Can. J. Math.Vol. 45 (5), 1993 pp. 1080–1093

ESSENTIALLY COMMUTING TOEPLITZ OPERATORS WITH HARMONIC SYMBOLS

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ABSTRACT In this paper we characterize the bounded harmonic functions f and g on the unit disk for which the Toeplitz operators T_f and T_g defined on the Bergman space of the unit disk are essentially commuting

1. **Introduction.** We use dA to denote the area measure on the open unit disk $\mathbf{D}$ in the complex plane, normalized so that $\mathbf{D}$ has measure 1. The Bergman space L^2_a is the set of analytic functions on $\mathbf{D}$ which are in $L^2(\mathbf{D}, dA)$. Since the Bergman space L^2_a is a closed subspace of $L^2(\mathbf{D}, dA)$ there is an orthogonal projection P from $L^2(\mathbf{D}, dA)$ onto L^2_a. For $f \in L^\infty(\mathbf{D}, dA)$, the Toeplitz operator with symbol f, denoted by T_f, is the operator on L^2_a defined by $T_f h = P(fh)$, $h \in L^2_a$. Recently Sheldon Axler and Željko Čučković characterized the bounded (complex-valued) harmonic functions on $\mathbf{D}$ for which the Toeplitz operators T_f and T_g commute. In [4] they proved that for bounded harmonic functions f and g on $\mathbf{D}$, the Toeplitz operators T_f and T_g commute if and only if (i) both f and g are analytic on $\mathbf{D}$, or (ii) both $\bar{f}$ and $\bar{g}$ are analytic on $\mathbf{D}$, or (iii) there are constants a and b, not both 0, such that $af + bg$ is constant on $\mathbf{D}$. In this paper we will characterize the bounded harmonic functions f and g on $\mathbf{D}$ for which the Toeplitz operators T_f and T_g are essentially commuting, that is, $T_fT_g - T_gT_f$ is compact: we will prove that this is the case if and only if f and g satisfy one of the above statements (i), (ii), or (iii) "locally". To make this precise we will need to introduce more notation.

Let H^∞ denote the algebra of bounded analytic functions on $\mathbf{D}$, and for $f \in H^\infty$ let $\|f\|_\infty$ denote the supremum of $|f|$ on $\mathbf{D}$. The maximal ideal space of H^∞, denoted by $\mathcal{M}$, is the set of multiplicative linear functionals on H^∞. Endowed with the weak-star topology it inherits as a subspace of the dual of H^∞, the space $\mathcal{M}$ is a compact Hausdorff space. Identifying a point in $\mathbf{D}$ with the functional of evaluation at this point, we may regard the disk $\mathbf{D}$ as a subset of $\mathcal{M}$. Carleson's Corona theorem says then that $\mathbf{D}$ is a dense subset of $\mathcal{M}$. Using the Gelfand transform we regard every function in H^∞ as a continuous function on $\mathcal{M}$. Furthermore, every bounded harmonic function on $\mathbf{D}$ can be uniquely extended to a continuous function on $\mathcal{M}$ ([6], Lemma 4.4). We will use the same notation to denote a bounded analytic or harmonic function on $\mathbf{D}$ and its continuous extension to $\mathcal{M}$.

The author was partially supported by grants from the University of Montana and the Montana University System

Received by the editors December 27, 1991

AMS subject classification Primary 47B05, 47B35

If $m_1, m_2 \in \mathcal{M}$, then the pseudohyperbolic distance $d(m_1, m_2)$ between m_1 and m_2 is defined by $d(m_1, m_2) = \sup\{|m_2(h)| : h \in H^\infty, \|h\|_\infty \leq 1, \text{ and } m_1(h) = 0\}$. For $m \in \mathcal{M}$, the Gleason part containing m, denoted by $\mathcal{P}(m)$, is defined by $\mathcal{P}(m) = \{m_1 \in \mathcal{M} : d(m, m_1) < 1\}$. The Gleason parts form a partition of $\mathcal{M}$, and for each $z \in \mathbf{D}$, the Gleason part containing z equals $\mathbf{D}$.

For $\lambda \in \mathbf{D}$, the Möbius function $\varphi_\lambda : \mathbf{D} \longrightarrow \mathbf{D}$ is defined by $\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$, $z \in \mathbf{D}$. It is easy to see that for $z, \lambda \in \mathbf{D}$ we have $d(z, \lambda) = |\varphi_\lambda(z)|$. For each $m \in \mathcal{M}$, K. Hoffman [6] constructed a canonical map L_m from $\mathbf{D}$ into $\mathcal{M}$. This map L_m is defined by taking a net $\{\lambda_\alpha\}$ in $\mathbf{D}$ such that $\lambda_\alpha \longrightarrow m$, and defining $L_m z(h) = \lim_\alpha h \circ \varphi_{\lambda_\alpha}(z)$, for $z \in \mathbf{D}$ and $h \in H^\infty$. The mapping L_m maps $\mathbf{D}$ onto the Gleason part $\mathcal{P}(m)$, and is one-to-one if $\mathcal{P}(m)$ consists of more than one point. For each $f \in H^\infty$ and $m \in \mathcal{M}$ the composition $f \circ L_m$ is in H^∞; in fact, we will see that if f is a continuous function on $\mathcal{M}$ and $\{\lambda_\alpha\}$ is a net in $\mathbf{D}$ tending to m in $\mathcal{M}$, then $f \circ \varphi_{\lambda_\alpha} \longrightarrow f \circ L_m$ uniformly on compact subsets of $\mathbf{D}$.

