \tau \in \overline{QT(A)}^w \tag{e7.15}$$

(or $f_{\varepsilon/8}((h_k)_+) \lesssim d_0$). Therefore, with $d_k = f_{\varepsilon/8}(h_k)$, for all $k \ge k_0$,

$$[f_\varepsilon(e_k)] \le [b_k] + [d_k] \ \text{ and } \ \sup\{d_\tau(d_k) : \tau \in \overline{QT(A)}^w\} < 1/m. \tag{e7.16}$$

Part (1) follows.

For (2), we keep the same d_k and h_k as described above. Since now $\{e_n\}$ is a permanent projection lifting, we may assume that $\lim_{n\to\infty} \omega(e_n) = 0$, by (2) of Proposition 6.2. Thus, we may assume that there exists $\eta_0 > 0$ and $n_1 \in \mathbb{N}$ such that

$$\sup\{d_\tau(g_\eta(e_n)) : \tau \in \overline{QT(A)}^w\} \le \sup\{d_\tau(g_{\eta_0}(e_n)) : \tau \in \overline{QT(A)}^w\} < \varepsilon/2m \tag{e7.17}$$

(see Notation 2.5) for all $n \ge n_0$ and $0 < \eta \le \eta_0$ (or we assume that $\omega^c(e_n) \to 0$, and then

$$g_\eta(e_n) \lesssim g_{\eta_0}(e_n) \lesssim d_0 \ \text{ for all } \ n \ge n_0 \ \text{ and } \ 0 < \eta \le \eta_0 \tag{e7.18}$$

for all $n \ge n_0$ and $0 < \eta \le \eta_0$).

There exists $n_2 \in \mathbb{N}$ such that (recall that $h_n \in (I_{\overline{QT(A)}^w})_+^{\mathbf{1}}$) (or $h_n \in (N_{cu})_+^{\mathbf{1}}$)

$$d_\tau(f_{\eta_0/8}((h_n))) < \varepsilon/2m \ \text{ for all } \ \tau \in \overline{QT(A)}^w \ \text{ and } \ n \ge n_2 \tag{e7.19}$$

(or $f_{\eta_0/8}((h_n)) \lesssim d_0$ for $n \ge n_2$). We have, by Lemma 7.1,

$$f_{\eta_0/2}(e_n) \lesssim \mathrm{diag}(f_{\eta_0/8}(b_n), f_{\eta_0/8}((h_n))). \tag{e7.20}$$

Put $k_0 = \max\{n_1, n_2\}$. Then, if $n \ge k_0$, we have

$$[e_n] \le [g_{\eta_0}(e_n)] + [f_{\eta_0/2}(e_n)] \le [g_{\eta_0}(e_n)] + [f_{\eta_0/8}(b_n)] + [f_{\eta_0/8}((h_n))] \tag{e7.21}$$

$$\le [b_n] + [d_n], \tag{e7.22}$$

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where $d_n = \mathrm{diag}(g_{\eta_0}(e_n), f_{\eta_0/8}((h_n))), n \in \mathbb{N}$. Recall (by (e7.17) and (e7.19)) that $d_\tau(d_n) < 1/m$ for all $\tau \in \overline{QT(A)}^w$ (or, in the second case, $[d_k] \le 2[d_0] \le [d]$). Part (2) then follows. ∎

Lemma 7.3 *Let A be a non-elementary algebraically simple C^*-algebra with $QT(A) \neq \emptyset$.*

(1) *Suppose that A has tracial approximate oscillation zero. Then, for any $a \in A_+^{\mathbf{1}}\backslash\{0\}$, there exists a sequence $0 \le e_n \le 1$ in* $\mathrm{Her}(a)$ *such that $\Pi_{cu}(\{e_n\})$ is full in $l^\infty(A)/N_{cu}(A)$ and*

$$\lim_{n\to\infty} \omega(e_n) = 0. \tag{e7.23}$$

(2) *Suppose that A has T-tracial approximate oscillation zero. Then, for any $a \in A_+^{\mathbf{1}}\backslash\{0\}$, there exists a sequence $0 \le e_n \le 1$ in* $\mathrm{Her}(a)$ *such that $\Pi_{cu}(\{e_n\})$ is full in $l^\infty(A)/N_{cu}(A)$ and*

$$\lim_{n\to\infty} \omega(e_n) = 0. \tag{e7.24}$$

Proof (1) Since A has tracial approximate oscillation zero, by Proposition 5.6, there exists a tracial approximate identity $\{e_n\}$ for $\mathrm{Her}(a)$ (with $\|e_n\| \le 1$) such that $\lim_{n\to\infty} \omega(e_n) = 0$. Note that

$$\Pi_{cu}(\iota(a)) = \Pi_{cu}(\iota(a^{1/2})\{e_n\}\iota(a^{1/2})). \tag{e7.25}$$

Since $A = \mathrm{Ped}(A)$, by Proposition 5.6 of [10], $\iota(a)$ is full in $l^\infty(A)$. Hence, $\Pi_{cu}(\iota(a))$ is full in $l^\infty(A)/N_{cu}(A)$, and so is $\Pi(\{e_n\})$.

The proof of (2) is exactly the same. We omit it. ∎

Lemma 7.4 *Let A be a non-elementary algebraically simple C^*-algebra with $QT(A) \neq \emptyset$. Suppose that A has T-tracial approximate oscillation zero. Then, for any $n \in \mathbb{N}$, there is a full projection $p \in l^\infty(A)/I_{\overline{QT(A)}^w}$ with $p = \Pi(\{e_j\})$ for some $e_j \in A_+^{\mathbf{1}}$ ($j \in \mathbb{N}$) such that*

$$\sup\{d_\tau(e_j) : \tau \in \overline{QT(A)}^w\} < 1/(n+1) \ \text{for all} \ j \in \mathbb{N}, \tag{e7.26}$$

and there is a sequence of mutually orthogonal full projections $p_1, p_2, \ldots, p_k, \ldots$ in $l^\infty(A)/I_{\overline{QT(A)}^w}$ such that $pp_j = 0, j \in \mathbb{N}$ and

$$2^{2k}[p_k] \le [p], \ k \in \mathbb{N}. \tag{e7.27}$$

Moreover, for each $k \in \mathbb{N}$, there are mutually orthogonal and mutually equivalent full projections $p_{k,1}, p_{k,2}, \ldots, p_{k,2^{k+1}}$ in $p_k(l^\infty(A)/I_{\overline{QT(A)}^w})p_k$.

Proof Fix $n \in \mathbb{N}$. Since A is a non-elementary simple C^*-algebra, we may choose two mutually orthogonal elements $a_1, a_2 \in \mathrm{Ped}(A)_+^{\mathbf{1}}\backslash\{0\}$ and $x \in A_+$ such that

$$x^*x = a_1, \ xx^* = a_2 \ \text{and} \ \sup\{d_\tau(a_1) : \tau \in \overline{QT(A)}^w\} < 1/(n+1). \tag{e7.28}$$

Find four mutually orthogonal and mutually Cuntz equivalent elements $a_{2,1}, \ldots, a_{2,4} \in \mathrm{Her}(a_2)_+^{\mathbf{1}}\backslash\{0\}$. By Lemma 7.3, there exists a sequence $\{e_n'\}$ in $\mathrm{Her}(a_1)^{\mathbf{1}}$ such that

$\Pi(\{e'_n\})$ is full and $\lim_{n\to\infty}\omega(e'_n)=0$. There exists, by (3) of Lemma 6.2, a sequence of integers $m(j)$ such that $p=\Pi(\{e'_j)^{1/m(j)}\})$ is a full projection. Note that

$$\sup\{\tau(e_j):\tau\in\overline{QT(A)}^w\}<1/(n+1), \tag{e7.29}$$

where $e_j=(e'_j)^{1/m(j)}$, $j\in\mathbb{N}$.

In $B=\mathrm{Her}(a_{2,1})$, one finds a sequence of mutually orthogonal nonzero positive elements $\{y_n\}$ such that

$$2^{2k}[y_{k+1}]\le[y_k],\ \ k\in\mathbb{N}. \tag{e7.30}$$

To see this, choose mutually orthogonal nonzero elements $b_1,b'_1\in B_+^{\mathbf{1}}$ and $c_1\in B$ such that $c_1^*c_1=b_1$ and $c_1c_1^*=b'_1$. Choose $y_1=b_1$. There are mutually orthogonal and mutually Cuntz equivalent nonzero elements $b_{2,i}\in\mathrm{Her}(b'_1)$ $(1\le i\le 2^2)$. Choose $y_2=b_{2,1}$. We then proceed to divide $b_{2,4}$. A standard induction argument produces the desired sequence $\{y_k\}$.

For each k, there are 2^{k+1} mutually orthogonal nonzero elements $y_{k,1},y_{k,2},\dots,y_{k,2^{k+1}}$ in $\mathrm{Her}(y_k)$ and elements $z_{k,1},z_{k,2},\dots,z_{k,2^{k+1}}$ in $\mathrm{Her}(y_k)$ such that

$$z_{k,j}^*z_{k,j}=y_{k,1}\ \text{ and }\ z_{k,j}z_{k,j}^*=y_{k,j},\ \ 1\le j\le 2^{k+1}. \tag{e7.31}$$

Applying Lemma 7.3, one obtains, for each $k\in\mathbb{N}$, a sequence $\{e_{k,1,n}\}\subset\mathrm{Her}(y_{k,1})_+^{\mathbf{1}}$ such that $\Pi(\{e_{k,1,n}\}_{n\in\mathbb{N}})$ is full in $l^\infty(A)/I_{\overline{QT(A)}^w}$ and $\lim_{n\to\infty}\omega(e_{k,1,n})=0$.

Since $\lim_{n\to\infty}\omega(e_{k,1,n})=0$, by (3) of Lemma 6.2, there is also, for each k, a sequence $m(n,k)\in\mathbb{N}$ such that

$$p_{k,1}:=\Pi(\{e_{k,1,n}^{1/m(n,k)}\}_{n\in\mathbb{N}}) \tag{e7.32}$$

is a full projection in $l^\infty(A)/I_{S,0}$. Write $z_{k,j}=u_{k,j}|z_{k,j}|$ as polar decomposition of $z_{k,j}$ in A^{**}, $(1\le j\le 2^{k+1})$. Put

$$v_{k,j}=\Pi(\{u_{k,j}e_{k,1,n}^{1/m(n,k)}\})\ \text{ and }\ p_{k,j}=v_{k,j}v_{k,j}^*,\ \ 1\le j\le 2^{k+1}. \tag{e7.33}$$

Then $v_{k,j}^*v_{k,j}=p_{k,1}$ (see (e7.31)). Thus, we obtain mutually orthogonal and mutually equivalent full projections $p_{k,j}$, $j=1,2,\dots,2^{k+1}$. By the construction, we also have $p_{k,j}p_{k',j'}=0$, if $k\ne k'$, as well as $pp_{k,j}=0$ for all $k,j\in N$. Put $p_k:=\sum_{j=1}^{2^{k+1}}p_{k,j}$. Note also

$$2^{2k}[p_k]\le[p].$$ ∎

The following two lemmas are variations of S. Zhang's halving projection lemma. We need some modification as we do not assume the C^*-algebra is simple.

Lemma 7.5 (Zhang [48, Theorem I(i)]) *Let A be a C^*-algebra of real rank zero and r a full projection of A. Suppose that $p\in A$ is a nonzero projection such that $[p]\not\le[r]$. Then, there are mutually orthogonal projections $p_1,p_2,p_3\in A$ such that*

$$p=p_1+p_2+p_3,\ \ p_1\sim p_2,\ \ \text{and}\ \ p_3\lesssim r. \tag{e7.34}$$

Proof We begin with the following claim which is extracted from the proof of [48, Theorem I(i)].

Claim: Let C be a C^*-algebra, and let $v_1, \ldots, v_{2^m} \in C$ be partial isometries such that $v_i v_i^* \perp v_j v_j^*$ $(i \neq j)$ and $v_i^* v_i \geq v_{i+1}^* v_{i+1}$ $(1 \leq i \leq 2^m - 1)$. Then there is a partial isometry $v \in C$ such that $v^* v \perp v v^*$ and $0 \leq \sum_{i=1}^{2^m} v_i v_i^* - (v^* v + v v^*) \leq v_1 v_1^*$.

Proof of the claim: We use induction on m. When $m = 1$, let $v := v_1 v_2^*$. Then $v^* v \perp v v^*$ because $v_1 v_1^* \perp v_2 v_2^*$, and $v^* v = v_2 v_1^* v_1 v_2^* = v_2 v_2^*$ because $v_1^* v_1 \geq v_2^* v_2$. Note that $v v^* = v_1 v_2^* v_2 v_1^*$. Thus,

$$v_1 v_1^* + v_2 v_2^* - (v^* v + v v^*) = v_1 v_1^* - v_1 v_2^* v_2 v_1^* = v_1 (1 - v_2^* v_2) v_1^* \leq v_1 v_1^*. \tag{e7.35}$$

The last equation above also shows that $v_1 v_1^* + v_2 v_2^* - (v^* v + v v^*)$ is positive. Hence, the claim holds for $m = 1$.

Assume that the claim holds for $m \geq 1$. Let $v_1, \ldots, v_{2^{m+1}}$ be partial isometries such that $v_i v_i^* \perp v_j v_j^*$ $(i \neq j)$ and $v_i^* v_i \geq v_{i+1}^* v_{i+1}$ $(1 \leq i \leq 2^{m+1} - 1)$. Define $e_i := v_i^* v_i$ and $w_i = v_i (e_i - e_{2^{m+1} - i + 1})$, $i = 1, \ldots, 2^m$. For $i = 1, \ldots, 2^m - 1$, we have

$$w_i^* w_i = (e_i - e_{2^{m+1} - i + 1}) e_i (e_i - e_{2^{m+1} - i + 1}) \tag{e7.36}$$

$$= e_i - e_{2^{m+1} - i + 1} \geq e_{i+1} - e_{2^{m+1} - i} = w_{i+1}^* w_{i+1}. \tag{e7.37}$$

Note that the above also shows that $w_i^* w_i$ are projections for all i. Since $w_i w_i^* \leq v_i v_i^*$, we have that $w_i w_i^*$ are mutually orthogonal.

Consider $w_i, 1 \leq i \leq 2^m$. By induction, there is a partial isometry $w \in C$ such that $w^* w \perp w w^*$ and

$$0 \leq \sum_{i=1}^{2^m} w_i w_i^* - (w^* w + w w^*) \leq w_1 w_1^* \ (\leq v_1 v_1^*). \tag{e7.38}$$

Hence, $ww = 0$. Note that

(e7.39)

$$w^* w + w w^* \leq \sum_{i=1}^{2^m} w_i w_i^* = \sum_{i=1}^{2^m} v_i (e_i - e_{2^{m+1} - i + 1}) v_i^*$$
$$= \sum_{i=1}^{2^m} v_i v_i^* - v_i e_{2^{m+1} - i + 1} v_i^* \in \left(\sum_{i=1}^{2^m} v_i e_{2^{m+1} - i + 1} v_i^* \right)^{\perp} \cap \left(\sum_{i=1}^{2^m} v_{2^{m+1} - i + 1} v_{2^{m+1} - i + 1}^* \right)^{\perp},$$

where $b^{\perp} = \{a \in A : ab = ba = 0\}$. Recall that $v_{2^{m+1} - i + 1}^* v_{2^{m+1} - i + 1} = e_{2^{m+1} - i + 1}$ $(1 \leq i \leq 2^m)$ and $v_i^* v_i \geq v_{i+1}^* v_{i+1}$ $(1 \leq i \leq 2^{m+1} - 1)$. Hence,

$$\sum_{i=1}^{2^m} v_i e_{2^{m+1} - i + 1} v_i^* = \sum_{i=1}^{2^m} v_i v_{2^{m+1} - i + 1}^* (v_i v_{2^{m+1} - i + 1}^*)^* \text{ and} \tag{e7.40}$$

$$\sum_{i=1}^{2^m} v_{2^{m+1} - i + 1} v_{2^{m+1} - i + 1}^* = \sum_{i=1}^{2^m} (v_i v_{2^{m+1} - i + 1}^*)^* v_i v_{2^{m+1} - i + 1}^*. \tag{e7.41}$$

Thus,

$$w w^* + w^* w \in \left(\sum_{i=1}^{2^m} v_i v_{2^{m+1} - i + 1}^* (v_i v_{2^{m+1} - i + 1}^*)^* \right)^{\perp} \cap \left(\sum_{i=1}^{2^m} (v_i v_{2^{m+1} - i + 1}^*)^* v_i v_{2^{m+1} - i + 1}^* \right)^{\perp}.$$

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It follows that, for $1 \le i \le 2^m$,

$$w \perp v_i v^*_{2^{m+1}-i+1}. \tag{e7.42}$$

Define $v := w + \sum_{i=1}^{2^m} v_i v^*_{2^{m+1}-i+1}$. By (e7.42) and the fact that $w^2 = 0$ together with $v_i v_i^* \perp v_j v_j^*$ $(i \neq j)$, we compute that

$$v^2 = \left(w + \sum_{i=1}^{2^m} v_i v^*_{2^{m+1}-i+1}\right)\left(w + \sum_{i=1}^{2^m} v_i v^*_{2^{m+1}-i+1}\right) = 0, \tag{e7.43}$$

$$v^*v = w^*w + \sum_{i=1}^{2^m} v_{2^{m+1}-i+1} v_i^* v_i v^*_{2^{m+1}-i+1} = w^*w + \sum_{i=1}^{2^m} v_{2^{m+1}-i+1} v^*_{2^{m+1}-i+1}, \quad \text{and} \tag{e7.44}$$

$$vv^* = ww^* + \sum_{i=1}^{2^m} v_i v^*_{2^{m+1}-i+1} v_{2^{m+1}-i+1} v_i^* = ww^* + \sum_{i=1}^{2^m} v_i e_{2^{m+1}-i+1} v_i^*. \tag{e7.45}$$

Then

$$\sum_{i=1}^{2^{m+1}} v_i v_i^* - (v^*v + vv^*) = \sum_{i=1}^{2^m} v_i v_i^* - \left(w^*w + ww^* + \sum_{i=1}^{2^m} v_i e_{2^{m+1}-i+1} v_i^*\right) \tag{e7.46}$$

$$= \sum_{i=1}^{2^m} (v_i v_i^* - v_i e_{2^{m+1}-i+1} v_i^*) - (w^*w + ww^*) \tag{e7.47}$$

$$\overset{(7.39)}{=} \sum_{i=1}^{2^m} w_i w_i^* - (w^*w + ww^*) \tag{e7.48}$$

$$\overset{(7.38)}{\le} w_1 w_1^* \le v_1 v_1^*. \tag{e7.49}$$

By induction, the claim holds for all $m \in \mathbb{N}$.

For the proof of the lemma, applying Lemma 1.1 of [47], we obtain partial isometries $v_1, v_2, \dots, v_n \in A$ such that

$$r \ge v_1^* v_1 \ge v_2^* v_2 \ge \cdots v_n^* v_n \quad \text{and} \tag{e7.50}$$

$$p = v_1 v_1^* \oplus v_2 v_2^* \oplus \cdots \oplus v_n v_n^*. \tag{e7.51}$$

Since $[p] \nleq [r]$, $n \ge 2$. By adding 0 if necessary, we may assume that $n = 2^m$ for some $m \in \mathbb{N}$. Then, by (e7.50), (e7.51), and the claim, there is a partial isometry $v \in A$ such that $v^*v \perp vv^*$, $0 \le p - (v^*v + vv^*) \le v_1 v_1^* \lesssim r$. Then the lemma holds (by choosing $p_1 = vv^*$ and $p_2 = v^*v$). ∎

Lemma 7.6 (S. Zhang) *Let A be a C^*-algebra of real rank zero and r be a full projection of A such that $B = (1-r)A(1-r)$ contains, for each $k \in \mathbb{N}$, a sequence of mutually orthogonal full projections $\{r'_{n,j} : 1 \le j \le 2^{n+1}, n \in \mathbb{N}\}$ such that $2^{k+n}[r'_n] \le [r]$, where $r'_n = \sum_{j=1}^{2^{n+1}} r'_{n,j}$, and $r'_{n,1}, r'_{n,2}, \dots, r'_{n,2^{n+1}}$ are mutually equivalent $(n \in \mathbb{N})$. Suppose that $p \in A$ is a nonzero projection such that $[p] \nleq [r]$. Then, for any $m \in \mathbb{N}$, there are mutually orthogonal projections $p_1, p_2, \dots, p_{2^m}, p_{2^m+1} \in A$ such that*

$$p = \sum_{j=1}^{2^m} p_j + p_{2^m+1}, \quad p_j \sim p_1, 1 \le j \le 2^m, \quad \text{and} \quad p_{2^m+1} \lesssim r + r', \tag{e7.52}$$

where r' is a finite sum of $r'_{n,j}$s and $2[r'] < r$.

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Proof We use the induction on m. If $m = 1$, the lemma follows from Lemma 7.5.

Suppose that the lemma holds for $m \geq 1$. Then there are mutually orthogonal projections $p_1, p_2, \ldots, p_{2^m}, p_{2^m+1} \in A$ such that

$$p = \sum_{j=1}^{2^m} p_j + p_{2^m+1}, \ \ p_j \sim p_1, 1 \leq j \leq 2^m, \ \text{ and } \ p_{2^m+1} \lesssim r + r'', \tag{e7.53}$$

where r'' is a finite sum of $r'_{n,i}$s and $2[r''] < [r]$. Choose $r'_{K,1}$ among $\{r'_{n,j}\}$ but not those which have been used for the sum of r''. We choose K such that $K > 2m$. Note that

$$2[r''] + 2^{K+1}[r'_{K,1}] < [r].$$

We also note that $[p_1] \nleq [r'_{K,1}]$. (Otherwise, $[p] \leq 2^m[r'_{K,1}] \leq 2^K[r_{K,1}] \leq [r]$, a contradiction.) Applying Lemma 7.5 to p_1 (as p) and the full projection $r'_{K,1}$, we may write

$$p_1 = p_{1,1} + p_{1,2} + p_{1,3}, \ \ p_{1,1} \sim p_{1,2} \ \text{ and } \ p_{1,3} \lesssim r'_{K,1}. \tag{e7.54}$$

Since $p_j \sim p_1$, we also have mutually orthogonal projections $p_{j,1}, p_{j,2}, p_{j,3}$ such that

$$p_j = p_{j,1} + p_{j,2} + p_{j,3}, \ \ p_{j,2} \sim p_{j,1} \sim p_{1,1} \ \text{ and } \ p_{j,3} \lesssim r'_{K,1}. \tag{e7.55}$$

Note that

$$p_{j,i} \sim p_{j',i'}, \ \ i, i' = 1, 2, \ \ j = 1, 2, \ldots., 2^m. \tag{e7.56}$$

Put $s := \sum_{j=1}^{2^m} p_{j,3}$. Then

$$s \lesssim r + r'' + \sum_{j=1}^{2^m} r'_{K,j}, \ \ 2\left[r'' + \sum_{j=1}^{2^m} r'_{K,j}\right] \leq [r] \ \text{ and } \ p = \sum_{j=1}^{2^m}(p_{j,1} + p_{j,2}) + s. \tag{e7.57}$$

This completes the induction. ∎

Recall that, if $C_n \subset A$ ($n \in \mathbb{N}$), then $l^\infty(\{C_n\}) = \{\{c_n\} \in l^\infty(A) : c_n \in C_n\}$.

Lemma 7.7 *Let A be a non-elementary, σ-unital, and algebraically simple C^*-algebra with $QT(A) \neq \emptyset$ and with T-tracial approximate oscillation zero. Let $\{e_k\}$ be a sequence in $A_+^{\mathbf{1}} \setminus \{0\}$ such that $\{e_k\} \notin I_{\overline{QT(A)}^w}$, and $\{e_k\}$ is a permanent projection lifting of the projection $p = \Pi(\{e_k\})$, and let $n > 1$ be an integer and $\varepsilon > 0$.*

Then, there are mutually orthogonal and mutually (Cuntz) equivalent elements

$$\{f_{k,1}\}, \{f_{k,2}\}, \ldots, \{f_{k,n}\} \in l^\infty(\{\mathrm{Her}(e_k)\}) \ \textit{such that}$$

$$\lim_{k\to\infty} \omega(f_{k,j}) = 0, \ 1 \leq j \leq n, \ \textit{and}, \tag{e7.58}$$

for $k \geq k_0$ (for some k_0),

$$\sup\{|d_\tau(e_k) - n d_\tau(f_{k,1})| : \tau \in \overline{QT(A)}^w\} < \varepsilon. \tag{e7.59}$$

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Proof Let $m \ge 1$ be an integer, and let $\varepsilon > 0$. Put $B = l^\infty(A)/I_{\overline{QT(A)}^w}$. Since A has T-tracial oscillation zero, by Theorem 6.4, B has real rank zero. Since $\{e_k\} \notin I_{\overline{QT(A)}^w}$,

$$\sigma_0 := \limsup_k (\sup\{\tau(e_k) : \tau \in \overline{QT(A)}^w\}) > 0. \tag{e7.60}$$

If $r = \Pi(\{r_n\})$ (is full) and $r_n \in A_+^{\mathbf{1}}$, and

$$\sup\{\tau(r_n) : \tau \in \overline{QT(A)}^w\} < \sigma_0/4 \text{ for all } n \in \mathbb{N}, \tag{e7.61}$$

then $[\Pi(\{e_k\})] \not\le [r]$. It then follows from Lemmas 7.4 and 7.6 that, there are $2^m + 1$ mutually orthogonal projections $p_1, p_2, ..., p_{2^m}, s \in B$ such that s is full, $s = \Pi(\{s_n\})$, where $s_n \in A_+^{\mathbf{1}}$,

$$p = \sum_{i=1}^{2^m} p_i + s, \ \ p_1 \sim p_j, 1 \le j \le 2^m \text{ and} \tag{e7.62}$$

$$\sup\{d_\tau(s_n) : \tau \in \overline{QT(A)}^w\} < \varepsilon/2. \tag{e7.63}$$

Recall that $\{e_k\}$ is a permanent projection lifting of p. Then, by (5) of Proposition 6.2, $pBp = l^\infty(\{\mathrm{Her}(e_k)\})/I_{\overline{QT(A)}^w}$. Define a homomorphism $\phi : C_0((0,1]) \otimes M_{2^m} \to pBp$ such that

$$\phi(\imath \otimes e_{i,i}) = p_i, \ \ i = 1, 2, ..., 2^m. \tag{e7.64}$$

Since $C_0((0,1]) \otimes M_{2^m}$ is semiprojective, there is a homomorphism

$$\psi : C_0((0,1]) \otimes M_{2^m} \to l^\infty(\{\mathrm{Her}(e_n)\}) \text{ such that } \Pi(\psi(e_{i,i})) = p_i, \ \ i = 1, 2, ..., 2^m. \tag{e7.65}$$

Write $\psi = \{\Psi_k\}$, where $\Psi_k : C_0((0,1]) \otimes M_{2^m} \to \mathrm{Her}(e_k)$ is a homomorphism. Put

$$g_{k,i} = f_{1/4}(\Psi_k(e_{i,i})) \in \mathrm{Her}(e_k), \ \ i = 1, 2, ..., 2^m, \ \ k \in \mathbb{N}.$$

Then $\{g_{k,1}\}, \{g_{k,2}\}, ..., \{g_{k,2^m}\}$ are mutually orthogonal and mutually equivalent. Note that $\Pi(g_{k,i})\} = p_i$ and $\{g_{k,i}\}$ is a permanent projection lifting of p_i, and by (2) of Lemma 6.2, $\lim_{k\to\infty} \omega(g_{k,i}) = 0$.

Then, by Lemma 7.2, there is $k_1 \in \mathbb{N}$, such that, for all $k \ge k_1$,

$$d_\tau(e_k) < d_\tau\left(\sum_{j=1}^{2^m} g_{k,j}\right) + d_\tau(s_k) + \varepsilon/2 \le 2^m d_\tau(g_{k,1}) + \varepsilon \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e7.66}$$

Also, by Lemma 7.2, we may assume that, for all $k \ge k_1$,

$$2^m d_\tau(g_{k,1}) \le d_\tau\left(\sum_{j=1}^{2^m} g_{k,j} + s_k\right) \le d_\tau(e_k) + \varepsilon \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e7.67}$$

We choose a large m such that $2^m = ln + m_0$, where $l \in \mathbb{N}$ and $m_0 \in \mathbb{N} \cup \{0\}$ such that $m_0/2^m < \varepsilon/4$. Note that, since $\{g_{k,1}\}, \{g_{k,2}\}, ..., \{g_{k,2^m}\}$ are mutually orthogonal, for any sum $f_{k,j}$ of some l many $\{g_{k,i}\}'s$, $\lim_{k\to\infty} \omega(f_{k,j}) = 0$. For each $1 \le j \le n$, by

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(e7.66) and (e7.67), and by choosing $f_{k,j}$ to be a sum of l many (different) $g'_{k,i}s$, we see (e7.58) and (e7.59) hold. ■

Corollary 7.8 *Let A be a non-elementary, σ-unital, and algebraically simple C^*-algebra with $QT(A) \neq \emptyset$ and T-tracial approximate oscillation zero. Let $e_k \in A_+^{\mathbf{1}} \setminus \{0\}$ $(k \in \mathbb{N})$ be such that $\{e_k\} \notin I_{\overline{QT(A)}^w}$ and $\{e_k\}$ is a permanent projection lifting of $p = \Pi(\{e_k\})$, and let $n > 1$ be an integer and $\varepsilon > 0$.*

Then, there is $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$ and for any $1 \leq i \leq n$, there exists $h_{k,i} \in$ $\mathrm{Her}(e_k)_+$ such that

$$\sup\left\{\left|\frac{i}{n}d_\tau(e_k) - d_\tau(h_{k,i})\right| : \tau \in \overline{QT(A)}^w\right\} < \varepsilon. \tag{e7.68}$$

Proof By the proof of Lemma 7.7, for $k \geq k_0$,

$$\sup\left\{\left|\frac{1}{n}d_\tau(e_k) - d_\tau(f_{k,1})\right| : \tau \in \overline{QT(A)}^w\right\} < \varepsilon/n. \tag{e7.69}$$

So, for any $1 \leq i \leq n$, choose $h_{k,i} = \sum_{j=1}^{i} f_{k,j}$. Then

$$\sup\left\{\left|\frac{i}{n}d_\tau(e_k) - d_\tau(h_{k,i})\right| : \tau \in \overline{QT(A)}^w\right\} < \varepsilon.$$

■

Definition 7.9 Let A be a C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$. Define

$$R_{\tau,f}(A) = \{\widehat{a} : a \in \mathrm{Ped}(A \otimes \mathcal{K})_+\} \subset \mathrm{Aff}_+(\widetilde{QT}(A)). \tag{e7.70}$$

Lemma 7.10 *Let A be a non-elementary, σ-unital, and simple C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$. Suppose that A has T-tracial approximate oscillation zero. Then the image of the canonical map Γ (see Definition 2.13) is dense in $R_{\tau,f}(A)$.*

Proof Fix a nonzero element $0 \leq e \leq 1$ in $\mathrm{Ped}(A)_+$. Let $A_1 = \mathrm{Her}(e)$. Then $A_1 =$ $\mathrm{Ped}(A_1)$. By Brown's stable isomorphism theorem [4], $A_1 \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. It suffices to show that the image of the map $\Gamma_1 : \mathrm{Cu}(A_1) = \mathrm{Cu}(A) \to \mathrm{LAff}_+(\overline{QT(A_1)}^w)$ is dense in

$$R_{\tau,f}(A_1) = \{\hat{a} : a \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+\} \subset \mathrm{Aff}_+(\overline{QT(A_1)}^w) \qquad \text{(see Definition2.13)}.$$

Fix $a \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+$. Let $\varepsilon > 0$. It suffices to show that there is $f \in \mathrm{Cu}(A_1)$ such that

$$\sup\{|\tau(a) - \widehat{f}(\tau)| : \tau \in \overline{QT(A_1)}^w\} < \varepsilon. \tag{e7.71}$$

Without loss of generality, we may assume that $0 \leq a \leq 1$. Since $a \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+$, there exists $r \geq 1$ such that $r[f_\delta(e)] \geq [a]$ for some $\delta \in (0, 1/4)$. Therefore, we may assume that $a \in M_r(A_1)$ for some integer $r \geq 1$.

Put $B := l^\infty(A_1)/I_{\overline{QT(A_1)}^w}$. Then, by Theorem 6.4, B has real rank zero.

Therefore, for any $\varepsilon > 0$, there are mutually orthogonal projections $p_1, p_2, \ldots, p_m \in$ $M_r(B)$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in (0, 1)$ such that

$$\left\|\Pi(\iota(a)) - \sum_{i=1}^{m} \lambda_i p_i\right\| < \varepsilon/16. \tag{e7.72}$$

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We may assume that $\lambda_i \in (0,1) \cap \mathbb{Q}$. There are mutually orthogonal elements $\{e_{n,i}\} \in l^\infty(M_r(A_1))$ $(i = 1, 2, \ldots, m)$ such that $\Pi(\{e_{n,i}\}) = p_i, i = 1, 2, \ldots, m$. By Proposition 6.2, we may assume that $\{e_{n,i}\}$ is a permanent projection lifting. By (2) of Proposition 6.2, $\lim_{n\to\infty} \omega(e_{n,i}) = 0$. Without loss of generality, we may assume that, for all $n \in \mathbb{N}$,

$$d_\tau(e_{n,i}) - \tau(e_{n,i}) < \varepsilon/16(m+1)^2 \text{ for all } \tau \in \overline{QT(A_1)}^w, \; i = 1, 2, \ldots, m. \tag{e7.73}$$

Applying Corollary 7.8, without loss of generality, we may also assume that, there are permanent projection liftings $\{f_{n,i}\}$ such that

$$\sup\{|\lambda_i d_\tau(e_{n,i}) - d_\tau(f_{n,i})| : \tau \in \overline{QT(A_1)}^w\} < \varepsilon/16(m+1)^2, \; i = 1, 2, \ldots, m. \tag{e7.74}$$

By (e7.72), there exists $\{c_n\} \in I_{\overline{QT(A_1)}^w}$ and $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\left\| a - \sum_{i=1}^m \lambda_i e_{i,n} + c_n \right\| < \varepsilon/8. \tag{e7.75}$$

Then, for $n \geq n_1$,

$$\sup\left\{ \left| \tau\left(a - \sum_{i=1}^m \lambda_i e_{n,i} + c_n \right) \right| : \tau \in \overline{QT(A_1)}^w \right\} < \varepsilon/8.$$

Since $\{c_n\} \in I_{\overline{QT(A_1)}^w}$, we have $\{|c_n|^{1/2}\} \in I_{\overline{QT(A_1)}^w}$. It follows that

$$\lim_{n\to\infty} \sup\{|\tau(c_n)| : \tau \in \overline{QT(A_1)}^w\} = 0.$$

Therefore, there exists $n_2 \geq n_1$ such that

$$\left| \tau\left(a - \sum_{i=1}^m \lambda_i e_{n,i} \right) | < |\tau(c_n) \right| + \varepsilon/8 < \varepsilon/4 \text{ for all } \tau \in \overline{QT(A_1)}^w \text{ and for all } n \geq n_2. \tag{e7.76}$$

Thus, by (e7.73) and (e7.74), for $n \geq n_2$,

$$\left| \tau(a) - \sum_{i=1}^m d_\tau(f_{n,i}) \right| < \varepsilon/2 \text{ for all } \tau \in \overline{QT(A_1)}^w. \tag{e7.77}$$

Put $e = \mathrm{diag}(f_{n,1}, f_{n,2}, \ldots, f_{n,m})$. Then

$$|\tau(a) - d_\tau(e)| < \varepsilon \text{ for all } \tau \in \overline{QT(A_1)}^w. \tag{e7.78}$$

This completes the proof. ∎

Theorem 7.11 *Let A be a non-elementary and σ-unital simple C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \neq \emptyset$ and strict comparison. Suppose that A has T-tracial approximate oscillation zero. Then Γ is surjective (see Definition 2.13).*

Proof We keep the same setting as in the proof of Theorem 7.10.

Let $b \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+$ with $0 \leq b \leq 1$. Choose $b_n = (1 - 1/(n+1))b$. Then $h_n := b_{n+1} - b_n \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+ \backslash \{0\}$. Put

$$\sigma_n := \inf\{\tau(h_n) : \tau \in \overline{QT(A_1)}^w\} > 0. \tag{e7.79}$$

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Applying Theorem 7.10, for each $n \in \mathbb{N}$, we obtain $f_n \in \mathrm{Ped}(A_1 \otimes \mathcal{K})_+$ such that

$$\eta_n := \sup\{|\tau(b_n) - d_\tau(f_n)| : \tau \in \overline{QT(A_1)}^w\} < \frac{\min\{\sigma_j : 1 \le j \le n+1\}}{2^{n+2}}. \tag{e7.80}$$

In particular, for all $n \in \mathbb{N}$,

$$d_\tau(f_n) < \eta_n + \tau(b_n) < \tau(b) \ \text{ for all } \ \tau \in \overline{QT(A_1)}^w. \tag{e7.81}$$

Then, for all $\tau \in \overline{QT(A_1)}^w$ (note that $b_n b_{n+1} = b_{n+1} b_n$),

$$d_\tau(f_{n+1}) - d_\tau(f_n) > (\tau(b_{n+1}) - \eta_{n+1}) - (\tau(b_n) + \eta_n) \tag{e7.82}$$

$$= \tau(h_n) - \eta_{n+1} - \eta_n > \tau(h_n) - \sigma_n/2 > 0. \tag{e7.83}$$

It follows from the strict comparison that $[f_n]$ is an increasing sequence in $\mathrm{Cu}(A)$. Let f be the supremum of $\{[f_n]\}$ in $\mathrm{Cu}(A)$. We also have, for all $\tau \in \overline{QT(A_1)}^w$,

$$d_\tau(f_{n+1}) - \tau(b_n) > \tau(b_{n+1}) - \eta_{n+1} - \tau(b_n) \tag{e7.84}$$

$$\ge \tau(h_n) - \sigma_n/2^{n+1} > 0. \tag{e7.85}$$

It follows that $\widehat{f}(\tau) \ge \tau(b_n)$ for all $\tau \in \overline{QT(A_1)}^w$ and for each $n \in \mathbb{N}$. Hence,

$$\widehat{f}(\tau) \ge \tau(b) \ \text{ for all } \ \tau \in \overline{QT(A_1)}^w.$$

Let $\varepsilon > 0$. By Theorem 7.10, there is $c_\varepsilon \in \mathrm{Cu}(A_1)$ such that

$$\sup\{|\tau(b + (\varepsilon/2)b) - \Gamma_1(c_\varepsilon)(\tau)| : \tau \in \overline{QT(A_1)}^w\} < \varepsilon\sigma_1/8. \tag{e7.86}$$

Then, for all $n \in \mathbb{N}$ (see also (e7.81)),

$$\Gamma_1(c_\varepsilon)(\tau) > \tau(b + \varepsilon/2b) - \varepsilon\sigma_1/8 > \tau(b) > d_\tau(f_n) \ \text{ for all } \ \tau \in \overline{QT(A_1)}^w. \tag{e7.87}$$

Hence, $[f] \le [c_\varepsilon]$. It follows that

$$\widehat{f}(\tau) \le \widehat{c_\varepsilon}(\tau) < \tau(b) + \varepsilon \ \text{ for all } \ \tau \in \overline{QT(A_1)}^w. \tag{e7.88}$$

Let $\varepsilon \to 0$. We conclude that $\widehat{f}(\tau) = \tau(b)$ for all $\tau \in \overline{QT(A_1)}^w$.

So far we have shown that, for any $b \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, there is $f \in \mathrm{Cu}(A)$ such that $\widehat{f}(\tau) = \tau(b)$ for all $\tau \in \overline{QT(A)}^w$. Note that, for any $a \in (A \otimes \mathcal{K})_+$, $(a - \|a\|/n)_+ \in \mathrm{Ped}(A \otimes \mathcal{K})$. Thus, there are $f_n \in \mathrm{Cu}(A)$ such that $\widehat{f_n}(\tau) = \tau((a - \|a\|/n)_+)$ for all $\tau \in \overline{QT(A)}^w$. Since $(a - \|a\|/n)_+ \nearrow a$, we conclude, using the similar argument used above, that there is $f \in \mathrm{Cu}(A)$ such that $\widehat{f}(\tau) = \tau(a)$ for all $\tau \in \overline{QT(A)}^w$. Applying Theorem 5.7 of [13] and repeating the argument above, we conclude that Γ is surjective. ∎

8 Tracially matricial property

Definition 8.1 Let A be a C^*-algebra and $S \subset \widetilde{QT}(A) \backslash \{0\}$ be a nonempty compact subset. C^*-algebra A is said to have property (TM) relative to S, if for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+$, any $\varepsilon > 0$, any $n \in \mathbb{N}$, there is a c.p.c. order zero map $\phi : M_n \to \mathrm{Her}(a)$ such that $\|a - \phi(1_n)a\|_{_{2,S}} < \varepsilon$.

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A σ-unital simple C^*-algebra A with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$ is said to have property (TM), if for some $e \in \mathrm{Ped}(A)_+^{\mathbf{1}}\setminus\{0\}$, $\mathrm{Her}(e)$ has property (TM) relative to $\overline{QT(A)}^w$.

From the definition, it is clear that, if A is a σ-unital simple C^*-algebra which has property (TM), then $A \otimes \mathcal{K}$ has property (TM), and by Brown's stable isomorphism theorem [4], every σ-unital hereditary C^*-subalgebra has property (TM). Since we only need to consider $\mathrm{Her}(a)$ for each $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$, it follows that every hereditary C^*-subalgebra has property (TM).

In the absence of strict comparison, one may also define the following:

A C^*-algebra A is said to have property (CM), if, for any $n \in \mathbb{N}$ and any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$, there is a c.p.c. order zero map $\phi : M_n \to l^\infty(C)/N_{cu}(C)$ such that

$$\iota(b)\phi(1_n) = \iota(b) \text{ for all } b \in C, \tag{e8.1}$$

where $C = \overline{a(A \otimes \mathcal{K})a}$.

Remark 8.2 In Definition 8.1, let $\psi : C_0((0,1]) \otimes M_n \to \mathrm{Her}(a)$ be the homomorphism induced by ϕ, i.e., $\psi(\iota \otimes e_{i,j}) = \phi(e_{i,j})$ $(1 \le i, j \le n)$. Then $\mathrm{Her}(a)$ has some "tracially large" matricial structure (see Proposition 8.3).

Let $x \in \mathrm{Ped}(A \otimes \mathcal{K})$. Then $a_0 = x^*x + xx^* \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. Note $x \in \mathrm{Her}(a_0)$. Let $\mathcal{F} \subset \mathrm{Her}(a_0)^{\mathbf{1}}$ be a finite set. For any $1 > \varepsilon > 0$, there is $a \in \mathrm{Her}(a_0)_+^{\mathbf{1}}$ such that

$$\|ay - y\| < (\varepsilon/4)^3 \text{ and } \|ya - y\| < (\varepsilon/4)^3 \text{ for all } y \in \mathcal{F}. \tag{e8.2}$$

If A has property (TM) relative to S such that $\|\tau\| \le 1$ for all $\tau \in S$, then there is a c.p.c. order zero map $\phi : M_n \to \mathrm{Her}(a_0)$ such that $\|a - \phi(1_n)a\|_{2,S} < (\varepsilon/2)^3$. Then, for all $y \in \mathcal{F}$,

$$\|y - \phi(1_n)y\|_{2,S}^{2/3} \le \|y - ya\|_{2,S}^{2/3} + \|ya - ya\phi(1_n)\|_{2,S}^{2/3} \tag{e8.3}$$

$$< (\varepsilon/2)^2 + \|y\|\|a - a\phi(1_n)\|_{2,S}^{2/3} < \varepsilon. \tag{e8.4}$$

Similarly,

$$\|y - \phi(1_n)y\|_{2,S}^{2/3} < \varepsilon. \tag{e8.5}$$

The following fact is well known. For completeness, we include a proof here.

Proposition 8.3 *Let A be a C^*-algebra, $n \in \mathbb{N}$, and $\phi : M_n \to A$ be a c.p.c. order zero map. Then* $\mathrm{Her}(\phi(1_n)) \cong \mathrm{Her}(\phi(e_{1,1})) \otimes M_n$.

Proof Let $\psi : C_0((0,1]) \otimes M_n \to A$ be the homomorphism defined by $\psi(\iota \otimes e_{ij}) = \phi(e_{i,j})$, where ι is the identity function on $(0,1]$, $i, j \le n$. In particular, $\psi(\iota \otimes 1_n) = \phi(1_n)$. Write $\psi(\iota \otimes e_{i,j}) = u_{i,j}r_j$ as a polar decomposition of $\phi(\iota \otimes e_{i,j})$ in A^{**}. Hence, $r_j = |\psi(\iota \otimes e_{i,j})|$ and $u_{i,j}$ is a partial isometry in A^{**}. Note that $au_{i,j}b \in A$ for all $a \in p_iA^{**}p_i \cap A$ and $b \in p_jA^{**}p_j \cap A$, where p_i is the open projection of r_i, $i, j = 1, 2, \ldots, n$. Since ψ is a homomorphism, we compute that $p_i = u_{i,i}$, $i = 1, 2, \ldots, n$, and $\{u_{i,j}\}_{1 \le i,j \le n}$ forms a system of matrix units for M_n.

Define $\Phi : M_n(\mathrm{Her}(\varphi(e_{1,1}))) \cong \mathrm{Her}(\phi(e_{1,1})) \otimes M_n \to \mathrm{Her}(\phi(1_n))$ by defining $\Phi(a \otimes e_{i,j}) = u_{i,1}au_{1,j}$ for all $a \in \mathrm{Her}(\varphi(e_{1,1}))$, $i, j = 1, 2, \ldots, n$. Then Φ is a homomorphism. Since Φ is the identity map on $\mathrm{Her}(\phi(e_{1,1}))$ and M_n is simple, the map Φ is injective. Put $B = \Phi(M_n(\mathrm{Her}(\varphi(e_{1,1}))))$. To see Φ is surjective, let $x \in A$, then, for any $i, j = 1, 2, ..., n$, $b_{i,j} = u_{1,i} r_i x r_j u_{j,1} \in \mathrm{Her}(\phi(e_{1,1}))$. Therefore,

$$\phi(1_n) x \phi(1_n) = \sum_{1 \le i,j \le n} u_{i,i} r_i x r_j u_{j,j} \tag{e8.6}$$

$$= \sum_{1 \le i,j \le n} u_{i,1}(u_{1,i} x r_j u_{j,1}) u_{1,j} = \sum_{1 \le i,j \le n} u_{i,1} b_{i,j} u_{1,j} \in B. \tag{e8.7}$$

It follows that $\mathrm{Her}(\phi(1_n)) = B$. The lemma follows. ∎

Lemma 8.4 *Let A be a simple C^*-algebra with $\widetilde{QT}(A) \setminus \{0\} \neq \emptyset$. Suppose that $A = \mathrm{Ped}(A)$, A has strict comparison and Γ is surjective. Suppose that $b \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}}$. Then, for any $\varepsilon > 0$ and any integer $n \ge 1$, there is a c.p.c. order zero map $\phi : M_n \to \mathrm{Her}(b)$ such that*

$$\|b - \phi(1_n) b\|_{2, \overline{QT(A)}^w} < \|b\| \sqrt{\omega(b) + \varepsilon}. \tag{e8.8}$$

Moreover, if $QT(A) = T(A)$, then

$$\|b - \phi(1_n) b\|_{2, \overline{T(A)}^w} < \min\{\|b\|, \|b\|_{2, \overline{T(A)}^w}\} \sqrt{\omega(b) + \varepsilon}. \tag{e8.9}$$

Proof Fix $\varepsilon \in (0, 1/2)$ and $n \in \mathbb{N}$. Since $\Gamma : \mathrm{Cu}(A) \to \mathrm{LAff}_+(\overline{QT(A)}^w)$ is surjective, $\Gamma|_{\mathrm{Cu}(A)_+}$ is also surjective (see (2.13)). Note that $(1/n)\widehat{[b]} \in \mathrm{LAff}_+(\overline{QT(A)}^w)$. We may choose $b_1 \in (A \otimes \mathcal{K})_+^{\mathbf{1}}$ such that $d_\tau(b_1) = (1/n) d_\tau(b)$ for all $\tau \in \overline{QT(A)}^w$ and b_1 is not Cuntz equivalent to a projection. We compute that $\omega(b_1) = (1/n)\omega(b)$ (see (e4.5)). Choose $\delta > 0$ such that

$$\sup\{d_\tau(b_1) - \tau(f_{2\delta}(b_1)) : \tau \in \overline{QT(A)}^w\} < \omega(b_1) + \varepsilon/4n. \tag{e8.10}$$

Put $b_2 = f_\delta(b_1)$ and $d_1 = \mathrm{diag}(\overbrace{b_1, b_1, ..., b_1}^{n}) \in A \otimes \mathcal{K}$. Then

$$f_\delta(d_1) = \mathrm{diag}(\overbrace{b_2, b_2, ..., b_2}^{n}) \in A \otimes \mathcal{K}. \tag{e8.11}$$

Since b_1 is not Cuntz equivalent to a projection, for any $0 < \eta < \delta/2$, $d_\tau(f_\eta(d_1)) < d_\tau(b)$ for all $\tau \in \overline{QT(A)}^w$. Since A has strict comparison, by [40, Proposition 2.4(iv)], there is $x \in A \otimes \mathcal{K}$ such that

$$x^* x = f_\delta(d_1) \quad \text{and} \quad x x^* \in \mathrm{Her}(b). \tag{e8.12}$$

Then one obtains an isomorphism

$$\psi : \mathrm{Her}(x^* x) \to \mathrm{Her}(x x^*) \subset \mathrm{Her}(b) \ \text{ such that } \ \psi(f(x^* x)) = f(x x^*) \ \text{ for all } \ f \in C_0((0, \|x\|^2]).$$

It induces a homomorphism $\phi_c : C_0(\mathrm{sp}(f_\delta(b_1))) \otimes M_n \to \mathrm{Her}(b)$ such that $\phi_c(\imath \otimes 1_n) = x x^*$, where $\imath \in C_0(f_\delta(b_1))$ is the identity function on $\mathrm{sp}(f_\delta(b_1))$. Define a c.p.c. order zero map $\phi : M_n \to \mathrm{Her}(b)$ by $\phi(e_{i,j}) = \phi_c(\imath \otimes e_{i,j})$ $(1 \le i, j \le 1)$.

Let p be the open projection in A^{**} corresponding to b which may be identified with the identity of $\widetilde{\mathrm{Her}(b)}$. We extend each $\tau \in \overline{QT(A)}^w$ to a 2-quasitrace on

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$\widetilde{\mathrm{Her}(b)}$ (see II.2.5 of [2]) such that $\tau(p) = \|\tau|_{\mathrm{Her}(b)}\| = d_\tau(b)$. Then, we have, for all $\tau \in \overline{QT(A)}^w$,

$$\tau((p - \phi(1_n))^2) \le \tau(p - \phi(1_n)) = \tau(p) - \tau(\phi(1_n)) \tag{e8.13}$$

$$= d_\tau(b) - \tau(f_\delta(d_1)) < \omega(b) + \varepsilon. \tag{e8.14}$$

It follows that

$$\|b - \phi(1_n)b\|_{2,\overline{QT(A)}^w} \le \|b\|\|p - \phi(1_n)\|_{2,\overline{QT(A)}^w} < \|b\|\sqrt{\omega(b) + \varepsilon}. \tag{e8.15}$$

In the case that $QT(A) = T(A)$, one can also apply Cauchy–Bunyakovsky–Schwarz inequality (and (e8.14)) to obtain (e8.9). ∎

Theorem 8.5 *Let A be a σ-unital simple non-elementary C^*-algebra with $QT(A)\backslash\{0\} \ne \emptyset$. Suppose that A has strict comparison and T-tracial approximate oscillation zero. Then A has property (TM).*

Proof Choose $e \in \mathrm{Ped}(A)_+\backslash\{0\}$ and define $A_1 = \mathrm{Her}(e)$. By Brown's stable isomorphism theorem [4], $A_1 \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. To show that A has property (TM), it suffices to show that A_1 has property (TM). To simplify notation, without loss of generality, we may assume that $A = \mathrm{Ped}(A)$.

Fix $a \in \mathrm{Ped}(A \otimes \mathcal{K})$ such that $0 \le a \le 1$. Since A has T-tracial approximate oscillation zero, $\Omega^T(a) = 0$. It follows that there is a sequence $c_k \in \mathrm{Her}(a)$ with $0 \le c_k \le 1$ such that

$$\Pi(\iota(a)) = \Pi(c) \ \text{ and } \ \lim_{k\to\infty} \omega(c_k) = 0, \tag{e8.16}$$

where $c = \{c_k\}$ and $\Pi : l^\infty(A) \to l^\infty(A)/I_{\overline{QT(A)}^w}$ is the quotient map. By Theorem 7.11, Γ is surjective. Then, applying Lemma 8.4, we obtain a c.p.c. order zero map $\phi_k : M_n \to \mathrm{Her}(c_k) \subset \mathrm{Her}(a)$ such that

$$\|c_k - \phi_k(1_n)c_k\|_{2,\overline{QT(A)}^w} \le \sqrt{\omega(c_k) + 1/k^2}, \ \ k = 1, 2, \ldots. \tag{e8.17}$$

Fix $1 > \varepsilon > 0$. Choose $k_0 \ge 1$ such that

$$\sqrt{\omega(c_k) + 1/k^2} < (\varepsilon/3)^3. \tag{e8.18}$$

Since $\Pi(\iota(a)) = \Pi(c)$, there exists $k_1 \ge k_0$ such that

$$\|a - c_k\|_{2,\overline{QT(A)}^w} < (\varepsilon/3)^3 \ \text{ for all } \ k \ge k_1. \tag{e8.19}$$

Choose $b = c_{k_1+1}$. Then, for $k \ge k_1$,

$$\begin{aligned}\|a - \phi_k(1_n)a\|^{2/3}_{2,\overline{QT(A)}^w} &\le \|a - b\|^{2/3}_{2,\overline{QT(A)}^w} + \|b - \phi_k(1_n)b\|^{2/3}_{2,\overline{QT(A)}^w} \\ &\quad + \|\phi_k(1_n)(b - a)\|^{2/3}_{2,\overline{QT(A)}^w} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 < \varepsilon.\end{aligned}$$

∎

Lemma 8.6 *Let A be a σ-unital algebraically simple C^*-algebra with $QT(A) \ne \emptyset$. Suppose that A has property (TM) and $a \in \mathrm{Ped}(A \otimes \mathcal{K})^{\mathbf{1}}_+\backslash\{0\}$. Then, for any integer $n, r \in \mathbb{N}$, any $k \in \mathbb{N}$ $(k \ge 2)$ and $\varepsilon > 0$, there exist an integer $m_k \ge r \in \mathbb{N}$ and*

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mutually orthogonal elements $b_{k,1}, b_{k,2}, \dots, b_{k,n} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ *such that* $[b_{k,i}] = [b_{k,1}]$, $i = 1, 2, \dots, n$,

$$d_\tau(f_{1/k}(a^{1/m_k})) < n d_\tau(f_{1/k}(b_{k,1})) + \varepsilon \text{ for all } \tau \in \overline{QT(A)}^w \tag{e8.20}$$

and an integer $l(k) \in \mathbb{N}$ *such that*

$$n d_\tau(f_{1/k}(b_{k,1})) \le d_\tau(f_{1/l(k)}(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.21}$$

Proof Fix $n \in \mathbb{N}$. Since A has property (TM), for each $m \in \mathbb{N}$, there exists a c.p.c. order zero map $\phi_m : M_n \to \mathrm{Her}(a)$ such that

$$\|a^{1/m} - a^{1/m}\phi_m(1_n)\|_{2,\overline{QT(A)}^w} < 1/2^m. \tag{e8.22}$$

Define $c := \Pi(\{a^{1/m}\}_{m\in\mathbb{N}})$ and $\phi : M_n \to l^\infty(A)/I_{\overline{QT(A)}^w}$ such that $\phi(f) = \Pi(\{\phi_m(f)\})$ for all $f \in M_n$. Then

$$c = c\phi(1_n) = \phi(1_n)c = c. \tag{e8.23}$$

It follows that $c = cf_{1/2}(\phi(1_n)) = f_{1/2}(\phi(1_n))c = c$. Thus, $c \le f_{1/2}(\phi(1_n))$.

Let $\varepsilon > 0$. By (1) of Lemma 7.2, for each integer $k \ge 2$, there exists $m_k \ge r \in \mathbb{N}$ such that, for all $m \ge m_k$,

$$[f_{1/k}(a^{1/m})] \le [f_{1/2}(\phi_m(1_n))] + [d_m] \tag{e8.24}$$

$$\le [f_{1/k}(\phi_m(e_{1,1})) + f_{1/k}(\phi_m(e_{2,2})) + \cdots + f_{1/k}(\phi_m(e_{n,n}))] + [d_m], \tag{e8.25}$$

where $\sup\{d_\tau(d_m) : \tau \in \overline{QT(A)}^w\} < \varepsilon/2$. Put $b_{k,i} = \phi_{m_k}(e_{i,i}) 1 \le i \le n$ and $k \in \mathbb{N}$. Then (e8.20) holds. On the other hand, since $\phi_{m_k}(1_n) \in \mathrm{Her}(a)$, for each k, there is $l(k) \in \mathbb{N}$ such that

$$\|f_{1/l(k)}(a)\phi_{m_k}(1_n)f_{1/l(k)}(a) - \phi_{m_k}(1_n)\| < 1/4k. \tag{e8.26}$$

It follows that (see Proposition 2.2 of [40]), for $k \ge 2$,

$$f_{1/k}(\phi_{m_k}(1_n)) \lesssim f_{1/l(k)}(a) \text{ and } n[f_{1/k}(b_{k,1})] = [f_{1/k}(\phi_{m_k}(1_n))]. \tag{e8.27}$$

Then (e8.21) holds. ∎

Theorem 8.7 *Let A be a σ-unital simple C^*-algebra with $QT(A) \ne \{0\}$. If A has strict comparison and property (TM), then Γ is surjective (see Definition 2.13).*

Proof It suffices to prove the proposition for the case that $A = \mathrm{Ped}(A)$.

We claim that, for any $a \in \mathrm{Ped}(A \otimes \mathcal{K})_+^{\mathbf{1}} \setminus \{0\}$ and any integer $n \in \mathbb{N}$, there exists $b \in (A \otimes \mathcal{K})_+$ such that

$$n\widehat{[b]} \le \widehat{[a]} \le (n+1)\widehat{[b]}. \tag{e8.28}$$

Case (1): 0 is an isolated point of $\mathrm{sp}(a)$. In this case, we may assume that $a = p$ for some projection p. Choose

$$\eta := \left(\frac{1}{(n+1)^2}\right)\inf\{\tau(p) : \tau \in \overline{QT(A)}^w\} > 0. \tag{e8.29}$$

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Note that $f_{1/k}(p) = p$ for all $k > 1$. Applying Lemma 8.6, we obtain $b \in \mathrm{Her}(a)_+$ such that

$$nd_\tau(b) \le d_\tau(p) < nd_\tau(b) + \eta \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.30}$$

Then we compute that

$$n\widehat{[b]} \le \widehat{[a]} \le (n+1)\widehat{[b]}. \tag{e8.31}$$

Case (2): 0 is not an isolated point of $\mathrm{sp}(a)$. We will use Lemma 8.6 for an induction argument. Put

$$\sigma_0 := \inf\{\tau(a) : \tau \in \overline{QT(A)}^w\} > 0. \tag{e8.32}$$

Fix $n \in N$. Since 0 is a not an isolated point of $\mathrm{sp}(a)$, for each integer k, there is a smallest integer $J(k) > k$ such that

$$f_{1/J(k)}(a) - f_{1/k}(a) \ne 0. \tag{e8.33}$$

Define, for each k,

$$\sigma_k := \inf\{d_\tau(f_{1/J(k)}(a)) - d_\tau(f_{1/k}(a)) : \tau \in \overline{QT(A)}^w\} > 0 \text{ and} \tag{e8.34}$$

$$\eta_k := \min\{\sigma_j : 0 \le j \le k+1\}/2^{k+1}(n+1). \tag{e8.35}$$

Applying Lemma 8.6, there are mutually orthogonal elements $b_{1,1}, b_{1,2}, \ldots, b_{1,n} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that $[b_{1,i}] = [b_{1,1}]$, $i = 1, 2, \ldots, n$, and, for some $m_1 \in \mathbb{N}$,

$$d_\tau(f_{1/2}(a^{1/m_1})) < nd_\tau(f_{1/2}(b_{1,1})) + \eta_1 \text{ for all } \tau \in \overline{QT(A)}^w, \tag{e8.36}$$

and an integer $l(1) \in \mathbb{N}$ such that

$$nd_\tau(f_{1/2}(b_{1,1})) \le d_\tau(f_{1/l(1)}(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.37}$$

Put $c_1 := f_{1/2}(b_{1,1})$. Then, for all $\tau \in \overline{QT(A)}^w$,

$$\widehat{[f_{1/2}(a^{1/m_1})]}(\tau) \le n\widehat{[c_1]}(\tau) + \eta_1 \text{ and } n\widehat{[c_1]}(\tau) \le \widehat{[f_{1/l(1)}(a)]}(\tau). \tag{e8.38}$$

Choose $k_2 > l(1)$ such that $k_2 \ge J(l(1))$. Applying Lemma 8.6, we obtain $m_2 \ge m_1$ and mutually orthogonal $b_{2,1}, b_{2,2}, \ldots, b_{2,n} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ with $b_{2,j} \sim b_{2,1}$ $(1 \le j \le n)$ and $l(k_2) \in \mathbb{N}$ such that

$$d_\tau(f_{1/k_2}(a^{1/m_2})) < nd_\tau(f_{1/k_2}(b_{2,1})) + \eta_{l(1)} \text{ and} \tag{e8.39}$$

$$nd_\tau(b_{2,1}) \le d_\tau(f_{1/l(2)}(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.40}$$

Put $c_2 = f_{1/k_2}(b_{2,1})$. Then, for all $\tau \in \overline{QT(A)}^w$,

$$\widehat{[f_{1/k_2}(a^{1/m_2})]}(\tau) \le n\widehat{[c_2]}(\tau) + \eta_{l(1)} \text{ and } n\widehat{[c_2]}(\tau) \le \widehat{[f_{1/l(2)}(a)]}(\tau). \tag{e8.41}$$

We compute that, for all $\tau \in \overline{QT(A)}^w$, by (e8.39), (e8.37), and (e8.34) (recall that $k_2 > J(l(1))$),

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(e8.42)
$$nd_\tau(f_{1/k_2}(b_{2,1})) > d_\tau(f_{1/k_2}(a^{1/m_2})) - \eta_{l(1)}$$
$$= d_\tau(f_{1/l(1)}(a)) + (d_\tau(f_{1/k_2}(a^{1/m_2})) - d_\tau(f_{1/l(1)}(a))) - \eta_{l(1)} \tag{e8.43}$$
$$\geq nd_\tau(f_{1/2}(b_{1,1})) + (d_\tau(f_{1/k_2}(a^{1/m_2})) - d_\tau(f_{1/l(1)}(a))) - \eta_{l(1)} \tag{e8.44}$$
$$> nd_\tau(f_{1/2}(b_{1,1})) + \sigma_{l(1)} - \eta_{l(1)} > nd_\tau(f_{1/2}(b_{1,1})). \tag{e8.45}$$

Since A has strict comparison, we obtain
$$[c_1] \leq [c_2]. \tag{e8.46}$$

Suppose that we have constructed integers $k_i, m_i, l(i) \in \mathbb{N}$ and $b_{i,1} \in \mathrm{Her}(a)_+^{\mathbf{1}}$, $1 \leq i \leq I$ such that, for all $2 \leq i \leq I$ and $\tau \in \overline{QT(A)}^w$ (with $l(0) = 1$ and $\eta_{l(0)} = \eta_1$),
$$k_i > J(l(i-1)) > l(i-1), \tag{e8.47}$$
$$\widehat{[f_{1/k_i}(a)]}(\tau) < n\widehat{[f_{1/k_i}(b_{i,1})]}(\tau) + \eta_{l(i-1)} \text{ and} \tag{e8.48}$$
$$n\widehat{[f_{1/k_i}(b_{i,1})]}(\tau) \leq \widehat{[f_{1/l(i)}(a)]}(\tau), \tag{e8.49}$$

and verified that $[f_{1/k_i}(b_{i,1})] \leq [f_{1/k_{i+1}}(b_{i+1,1})]$, $1 \leq i \leq I-1$.

Define $c_i = f_{1/k_i}(b_{i,1})$, $1 \leq i \leq I$. By applying Lemma 8.6, there is $k_{I+1} > J(l(i)) > l(i)$, $l(I+1) \geq k_{I+1}$, $m_{I+1} \geq m_I$, and $b_{I+1,1} \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that, for all $\tau \in \overline{QT(A)}^w$,
$$\widehat{[f_{1/k_{I+1}}(a^{1/m(I+1)})]}(\tau) < n\widehat{[f_{1/k_{I+1}}(b_{I+1,1})]}(\tau) + \eta_{l(I)} \text{ and} \tag{e8.50}$$
$$n\widehat{[f_{1/k_{I+1}}(b_{I+1,1}]}(\tau) \leq \widehat{[f_{1/l(I+1)}(a)]}(\tau). \tag{e8.51}$$

Then, for all $\tau \in \overline{QT(A)}^w$,
$$nd_\tau(f_{1/k_{I+1}}(b_{I+1,1})) > d_\tau(f_{1/k_{I+1}}(a^{1/m_{I+1}})) - \eta_{l(I)}$$
$$= d_\tau(f_{1/l(I)}(a)) + (d_\tau(f_{1/k_{I+1}}(a^{1/m_{I+1}})) - d_\tau(f_{1/l(I)}(a))) - \eta_{l(I)}$$
$$\geq nd_\tau(f_{1/k_I}(b_{I,1})) + (d_\tau(f_{1/k_{I+1}}(a^{1/m_{I+1}})) - d_\tau(f_{1/l(I)}(a))) - \eta_{l(I)}$$
$$> nd_\tau(f_{1/k_I}(b_{I,1})) + \sigma_{l(I)} - \eta_{l(I)} > nd_\tau(f_{1/k_I}(b_{I,1})). \tag{e8.52}$$

Put $c_{I+1} = f_{1/k_{I+1}}(b_{I+1,1})$. Then, by the strict comparison, estimates above imply that
$$[c_I] \leq [c_{I+1}]. \tag{e8.53}$$

Thus, by induction, we obtain an increasing sequence $c_i \in \mathrm{Her}(a)_+^{\mathbf{1}}$ such that, for all $\tau \in \overline{QT(A)}^w$,
$$\widehat{[f_{1/k_i}(a)]}(\tau) < n\widehat{[c_i]}(\tau) + \eta_{l(i-1)} \text{ and} \tag{e8.54}$$
$$n\widehat{[c_i]}(\tau) \leq \widehat{[f_{1/l(i)}(a)]}(\tau), \quad i \geq 2. \tag{e8.55}$$

Let $c \in (A \otimes \mathcal{K})_+$ be such that $[c]$ is the supremum of $\{[c_i]\}$. Then, by (e8.54), for all i,

(e8.56)
$$d_\tau(f_{1/k_i}(a)) < n\widehat{[c]}(\tau) + \eta_{l(i-1)} \leq n\widehat{[c]}(\tau) + \sigma_0/2^{i+1}(n+1) \text{ for all } \tau \in \overline{QT(A)}^w.$$

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Thus (recall that A is simple), for all sufficiently large i,

$$d_\tau(f_{1/k_i}(a)) < (n+1)\widehat{[c]}(\tau) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.57}$$

It follows that (let $i \to \infty$)

$$d_\tau(a) \le (n+1)\widehat{[c]}(\tau) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.58}$$

On the other hand, by (e8.55),

$$n\widehat{[c_i]} \le \widehat{[a]} \text{ for all } i \in \mathbb{N}. \tag{e8.59}$$

For any $\varepsilon > 0$, choose a nonzero element $e \in A_+$ such that $d_\tau(e) < \varepsilon/2$ for all $\tau \in \overline{QT(A)}^w$. Then, by the strict comparison,

$$n[c_i] \le [a] + [e] \text{ for all } i \in \mathbb{N}. \tag{e8.60}$$

If follows that $n[c] \le [a] + [e]$. Hence,

$$n\widehat{[c]} < d_\tau(a) + \varepsilon \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.61}$$

Let $\varepsilon \to 0$. We also obtain

$$n\widehat{[c]} \le \widehat{[a]} \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.62}$$

Combining (e8.62) and (e8.58), the claim also holds for Case (2).

We now show that the proved claim implies that $\mathrm{Cu}(A)$ has the property D stated in the proof of Proposition 6.21 of [37]. Let $x' \ll x$, where $x = [a]$ for some $a \in (A \otimes \mathcal{K})_+$. Since

$$x = \sup\{[(a - \|a\|/k)_+] : k \in \mathbb{N}\},$$

then

$$x' \le [(a - \|a\|/k)_+] \text{ for } k \in \mathbb{N}.$$

Note $(a - \|a\|/k)_+ \in \mathrm{Ped}(A \otimes \mathcal{K})_+$. Then the claim implies that there is $y \in \mathrm{Cu}(A)$ such that

$$\widehat{x'} \le \widehat{[(a - \|a\|/k)_+]} \le (n+1)\widehat{y} \text{ and } n\widehat{y} \le \widehat{[(a - \|a\|/k)_+]} \le \widehat{[a]} = \widehat{x}. \tag{e8.63}$$

Therefore, as observed by L. Robert (see Property D in the proof of Proposition 6.21 of [37]), following Corollary 5.8 of [13], Γ is surjective. ∎

Lemma 8.8 *Let A be a C^*-algebra and $a, b \in A^{\mathbf{1}}_+$. Suppose that there is $x \in A$ such that*

$$x^*x = a \text{ and } xx^* \in \mathrm{Her}(b). \tag{e8.64}$$

Then, for any $\varepsilon > 0$, there exists a unitary $U \in M_2(\widetilde{A})$ such that

$$U^*\mathrm{diag}(f_\varepsilon(a), 0)U \in \mathrm{Her}(\mathrm{diag}(b, 0)). \tag{e8.65}$$

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Proof First, we claim that, for any $y \in A$, $\mathrm{dist}(\mathrm{diag}(y,0), GL(M_2(\widetilde{A}))) = 0$. To see this, let $\varepsilon > 0$. Choose $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is a unitary in $M_2(\mathbb{C})$. Then

$$Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix},$$

which is a nilpotent. Therefore, $Y + \varepsilon \cdot 1_2 \in GL(M_2(\widetilde{A}))$. Then

$$\mathrm{diag}(y,0) \approx_\varepsilon V^*(Y+\varepsilon) \in GL(M_2(\widetilde{A})).$$

This proves the claim.

To prove the lemma, we will combine the claim with an argument of Rørdam. By Proposition 2.4 of [40], for any $\varepsilon > 0$, there exist $\delta > 0$ and $r \in A$ such that

$$f_{\varepsilon/2}(a) = r f_\delta(b) r^*. \tag{e8.66}$$

Put $z = r f_\delta(b)^{1/2}$ and $Z = \mathrm{diag}(z,0)$. By the claim and Theorem 5 of [32], there is a unitary $U \in M_2(\widetilde{A})$ such that

(e8.67)
$$U^* f_{1/2}(ZZ^*)U = U^* f_{1/2}(\mathrm{diag}(zz^*,0))U = \mathrm{diag}(f_{1/2}(z^*z),0) = f_{1/2}(Z^*Z).$$

Note that $Z^*Z \in \mathrm{Her}(\bar{b})$, where $\bar{b} = \mathrm{diag}(b,0)$. Moreover (with $\bar{a} = \mathrm{diag}(a,0)$),

$$U^* f_\varepsilon(\bar{a})U \le U^* f_{1/2}(f_{\varepsilon/2}(\bar{a}))U = U^* f_{1/2}(ZZ^*)U = f_{1/2}(Z^*Z). \qquad \blacksquare$$

Lemma 8.9 *Let A be an algebraically simple C^*-algebra which has strict comparison. Suppose that $QT(A) \neq \emptyset$ and the canonical map Γ is surjective.*

Suppose that $a, a' \in \mathrm{Ped}(A)^{\mathbf{1}} \backslash \{0\}$ with $a \in \mathrm{Her}(a')$. Then, there exists $1/2 > \varepsilon_0$ satisfying the following: For any $0 < \eta < \varepsilon < \varepsilon_0$, any $\sigma > 0$, there exist $c \in \mathrm{Her}(f_\eta(a))_+^{\mathbf{1}}$ with $\|c\| \le \|a\|$ and unitary $U \in M_2(\mathrm{Her}(a')^\sim)$ such that (with $b = U^*\mathrm{diag}(c,0)U$)

$$(1) \quad \mathrm{diag}(f_\varepsilon(a),0) \le b, \tag{e8.68}$$

$$(2) \quad d_\tau(f_\varepsilon(a)) \le d_\tau(b) \le d_\tau(f_\eta(a)) \ \text{ for all } \ \tau \in \overline{QT(A)}^w, \tag{e8.69}$$

and, for some $1 > \delta > 0$,

$$(3) \quad |d_\tau(b) - \tau(f_\delta(b))| < \sigma \ \text{ for all } \ \tau \in \overline{QT(A)}^w. \tag{e8.70}$$

Moreover,

$$(4) \quad U^*\mathrm{diag}(g_{\eta/2}(a), a')U \in B, \tag{e8.71}$$

where $B := \mathrm{Her}(b)^\perp \cap \mathrm{Her}(\mathrm{diag}(a,a'))$ and $g_\eta(t) \in C_0((0,1])$ is defined as in Notation 2.5.

Consequently, if e is a strictly positive element in $(\mathrm{Her}(b)^\perp \cap \mathrm{Her}(\mathrm{diag}(a,a')))$, then

$$d_\tau(e) > d_\tau(a') + d_\tau(g_\eta(a)) \ \text{ for all } \ \tau \in \overline{QT(A)}^w. \tag{e8.72}$$

Proof Without loss of generality, we may assume that $\|a\| \le 1$. Let us first assume that $[0, \varepsilon_0) \subset \mathrm{sp}(a)$ for some $\varepsilon_0 > 0$.

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Fix $0 < \varepsilon < \varepsilon_0$. Note that, without loss of generality, we may assume that

(e8.73)
$$d_\tau(f_\varepsilon(a)) < \tau(f_{\delta_1}(a)) < d_\tau(f_{\eta_1}(a)) < \tau(f_{\delta_2}(a)) < d_\tau(f_\eta(a)) \text{ for all } \tau \in \overline{QT(A)}^w,$$

where $\varepsilon/2 > \delta_1, \delta_1/2 > \eta_1, \eta_1/2 > \delta_2, \delta_2/2 > \eta$. Put $h_i(\tau) = \tau(f_{\delta_i}(a))$ for all $\tau \in \overline{QT(A)}^w$, $i = 1, 2$. Then $h_i \in \mathrm{Aff}_+(\overline{QT(A)}^w)$, $i = 1, 2$.

Since Γ is surjective, there is $c_0 \in (A \otimes \mathcal{K})_+$ such that $d_\tau(c_0) = h_2(\tau)$ for all $\tau \in \overline{QT(A)}^w$. Choose $\delta_0 > 0$ such that (as h_2 is continuous)

$$d_\tau(c_0) - \tau(f_{\delta_0}(c_0)) < \sigma/2 \text{ for all} \tau \in \overline{QT(A)}^w. \tag{e8.74}$$

Since $h_1 < h_2$ are continuous, we may also assume, by choosing smaller δ_0, that

$$d_\tau(f_{\delta_0}(c_0)) > d_\tau(f_{\eta_1}(a)) > d_\tau(f_\varepsilon(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.75}$$

Since A has strict comparison, by (e8.73), there is $x \in A \otimes \mathcal{K}$ such that

$$x^* x = f_{\delta_0/4}(c_0) \text{ and } xx^* \in \mathrm{Her}(f_\eta(a)). \tag{e8.76}$$

Choose $c = xx^*$. Then $0 \le c \le 1$ and $d_\tau(c) = d_\tau(f_{\delta_0/4}(c_0))$ for all $\overline{QT(A)}^w$. Let $C = \mathrm{Her}(f_\eta(a))$. By (e8.75), the strict comparison and Lemma 8.8, we obtain a unitary $U \in M_2(\widetilde{C})$ such that

$$U \mathrm{diag}(f_{\eta_1}(a), 0) U^* \in \mathrm{Her}(\bar{c}), \tag{e8.77}$$

where $\bar{c} = \mathrm{diag}(c, 0)$. Let $b = U^* \bar{c} U$. Then

$$\mathrm{diag}(f_\varepsilon(a), 0) \le \mathrm{diag}(f_{\eta_1}(a), 0) \le b. \tag{e8.78}$$

(so (1) holds). Moreover, $d_\tau(b) = d_\tau(c)$ for all $\tau \in \overline{T(A)}^w$. Consequently,

$$d_\tau(f_{\eta_1}(a)) \le d_\tau(b) \le d_\tau(f_\eta(a)) \text{ for all } \tau \in \overline{QT(A)}^w \tag{e8.79}$$

(so (2) holds). Moreover, there is $1 > \delta > 0$ such that

$$d_\tau(f_\delta(b)) \ge \tau(f_{\delta_0}(c)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.80}$$

It then follows from (e8.74) that

$$|d_\tau(b) - \tau(f_\delta(b))| = |d_\tau(f_{\delta_0/4}(c_0)) - \tau(f_\delta(b))| < \sigma \text{ for all } \tau \in \overline{QT(A)}^w \tag{e8.81}$$

(so (3) holds). To show the "Moreover" part, put

$$B = \mathrm{Her}(b)^\perp \cap \mathrm{Her}(\mathrm{diag}(a, a')) \text{ and } e' = U^* \mathrm{diag}(g_{\eta/2}(a), a') U.$$

Since $g_{\eta/2}(a) \perp f_\eta(a)$, we have

$$\mathrm{diag}(g_{\eta/2}(a), a') \perp \mathrm{diag}(f_\eta(a), 0) \text{ and } e' \perp b.$$

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It follows that $U^*\mathrm{diag}(g_{\eta/2}(a), a')U \in B$. If e is a strictly positive element in $\mathrm{Her}(b)^\perp \cap \mathrm{Her}(\mathrm{diag}(a, a'))$, then

$$d_\tau(e) \ge d_\tau(e') = d_\tau(a') + d_\tau(g_{\eta/2}(a)) \text{ for all } \tau \in \overline{QT(A)}^w. \tag{e8.82}$$

This proves the case that $[0, \varepsilon_0] \subset \mathrm{sp}(a)$.

If there exists $r_n \in (0, 1]$ with

$$r_n > r_{n+1} \text{ and } \lim_{n\to\infty} r_n = 0 \text{ such that } r_n \notin \mathrm{sp}(a),$$

then $b_n = f_{2r_n}(a)$ has the property that $\omega(b_n) = 0$. Then the lemma follows by choosing $U = \mathrm{diag}(1, 1)$ and $b = b_n$ for some sufficiently large n. ■

Lemma 8.10 *Let A be a σ-unital algebraically simple C^*-algebra with $QT(A)\neq\emptyset$. Suppose that A has strict comparison and Γ is surjective. Suppose that $a = \mathrm{diag}(0, a_1, a_2, ..., a_n)$ in $M_{n+1}(A)^{\mathbf{1}}_+$ for some integer $n \ge 1$. Then, for any $1/2 > \varepsilon > 0$ and $1/2 > \sigma > 0$, there exists $d \in M_{n+1}(A)^{\mathbf{1}}_+$ such that*

$$f_\varepsilon(a) \le d \le 1 \text{ and } \omega(d) < \sigma. \tag{e8.83}$$

Proof For $n = 1$, this follows immediately from Lemma 8.9.

Assume that the lemma holds for $n \ge 1$.

Let $0 \le e_A \le 1$ be a strictly positive element of A. Fix $1/2 > \varepsilon > 0$. Choose $\eta = \varepsilon/4(n + 2)$ and $\sigma_0 := \sigma/2(n + 2)$. We will apply Lemma 8.9 with $a_j \in \mathrm{Her}(e_A)$ $(1 \le j \le n + 1)$, η as above, and σ_0 (in place of σ).

By Lemma 8.9, there is $c_1 \in \mathrm{Her}(f_\eta(a_1))^{\mathbf{1}}_+$, a unitary $U_1 \in M_2(\widetilde{A})$, and $b_1' = U_1^*\mathrm{diag}(0, c_1)U_1$ such that

$$\mathrm{diag}(0, f_\varepsilon(a_1)) \le b_1', \tag{e8.84}$$

$$d_\tau(f_\varepsilon(a_1)) \le d_\tau(b_1') \le d_\tau(f_\eta(a_1)) \text{ for all } \tau \in \overline{QT(A)}^w, \tag{e8.85}$$

$$\omega(b_1) < \sigma_0 \text{ and } U_1^*\mathrm{diag}(e_A, g_{\eta/2}(a_1))U_1 \in B_1, \tag{e8.86}$$

where $B_1 := (\mathrm{Her}(b_1')^\perp \cap \mathrm{Her}(\mathrm{diag}(a_1, e_A)))_+$. Put

$$V_1 = \mathrm{diag}(U_1, \overbrace{1_{\widetilde{A}}, ..., 1_{\widetilde{A}}}^{n-1}), \ \alpha_2 = V_1^*\mathrm{diag}(0, 0, a_2, ..., a_{n+1})V_1 \text{ and} \tag{e8.87}$$

$$b_1 = V_1^*(0, c_1, 0, ..., 0)V_1. \tag{e8.88}$$

Define $C_1 = \mathrm{Her}(V_1^*(e_A, 0, e_A, ..., e_A)V_1)$. Then

$$b_1 \in C_1^\perp \text{ and } C_1 \cong M_{n+1}(A). \tag{e8.89}$$

In C_1, we may write $\alpha_2 = \mathrm{diag}(0, a_2, a_3, ..., a_{n+1})$ (the number of possible nonzero elements is now reduced to n).

By the induction assumption, there is $b_2 \in C_1$ with $0 \le b_2 \le 1$ such that

$$f_\varepsilon(\alpha_2) \le b_2 \text{ and } \omega(b_2) < \sigma_0. \tag{e8.90}$$

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Define $d := b_1 + b_2$. Note that $b_1 \perp b_2$. Then

$$f_\varepsilon(a) = \mathrm{diag}(0, f_\varepsilon(a_1), f_\varepsilon(a_2), \ldots, f_\varepsilon(a_{n+1})) \le b_1 + b_2, \tag{e8.91}$$

and (by (2) of 4.4)

$$\omega(d) \le \omega(b_1) + \omega(b_2) < 2\sigma_0 < \sigma. \tag{e8.92}$$

This completes the induction. The lemma follows. ∎

Theorem 8.11 *Let A be a σ-unital simple C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \ne \emptyset$. Suppose that A has strict comparison and property (TM). Then A has T-tracial approximate oscillation zero.*

Proof Choose $e \in \mathrm{Ped}(A)_+\backslash\{0\}$ and $A_1 = \mathrm{Her}(e)$. Then $\mathrm{Ped}(A_1) = A_1$. To prove the theorem, without loss of generality, we may assume that $A = \mathrm{Ped}(A)$. By Theorem 8.7, Γ is surjective.

Fix $a \in \mathrm{Ped}(A \otimes \mathcal{K})^{\mathbf{1}}_+$. We claim that $\mathrm{Her}(a)$ has a T-tracial approximate identity $\{d_n\}$ such that $\lim_{n\to\infty} \omega(d_n) = 0$.

Put $B = \mathrm{Her}(a)$. Then $\mathrm{Ped}(B) = B$ and $B \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Therefore, to simplify notation, without loss of generality, we may also assume that $a \in A$.

Let $\varepsilon > 0$ and let $n \in \mathbb{N}$ such that $1/n < (\varepsilon/8)^2$. Since A has property (TM), there is a c.p.c. order zero map $\phi : M_{n+1} \to \mathrm{Her}(a)$ such that

$$\|a - \phi(1_{n+1})a\|_{2,\overline{QT(A)}^w} < (\varepsilon/8)^3. \tag{e8.93}$$

By Proposition 8.3, let $C = \mathrm{Her}(\phi(1_{n+1})) \cong M_{n+1}((\mathrm{Her}(\phi(e_{1,1})))$. Write

$$\phi(1_{n+1}) = \mathrm{diag}(c, c, \ldots, c) \in M_{n+1}(\mathrm{Her}(\phi(e_{1,1}))). \tag{e8.94}$$

Choose $0 < \eta < (\varepsilon/16)^2$. Put $c_n = \mathrm{diag}(0, c, c, \ldots, c)$. It follows from Lemma 8.10 that there exists $d \in C^{\mathbf{1}}_+$ such that

$$f_\eta(c_n) \le d \le 1 \text{ and } \omega(d) < 1/2^n. \tag{e8.95}$$

Hence,

$$0 \le (c_n - 2\eta)_+(1-d)(c_n - 2\eta)_+ \le (c_n - 2\eta)_+(1 - f_\eta(c_n))(c_n - 2\eta)_+ = 0. \tag{e8.96}$$

Hence, $d(c_n - 2\eta)_+ = (c_n - 2\eta)_+ = (c_n - 2\eta)_+ d$. It follows that (we now working in a commutative C^*-subalgebra)

$$(1-d)^2 \le (1 - (c_n - 2\eta)_+)^2. \tag{e8.97}$$

Note that

$$\|\phi(1_{n+1}) - c_n\|_{2,\overline{QT(A)}^w} < \frac{1}{n+1}. \tag{e8.98}$$

Then (see also (2.16)), by (e8.97), (e8.98), and (e8.93),

$$\begin{aligned}
\|a - da\|^{2/3}_{2,\overline{QT(A)}^w} &= (\sup_{\tau\in\overline{QT(A)}^w}\{\tau(a(1-d)^2a)\})^{1/3} \\
&\le \sup_{\tau\in\overline{QT(A)}^w}\{\tau(a(1-(c_n-2\eta)_+)^2a)\})^{1/3} = \|a - (c_n - 2\eta)_+a\|^{2/3}_{2,\overline{QT(A)}^w} \\
&\le \|a - c_n a\|^{2/3}_{2,\overline{QT(A)}^w} + \|(c_n - (c_n - 2\eta)_+)a\|^{2/3}_{2,\overline{QT(A)}^w}
\end{aligned}$$

(e8.99) $$< \|a - \phi(1_{n+1})a\|^{2/3}_{2,\overline{QT(A)}^w} + \left(\frac{1}{n+1}\right)^{2/3} + (2\eta)^{2/3}$$

(e8.100) $$< (\varepsilon/8)^2 + (\varepsilon/8)^2 + (\varepsilon/8)^2 = 3(\varepsilon/8)^2.$$

Since $\omega(d) < 1/2^n$, this proves the claim. The theorem then follows from the claim and Lemma 5.6. ■

9 Stable rank one

Let A be a C^*-algebra and $n \in \mathbb{N}$. Recall that we view $M_n(A)$ as a C^*-subalgebra of $M_{n+1}(A)$ in the canonical way, i.e., $M_n(A)$ is the upper left block of $M_{n+1}(A)$.

Recall an element $a = (a_{i,j})_{n\times n}$ in $M_n(A)$ is called upper (resp. lower) triangular, if $a_{i,j} = 0$ whenever $i < j$ (resp. $i > j$), and a is called strictly upper (resp. lower) triangular, if $a_{i,j} = 0$ whenever $i \le j$ (resp. $i \ge j$).

The following proposition is a generalization of an elementary fact in linear algebra.

Proposition 9.1 *Let A be a C^*-algebra such that $A \subset \overline{GL(\widetilde{A})}$. Then, for any $n \in \mathbb{N}$, any $a \in M_n(A)$, and any $\varepsilon > 0$, there is an upper triangular matrix $x \in M_n(A)$ and a lower triangular matrix $y \in M_n(A)$ such that $a \approx_\varepsilon xy$.*

Proof We prove this by induction on n. For $n = 1$, let $a \in M_n(A) = A$ and $\varepsilon > 0$. By the existence of approximate identity, there is $e \in A_+$ such that $a \approx_\varepsilon ae$. Note that a and e are triangular matrices in $M_n(A)$. Thus, the proposition holds for $n = 1$.

Assume the proposition holds for $n \ge 1$. Let $a = \sum_{i,j=1}^{n+1} a_{i,j} \otimes e_{i,j} \in A \otimes M_{n+1}$, where $a_{i,j} \in A$ and $\{e_{i,j}\}$ is the matrix units of M_{n+1}, $i, j = 1, \ldots, n+1$. Let $\varepsilon > 0$. Since $A \subset \overline{GL(\widetilde{A})}$, there is $\tilde{a} \in GL(\widetilde{A})$ such that

$$a_{n+1,n+1} \approx_{\varepsilon/2} \tilde{a}.$$

In what follows in this proof, 1 is the identity of $\widetilde{A}$ and 1_{n+1} is the identity of $M_{n+1}(\widetilde{A})$.

Let $b^{(0)} := \sum_{i=1}^{n} a_{i,n+1}\tilde{a}^{-1} \otimes e_{i,n+1}$ and $c^{(0)} := \sum_{j=1}^{n} \tilde{a}^{-1}a_{n+1,j} \otimes e_{n+1,j}$. Then $b^{(0)}$ and $c^{(0)}$ are nilpotents. Put

(e9.1) $$a' := a + (\tilde{a} - a_{n+1,n+1}) \otimes e_{n+1,n+1},$$

(e9.2) $$b := 1_{n+1} - b^{(0)} \quad \text{and} \quad c := 1_{n+1} - c^{(0)}.$$

Let $s := ba'c$. Note that a', b, c, s are in $M_{n+1}(\widetilde{A})$. Let $a'_{i,j}$ (resp. $b_{i,j}, c_{i,j}, s_{i,j}$) be the (i,j)th entry of a' (resp. b, c, s), $1 \le i, j \le n+1$. Note that

(e9.3) $$s_{i,j} = \sum_{m=1}^{n+1}\left(\sum_{k=1}^{n+1} b_{i,k}a'_{k,m}c_{m,j}\right) \quad (1 \le i, j \le n+1).$$

If $A = \widetilde{A}$, then $s_{i,j} \in A$. Otherwise, denote by $\pi : M_{n+1}(\widetilde{A}) \to M_{n+1}$ the quotient map. Then

$$\pi(b) = \pi(1_{n+1}) = \pi(c) \text{ and } \pi(a') = \pi(\tilde{a} \otimes e_{n+1,n+1}).$$

It follows that $\pi(s) = \pi(\tilde{a} \otimes e_{n+1,n+1})$. Thus,

(e9.4) $$s_{i,j} \in A \quad (1 \le i, j \le n).$$

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Note that $b_{n+1,k} = 0$ for $1 \le k \le n$ and $c_{m,j} = 0$ for $m \notin \{j, n+1\}$, if $1 \le j \le n$. By (e9.3), we have, for $1 \le j \le n$,

$$s_{n+1,j} = b_{n+1,n+1}a'_{n+1,j}c_{j,j} + b_{n+1,n+1}a'_{n+1,n+1}c_{n+1,j} \tag{e9.5}$$

$$= a_{n+1,j} + \tilde{a} \cdot (-\tilde{a}^{-1}a_{n+1,j}) = 0. \tag{e9.6}$$

If $1 \le i \le n$, then $b_{i,k} = 0$ for $k \notin \{i, n+1\}$ and $c_{m,n+1} = 0$ for $1 \le m \le n$. By (e9.3), we compute

$$s_{i,n+1} = b_{i,i}a'_{i,n+1}c_{n+1,n+1} + b_{i,n+1}a'_{n+1,n+1}c_{n+1,n+1} \tag{e9.7}$$

$$= a_{i,n+1} + (-a_{i,n+1}\tilde{a}^{-1}) \cdot \tilde{a} = 0. \tag{e9.8}$$

We also have

$$s_{n+1,n+1} = \sum_{m=1}^{n+1}\left(\sum_{k=1}^{n+1} b_{n+1,k}a'_{k,m}c_{m,n+1}\right) = b_{n+1,n+1}a'_{n+1,n+1}c_{n+1,n+1} = \tilde{a}. \tag{e9.9}$$

Therefore (e9.4), (e9.6), (e9.8), and (e9.9) show that

$$ba'c = d + \tilde{a} \otimes e_{n+1,n+1},$$

where $d \in M_n(A)$. Note that b and c are invertible in $M_{n+1}(\widetilde{A})$, as both $b^{(0)}$ and $c^{(0)}$ are nilpotents. Let $\varepsilon_1 = \frac{\varepsilon}{4(1+\|b^{-1}\|\cdot\|c^{-1}\|)}$. By our assumption, there is an upper triangular matrix x_1 and a lower triangular matrix y_1 in $M_n(A)$ such that

$$d \approx_{\varepsilon_1} x_1 y_1. \tag{e9.10}$$

Let $e \in M_{n+1}(A)_+^{\mathbf{1}}$ be a diagonal matrix such that $a \approx_{\varepsilon/4} eae$.

Note that $b^{-1} = 1_{n+1} + b^{(0)}$ and $x := eb^{-1}(x_1 + \tilde{a} \otimes e_{n+1,n+1})$ are upper triangular matrix in $M_{n+1}(A)$. Similarly, $c^{-1} = 1_{n+1} + c^{(0)}$ is a lower triangular matrix in $M_{n+1}(\widetilde{A})$, and

$$y := (y_1 + 1 \otimes 1_{n+1,n+1})c^{-1}e$$

is a lower triangular matrix in $M_{n+1}(A)$. Then

$$a \approx_{\varepsilon/4} eae \approx_{\varepsilon/2} ea'e = eb^{-1}ba'cc^{-1}e = eb^{-1}(d + \tilde{a} \otimes e_{n+1,n+1})c^{-1}e \tag{e9.11}$$

$$\approx_{\varepsilon/4} eb^{-1}(x_1y_1 + \tilde{a} \otimes e_{n+1,n+1})c^{-1}e \tag{e9.12}$$

$$= eb^{-1}(x_1 + \tilde{a} \otimes e_{n+1,n+1}) \cdot (y_1 + 1 \otimes e_{n+1,n+1})c^{-1}e = xy. \tag{e9.13}$$

Thus, the proposition holds for $n + 1$. By induction, the proposition holds. ∎

Proposition 9.2 *Let A be a C^*-algebra such that $A \subset \overline{GL(\widetilde{A})}$ and let $n \in \mathbb{N}$. Then, for any $a \in M_n(A)$ and any $\varepsilon > 0$, there is a strictly upper triangular matrix $x \in M_{n+1}(A)$ and a strictly lower triangular matrix $y \in M_{n+1}(A)$ such that $a \approx_\varepsilon xy$.*

In particular, any element in $M_n(A)$ can be approximated in norm by product of two nilpotent elements in $M_{n+1}(A)$.

Proof By Proposition 9.1, there is an upper triangular matrix $x_1 \in M_n(A)$ and a lower triangular matrix $y_1 \in M_n(A)$ such that $a \approx_\varepsilon x_1y_1$. Let $v = \sum_{i=1}^n 1_{\widetilde{A}} \otimes e_{i,i+1} \in M_{n+1}(\widetilde{A})$. Then $x = x_1v \in M_{n+1}(A)$ is a strict upper triangular matrix and $y = v^*y_1 \in M_{n+1}(A)$ is

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a strict lower triangular matrix, and $xy = x_1vv^*y_1 = x_1y_1 \approx_\varepsilon a$. (Recall that we identify $M_n(A)$ with the upper left $n \times n$ corner of $M_{n+1}(A)$.)

The last part of the proposition follows from the fact that strictly triangular matrices are nilpotents. ∎

Lemma 9.3 *Let A be a σ-unital algebraically simple non-elementary C^*-algebra with $QT(A) \neq \emptyset$ which has strict comparison. Suppose that A also has the property (TM). Let $a \in A$. If there are $b_1, b_2 \in A_+\setminus\{0\}$ such that $a^*a + aa^*, b_1, b_2$ are mutually orthogonal, then, for any $\varepsilon > 0$, there are two nilpotents $x, y \in A$ such that $\|a - xy\| < \varepsilon$.*

Proof Let $B = l^\infty(A)/I_{\overline{QT(A)}^w}$. Recall that $\Pi : l^\infty(A) \to B$ is the quotient map and $\iota : A \to l^\infty(A)$ is the canonical embedding. Denote $\bar\iota := \Pi \circ \iota$. Fix $a \in A$. Without loss of generality, we may assume $\|a\| \le 1$. Put $a_0 = a^*a + aa^*$. Assume that there are $b_1, b_2 \in A_+$ such that $0 = b_1b_2 = ab_1 = b_1a = ab_2 = b_2a$. Let $\varepsilon > 0$. Since A is simple and non-elementary, one can choose $n \in \mathbb{N}$ such that

$$1/n < \inf\{d_\tau(b_i) : \tau \in \overline{QT(A)}^w\}, \quad i = 1, 2.$$

Since A has property (TM), by Theorem 8.11, A has T-tracial approximate oscillation zero. Then, by Theorem 6.6, B has stable rank one. Also, by the last part of Remark 8.2, for each $m \in \mathbb{N}$, there is a c.p.c. order zero map $\phi_m : M_n \to \mathrm{Her}(a_0)$ such that

$$\|a - \phi_m(1_n)a\|_{2,\overline{QT(A)}^w} < 1/m. \tag{e9.14}$$

Let $\phi : M_n \to l^\infty(A)$ be the map induced by $\{\phi_m\}_{m\in\mathbb{N}}$ and $\bar\phi := \Pi \circ \phi$. Then (e9.14) shows that

$$\bar\phi(1_n)\bar\iota(a) = \bar\iota(a). \tag{e9.15}$$

Denote by $\{e_{i,j} : 1 \le i, j \le n\}$ a system of matrix units for M_n and $\{e_{i,j} : 1 \le i, j \le n+1\}$ an expanded system of matrix units for M_{n+1}. In particular, we view M_n generated by $\{e_{i,j} : 1 \le i, j \le 1\}$ as a C^*-subalgebra of M_{n+1} generated by $\{e_{i,j} : 1 \le i, j \le n+1\}$.

Since A has strict comparison, and for all $m \in \mathbb{N}$,

$$\sup\{d_\tau(\phi_m(e_{1,1})) : \tau \in \overline{QT(A)}^w\} \le 1/n < \inf\{d_\tau(b_2) : \tau \in \overline{QT(A)}^w\}, \tag{e9.16}$$

we have $\phi_m(e_{1,1}) \lesssim b_2$ for all $m \in \mathbb{N}$. By [40, Proposition 2.4(iv)], there are $v_m \in A$

$$v_m^*v_m = (\phi_m(e_{1,1}) - 1/m)_+ \quad \text{and} \quad v_mv_m^* \in \mathrm{Her}_A(b_2) \quad (m \in \mathbb{N}) \tag{e9.17}$$

(see (e3.5)) and (e3.6)). Note that

$$\|v_m\|^2 = \|v_m^*v_m\| = \|(\phi_m(e_{1,1}) - 1/m)_+\| \le 1.$$

Let $v = \{v_1, v_2, \dots\} \in l^\infty(A)$. Since $\|(\phi_m(e_{1,1}) - 1/m)_+ - \phi_m(e_{1,1})\| \le 1/m$ $(m \in \mathbb{N})$, we have

$$\Pi(v^*v) = \bar\phi(e_{1,1}). \tag{e9.18}$$

The facts that $\phi_m(1_n) \in \mathrm{Her}(a_0)$, $\mathrm{Her}(a_0) \perp \mathrm{Her}(b_2)$, and $v_mv_m^* \in \mathrm{Her}(b_2)$ show that $\phi_m(1_n) \perp v_mv_m^*$ for all $m \in \mathbb{N}$. Hence,

$$\Pi(vv^*)\bar\phi(1_n) = 0. \tag{e9.19}$$

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Let $h : C_0((0,1]) \otimes M_n \to B$ be the homomorphism defined by $h(\imath \otimes e_{i,j}) = \bar{\phi}(e_{i,j})$ $(1 \le i, j \le n)$. Extend $\widetilde{h} : C_0((0,1]) \otimes M_{n+1} \to B$ by $\widetilde{h}(\imath \otimes e_{i,j}) = h(\imath \otimes e_{i,j})$ and $\widetilde{h}(\imath \otimes e_{1,n+1}) = v^*$. By (e9.18) and (e9.19), $\widetilde{h}$ is indeed a homomorphism. Define $\widetilde{\phi}(e_{i,j}) = \widetilde{h}(\imath \otimes e_{i,j})$ for $1 \le i, j \le n+1$.

As we view M_n as a C^*-subalgebra of M_{n+1}, $\widetilde{\phi}$ is an extension of $\bar{\phi}$, i.e., $\widetilde{\phi}|_{M_n} = \bar{\phi}$. By Proposition 8.3,

$$\mathrm{Her}_B(\widetilde{\phi}(1_{n+1})) \cong \mathrm{Her}_B(\widetilde{\phi}(e_{1,1})) \otimes M_{n+1} = \mathrm{Her}_B(\bar{\phi}(e_{1,1})) \otimes M_{n+1}, \tag{e9.20}$$

and

$$\mathrm{Her}_B(\bar{\phi}(1_n)) \cong \mathrm{Her}_B(\bar{\phi}(e_{1,1})) \otimes M_n. \tag{e9.21}$$

Moreover, as $\{e_{i,j} : 1 \le i, j \le n\} \subset \{e_{i,j} : 1 \le i, j \le n+1\}$, we also write

$$\mathrm{Her}_B(\bar{\phi}(1_n)) \subset \mathrm{Her}_B(\widetilde{\phi}(1_{n+1})).$$

Since B has stable rank one, by [5, Corollary 3.6], $\mathrm{Her}_B(\widetilde{\phi}(1_{n+1}))$ also has stable rank one. Note, by (e9.15), $\bar{\imath}(a) \in \mathrm{Her}_B(\bar{\phi}(1_n)) \cong \mathrm{Her}_B(\bar{\phi}(e_{1,1})) \otimes M_n$, Then, by Proposition 9.2, there are nilpotents

$$x_1, y_1 \in \mathrm{Her}_B(\widetilde{\phi}(1_{n+1})) \cong \mathrm{Her}_B(\bar{\phi}(e_{1,1})) \otimes M_{n+1} \text{ such that } \|\bar{\imath}(a) - x_1 y_1\| < \varepsilon/8.$$

Recall that $\Pi(vv^* + \phi(1_n)) = \widetilde{\phi}(1_{n+1})$. Thus,

$$\mathrm{Her}_B(\widetilde{\phi}(1_{n+1})) = \Pi(\mathrm{Her}_{l^\infty(A)}(vv^* + \phi(1_n))). \tag{e9.22}$$

Also, note that $(vv^* + \phi(1_n)) \perp \iota(b_1)$. Thus,

$$\mathrm{Her}_{l^\infty(A)}(vv^* + \phi(1_n)) \subset \{\iota(b_1)\}^\perp. \tag{e9.23}$$

By (e9.22) and the fact that nilpotents can be lifted (see [29, Theorem 6.7]), there are nilpotents $x_2, y_2 \in \mathrm{Her}_{l^\infty(A)}(vv^* + \phi(1_n))$ such that

$$\Pi(x_2) = x_1, \quad \Pi(y_2) = y_1. \tag{e9.24}$$

It follows from (e9.23) that we also have

$$x_2 \perp \iota(b_1), \quad \text{and} \quad y_2 \perp \iota(b_1). \tag{e9.25}$$

Since $\|\bar{\imath}(a) - x_1 y_1\| < \varepsilon/8$, there is $\bar{z} \in I_{\overline{QT(A)}^w}$ such that

$$\iota(a) - x_2 y_2 \approx_{\varepsilon/8} \bar{z}.$$

Note that $\iota(a) - x_2 y_2 \in \{\iota(b_1)\}^\perp$. Hence, there is $d \in \{\iota(b_1)\}^\perp$ such that

$$\iota(a) - x_2 y_2 \approx_{\varepsilon/8} d(\iota(a) - x_2 y_2) d \approx_{\varepsilon/8} d\bar{z}d.$$

Let

$$z := d\bar{z}d \in \{\iota(b_1)\}^\perp \cap I_{\overline{QT(A)}^w}. \tag{e9.26}$$

Then

$$\|\iota(a) - (x_2 y_2 + z)\| < \varepsilon/4.$$

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Choose $\delta > 0$ such that $\|zf_\delta(|z|) - z\| < \varepsilon/8$. Then

$$\|\iota(a) - (x_2y_2 + zf_\delta(|z|))\| < \varepsilon/2. \tag{e9.27}$$

Write $z = \{z_1, z_2, \ldots\}$ with $z_i \perp b_1$ $(i \in \mathbb{N})$. Since $z \in I_{\overline{QT(A)}^w}$, there is $i \in \mathbb{N}$ such that (note that the first inequality of the following always holds)

$$\sup_{\tau \in \overline{QT(A)}^w} \{d_\tau(f_{\delta/2}(|z_i|))\} \le \frac{4}{\delta} \sup_{\tau \in \overline{QT(A)}^w} \tau(|z_i|) < \inf_{\tau \in \overline{QT(A)}^w} \{d_\tau(b_1)\}. \tag{e9.28}$$

Since A has strict comparison, $f_{\delta/2}(|z_i|) \lesssim b_1$. By [40, Proposition 2.4(iv)], there is $r \in A$ such that

$$r^*r = f_\delta(|z_i|) \quad \text{and} \quad rr^* \in \mathrm{Her}(b_1). \tag{e9.29}$$

Write $x_2 = \{x_{2,j}\}_{j \in \mathbb{N}}$ and $y_2 = \{y_{2,j}\}_{j \in \mathbb{N}}$. By (e9.25), $x_{2,i} \perp b_1$ and $y_{2,i} \perp b_1$. Together with (e9.29), we have

$$x_{2,i}r = r^*x_{2,i} = y_{2,i}r = r^*y_{2,i} = 0. \tag{e9.30}$$

Thus,

$$x_{2,i}y_{2,i} + z_if_\delta(|z_i|) = x_{2,i}y_{2,i} + z_ir^*r = (x_{2,i} + z_ir^*)(y_{2,i} + r). \tag{e9.31}$$

Since x_2 and y_2 are nilpotents, so are $x_{2,i}, y_{2,i}$. By (e9.26) and (e9.29), $r^*z_i = 0$. Hence,

$$(z_ir^*)^2 = z_ir^*z_ir^* = 0.$$

By (e9.29), $r^2 = 0$. By (e9.30),

$$(z_ir^*)x_{2,i} = 0 \ \text{ and } \ y_{2,i}r = 0.$$

Let $\alpha_1 := x_{2,i}, \alpha_2 := z_ir^*, \beta_1 := y_{2,i}$, and $\beta_2 := r$. Then the last paragraph shows that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all nilpotents, and $\alpha_2\alpha_1 = \beta_1\beta_2 = 0$. Then it is standard to conclude that $x := \alpha_1 + \alpha_2$ and $y := \beta_1 + \beta_2$ are nilpotents (see the proof of Claim 1 in the proof of [16, Lemma 5.6]).

By (e9.27) and (e9.31),

$$a \approx_{\varepsilon/2} x_{2,i}y_{2,i} + z_if_\delta(|z_i|) = (x_{2,i} + z_ir^*)(y_{2,i} + r) = xy. \tag{e9.32}$$

The lemma follows. ∎

Theorem 9.4 *Let A be a σ-unital simple C^*-algebra with $\widetilde{QT}(A)\backslash\{0\} \neq \emptyset$. If A has strict comparison and has T-tracial approximate oscillation zero, then A has stable rank one.*

Proof We may assume that A is non-elementary. There are two cases.

Case 1. A has a nonzero projection p.

Set $A_1 := pAp$. Then A_1 is unital, simple, has nonempty $QT(A_1)$, and has strict comparison as well as T-tracial approximate oscillation zero. Hence, $l^\infty(A_1)/I_{\overline{QT(A_1)}}$ has stable rank one (see Theorem 6.6). Let $a \in A_1$ be a non-invertible element, and let $\varepsilon > 0$. Since A_1 is simple and finite, by [39, Proposition 3.2 and Lemma 3.5], there is a unitary $u \in U(A_1)$, an element $\bar{a} \in A_1$, and a positive element $b \in (A_1)_+\backslash\{0\}$ such that

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$\|a - \bar{a}\| < \varepsilon/4$ and $b(u\bar{a}) = (u\bar{a})b = 0$. Note that $\mathrm{Her}(b)$ is also infinite-dimensional. Hence, there are two nonzero orthogonal positive elements $b_1, b_2 \in \mathrm{Her}(b)$.

By Theorem 8.5, A has property (TM). We then apply Lemma 9.3 to obtain two nilpotent elements x, y such that

$$u\bar{a} \approx_{\varepsilon/4} xy.$$

Let $\delta > 0$ be such that

$$xy \approx_{\varepsilon/4} (x+\delta)(y+\delta).$$

Note that $x+\delta$ and $y+\delta$ are invertible since x, y are nilpotents. Then that

$$a \approx_{\varepsilon/4} u^*u\bar{a} \approx_{\varepsilon/2} u^*(x+\delta)(y+\delta)$$

shows that a can be approximated by an invertible element $u^*(x+\delta)(y+\delta)$ up to the tolerance ε. Hence, A_1 has stable rank one. It follows that A also has stable rank one.

Case 2. A has no nonzero projections. By Theorem 7.11, the canonical map Γ is surjective. Choose $e \in (A \otimes \mathcal{K})_+$ with $0 \le e \le 1$ such that $\widehat{[e]}$ is continuous on $\widetilde{QT}(A)$. By Theorem 2.19, $C = \mathrm{Her}(e)$ has continuous scale. By Brown's stable isomorphism theorem [4], $C \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Therefore, it suffices to show that C has stable rank one (see [36, Theorem 6.4]). Hence, without loss of generality, we may assume that A has continuous scale (and $QT(A) \neq \emptyset$).

Let A_1 be a σ-unital hereditary C^*-subalgebra of A. Let $a \in A_1^{\mathbf{1}}$ and let $\varepsilon > 0$. Let $a_1 := a^*a + aa^*$. Then there is $\delta > 0$ such that

$$\|a - f_\delta(a_1)af_\delta(a_1)\| < \varepsilon/2.$$

Let $\bar{a} := f_\delta(a_1)af_\delta(a_1)$. Since A_1 has no nonzero projections, we may assume that $[0, \delta] \subset \mathrm{sp}(a_1)$. Let $g \in C_0((0,1])_+$ with $\mathrm{supp}(g) \subset [\delta/4, \delta/2]$, then

$$b := g(a_1) \neq 0 \quad \text{and} \quad b\bar{a} = \bar{a}b = 0.$$

Let us consider $A_2 = \mathrm{Her}_{A_1}(f_{\delta/8}(a_1))$. Note that A_2 is simple, $A_2 = \mathrm{Ped}(A_2)$ and $QT(A_2) \neq \emptyset$ and has strict comparison. Moreover, by Proposition 5.4, A_2 has T-tracial approximate oscillation zero. Hence, by Theorem 6.6, $l^\infty(A_2)/I_{\overline{QT(A_2)}^w}$ has stable rank one. Note that $\bar{a}, b \in A_2$. Note also that, since A is non-elementary, $\mathrm{Her}_{A_2}(b)$ is infinite-dimensional. It follows that there are $b_1, b_2 \in \mathrm{Her}_{A_2}(b)_+\backslash\{0\}$ such that $b_1 \perp b_2$. Since $\bar{a}^*\bar{a} + \bar{a}\bar{a}^*, b_1, b_2$ are mutually orthogonal, applying Lemma 9.3, we get two nilpotents $x, y \in A_2 \subset A_1$ such that $\|\bar{a} - xy\| < \varepsilon/2$. It follows that $\|a - xy\| < \varepsilon$.

Therefore, for any σ-unital hereditary C^*-subalgebra $A_1 \subset A$, any $a \in A_1$, and any $\varepsilon > 0$, there are nilpotents $x, y \in A_1$ such that $\|a - xy\| < \varepsilon$. Together with the facts that A is projectionless and assumed to have continuous scale, applying [14, Theorem 6.4], we conclude that A has stable rank one. ∎

The proof of Theorem 1.1

Proof For (1) ⇒ (2), applying Theorem 7.11, we know that Γ is surjective. Then (2) follows from Theorem 9.4.

Both (2) ⇒ (3) and (2) ⇒ (4) are obvious. That (3) ⇒ (2) follows from [1].

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For (4) ⇒ (1), we apply Theorem 5.10. That (1) ⇔ (5) follows from Theorems 8.5 and 8.11. This ends the proof of Theorem 1.1. ∎

Note that the separability condition is only used in the implication of (3) ⇒ (2). We learned that the following is also obtained by S. Geffen and W. Winter.

Corollary 9.5 *Let A be a σ-unital stably finite simple C^*-algebra of real rank zero which has strict comparison. Then Γ is surjective and A has stable rank one.*

Proof Let $p \in A$ be a nonzero projection and $B = pAp$. It suffices to show the statement holds for B. Note that B is unital and stably finite. By the paragraph right after the proof of Theorem 3.3 of [3], B has a 2-quasitrace (see also 1.3(III) of [2]). By Proposition 5.8, A has tracial approximate oscillation zero. Then the corollary follows from Theorem 9.4. ∎

Let A be a separable simple C^*-algebra with $\widetilde{QT}(A)\setminus\{0\} \neq \emptyset$. Let $e \in \mathrm{Ped}(A)_+\setminus\{0\}$. Recall that $T_e = \{\tau \in \widetilde{QT}(A) : \tau(e) = 1\}$ is a compact convex set which is also a basis for the cone $\widetilde{QT}(A)$.

Corollary 9.6 *Let A be a σ-unital simple C^*-algebra with $QT(A) \neq \emptyset$ which has strict comparison. Suppose that, for some $e \in \mathrm{Ped}(A)_+^{\mathbf{1}}\setminus\{0\}$, $\partial_e(T_e)$ has countably many points. Then Γ is surjective, A has stable rank one, property (TM) and T-tracial approximate oscillation zero.*

Proof It follows from Theorem 5.9 that A has norm approximate oscillation zero. Thus, the corollary follows from Theorem 9.4 immediately. ∎

The following is perhaps known, but we are not able to locate it in the literature.

Proposition 9.7 *Let A be a separable C^*-algebra which has local finite nuclear dimension. Then every hereditary C^*-subalgebra $B \subset A$ also has local finite nuclear dimension.*

Proof Let B be a hereditary C^*-subalgebra of A. Let $\varepsilon > 0$ and $\mathcal{F} \subset B$ be a finite subset. To simplify notation, without loss of generality, we may assume that $\mathcal{F} \subset B^{\mathbf{1}}$ and there is $e_B \in B_+^{\mathbf{1}}$ such that $e_B x = x e_B = x$ for all $x \in \mathcal{F}$.

Choose $\delta > 0$ as in Lemma 3.3 of [10] associated with $\varepsilon/4$ (in place of ε) and $\sigma = \varepsilon/4$. We may assume that $\delta < \varepsilon/4$.

Since A has local finite nuclear dimension, there is a C^*-subalgebra $C \subset A$ with finite nuclear dimension, say k ($k \in \mathbb{N}\cup\{0\}$), such that

$$x \in_{\delta/2} C \text{ for all } x \in \mathcal{F} \cup \{e_B\}. \tag{e9.33}$$

Choose $d \in C_+^{\mathbf{1}}$ such that $\|e_B - d\| < \delta$. Then, by Lemma 3.3 of [10], there is a partial isometry $w \in A^{**}$ such that

$$ww^* f_{\varepsilon/4}(d) = f_{\varepsilon/4}(d)ww^* = f_{\varepsilon/4}(d), \; w^*cw \in \mathrm{Her}(e_B) \subset B \text{ and} \tag{e9.34}$$

$$\|w^*cw - c\| < (\varepsilon/4)\|c\| \text{ for all } c \in \overline{f_{\varepsilon/4}(d)Af_{\varepsilon/4}(d)}. \tag{e9.35}$$

Set $C_1 = w^*\overline{f_{\varepsilon/4}(d)Cf_{\varepsilon/4}(d)}w \subset B$. By Proposition 2.5 of [46], $\overline{f_{\varepsilon/4}(d)Cf_{\varepsilon/4}(d)}$ has nuclear dimension k. Since $C_1 \cong \overline{f_{\varepsilon/4}(d)Cf_{\varepsilon/4}(d)}$, C_1 has nuclear dimension k. We then estimate that

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$$x \in_{\varepsilon} C_1 \text{ for all } x \in \mathcal{F}. \tag{e9.36}$$

Thus, B has local finite nuclear dimension. ∎

As in [42], we have the following (note that, by [17], since A is exact, $\widetilde{T}(A) = \widetilde{QT}(A)$).

Corollary 9.8 *Let A be a separable exact simple C^*-algebra with $\widetilde{T}(A)\backslash\{0\} \neq \emptyset$. Suppose that A has strict comparison, T-tracial approximate oscillation zero and has local finite nuclear dimension. Then $A \otimes \mathcal{Z} \cong A$.*

Proof Choose $e \in \mathrm{Ped}(A)_+^{\mathbf{1}}\backslash\{0\}$ and $B = \mathrm{Her}(e)$, Then $\mathrm{Ped}(B) = B$. It suffices to show (see Corollary 3.1 of [44]) that B is $\mathcal{Z}$-stable. Note that B has strict comparison, and, by Proposition 9.7, has local finite nuclear dimension. Since B also has T-tracial approximate oscillation zero (see Proposition 5.4), by Theorem 7.11, Γ is surjective. It follows that B has m-almost divisibility for some m (in fact m can be zero). By [43, Theorem 8.5(iii)], B is $\mathcal{Z}$-stable. ∎

Remark 9.9 At least in the unital case, the condition that A has local finite nuclear dimension in Corollary 9.8 can be further weakened to that A is amenable and has weak tracial finite nuclear dimension (see Definition 8.1 and Theorem 8.3 of [24]).

Remark 9.10 (1) Note that, in Theorem 1.1 and Corollary 9.6, we do not assume that A is amenable or even exact.

(2) Usually, the condition that A has strict comparison implies that A has at least one densely defined nonzero 2-quasitrace. However, one may insist that the condition that A has strict comparison means that, if A has no nonzero 2-quasitraces, A is purely infinite. In that case, the assumption in Theorem 1.1 (part of the assumption of Corollary 9.8) may be replaced by that A is finite and has strict comparison.

(3) On the other hand, if one assumes that $\mathrm{Cu}(A)$ is almost unperforated and A is not purely infinite, then, by [41], A has strict comparison (in the usual sense) (see also Remark 2.5 and Proposition 4.9 of [16]). Conversely, if A has strict comparison (in usual sense), $\mathrm{Cu}(A)$ is almost unperforated. Therefore, if one prefers not to mention 2-quasitraces in Theorem 1.1, one could use the condition that A is finite and $\mathrm{Cu}(A)$ is almost unperforated.

(4) If A is a unital stably finite simple C^*-algebra, then, by [40, Theorem 6.1] (see also [8, Corollary 4.7] and [2, Theorem II.2.2]), A has at least one nontrivial 2-quasitrace. So, in the unital case, we may assume that A is stably finite instead assume that A has a nontrivial 2-quasitrace. This also works for the case that A is not unital but $K_0(A)_+ \neq \{0\}$. However, when A is stably projectionless, the situation is somewhat different. Nevertheless, we may proceed this as follows:

We assume that A is a separable simple C^*-algebra. Recall that an element $a \in \mathrm{Ped}(A)_+$ is infinite, if there are nonzero elements $b, c \in \mathrm{Ped}(A)_+$ such that $bc = 0, b + c \lesssim c$ and $c \lesssim a$.A is said to be finite, if there are no infinite elements in $\mathrm{Ped}(A)_+$.A is said to be stably finite, if $M_n(A)$ is finite for each n (see Definition 1.1 of [27] and Definition 4.7 of [16], for example).

Choose $e \in \mathrm{Ped}(A)_+\backslash\{0\}$ and consider $B = \mathrm{Her}(e)$. Without loss of generality, we may well assume that $A = B$ for convenience. Define $K_0^*(A)$ (using $W(A)$ not

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Cu(A)) exactly the same way as in Section 4 of [8]. Note that Lemma 4.1 of [8] holds automatically with the definition above. The same definition of order there (before Proposition 4.2 of [8]) also works in this case. In other words, so defined $K_0^*(A)$ is a (directed) ordered group and the stably finiteness ensures that $K_0^*(A)$ is not zero. Since A is simple, Proposition 4.2 of [8] still holds. We now return to the paragraph right after the proof Theorem 3.3 of [3]. Note that $(K_0^*(A), K_0^*(A)_+, [e])$ is a scaled ordered group which has a state, and which gives a dimension function. By [2, II.2.2], the dimension function just mentioned gives a 2-quasitrace on A. Therefore, we may replace the condition that $\widetilde{QT}(A)\backslash\{0\} \neq \emptyset$ by the condition that A is stably finite (recall that we assume that A is simple) in Theorem 1.1 (see also [18] for the case that A is exact).

(5) One may notice that the condition that A has strict comparison and Γ is surjective implies that there is an isomorphism $\Gamma^{\sim} : \mathrm{Cu}(A) \to V(A) \sqcup (\mathrm{LAff}_+(\widetilde{QT}(A))\backslash\{0\})$ (see also Definition 2.13).

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