If $\mathcal{P}$ is a Gleason part in $\mathcal{M}$ and f is a function on $\mathcal{M}$, then we will say that f is analytic on $\mathcal{P}$ if the function $f \circ L_m$ is analytic on $\mathbf{D}$, where $m \in \mathcal{P}$. Note that this definition does not depend on the chosen representative m from $\mathcal{P}$: if $m_1 \in \mathcal{P}$ is distinct from m, then $m_1 = L_m(\lambda)$ for some $\lambda \in \mathbf{D}$, and by Schwarz's lemma there is a unimodular constant ζ for which $L_{m_1}(w) = L_m \circ \varphi_\lambda(\zeta w)$, for all $w \in \mathbf{D}$, so that $f \circ L_{m_1}$ is analytic on $\mathbf{D}$ if and only if $f \circ L_m$ is.

Our main result is the following theorem.

THEOREM 1. *Let f and g be bounded harmonic functions on $\mathbf{D}$. Then the following statements are quivalent:*

(a) T_f and T_g are essentially commuting;

(b) on each Gleason part $\mathcal{P}$ of $\mathcal{M}$ except $\mathbf{D}$:

(i) both f and g are analytic on $\mathcal{P}$, or

(ii) both $\bar{f}$ and $\bar{g}$ are analytic on $\mathcal{P}$, or

(iii) there are constants a and b, not both 0, such that $af + bg$ is constant on $\mathcal{P}$;

(c) $\lim_{|\lambda| \to 1^-} (1 - |\lambda|^2)^2 \{\frac{\partial f}{\partial \bar{z}}(\lambda) \frac{\partial g}{\partial z}(\lambda) - \frac{\partial f}{\partial z}(\lambda) \frac{\partial g}{\partial \bar{z}}(\lambda)\} = 0.$

After some preliminaries in the next section, we will prove the implications "(a) $\Rightarrow$ (c)" and "(c) $\Rightarrow$ (b)" of the above theorem in §3. To prove the implication "(b) $\Rightarrow$ (a)" we will introduce Hankel operators in §4, and obtain a sufficient condition for compactness of certain operators involving these Hankel operators. In §5 we will then complete the proof of the above theorem. For the operators of §4 we will get more descriptions of compactness in §6. In the final section of the paper we discuss some open questions.

2. **Preliminaries.** For u in $L^1(\mathbf{D}, dA)$ define its Berezin transform $\tilde{u}$ by

$$\tilde{u}(z) = \int_{\mathbf{D}} u\big(\varphi_z(w)\big)\, dA(w), \quad z \in \mathbf{D}.$$

Note that an integrable function on the unit disk satisfies the so called "area version of the invariant mean value property" if and only if it is invariant under the Berezin transform. In particular, $\tilde{u} = u$ if u is harmonic on $\mathbf{D}$.

For a continuous function u on $\mathbf{D}$ let $\mathcal{R}(u)$ denote the radialization of the function u, that is,

$$\mathcal{R}(u)(z) = \int_0^{2\pi} u(ze^{i\theta})\frac{d\theta}{2\pi}, \quad z \in \mathbf{D}.$$

Note that if u is a harmonic function on $\mathbf{D}$ and $z \in \mathbf{D}$, then the radialization of $u \circ \varphi_z$ on $\mathbf{D}$ is constant and thus extends to a continuous function on $\overline{\mathbf{D}}$. Axler and Čučković [4] discovered that these properties characterize harmonicity; more precisely, they obtained the following lemma.

LEMMA 2. *Suppose $u \in L^1(\mathbf{D}, dA)$ is continuous on $\mathbf{D}$. Then u is harmonic if and only if $\tilde{u} = u$ and for each $z \in \mathbf{D}$ the function $\mathcal{R}(u \circ \varphi_z)$ extends continuously to $\overline{\mathbf{D}}$.*

For an analytic function f on $\mathbf{D}$, the Bloch norm $\|f\|_{\mathcal{B}}$ is defined by

$$\|f\|_{\mathcal{B}} = \sup\{(1-|z|^2)|f'(z)| : z \in \mathbf{D}\}.$$

If $1 < p < \infty$ and $\| \; \|_p$ denotes the usual p-norm on $L^p(\mathbf{D}, dA)$, then there exists a finite positive constant C_p such that for every analytic function f on $\mathbf{D}$

$$C_p^{-1}\|f\|_{\mathcal{B}} \leq \sup\{\|f \circ \varphi_\lambda - f(\lambda)\|_p : \lambda \in \mathbf{D}\} \leq C_p\|f\|_{\mathcal{B}}.$$

(See [2]). The BMOA norm $\|f\|_{\mathrm{BMOA}}$ of an analytic function f on $\mathbf{D}$ is defined by

$$\|f\|_{\mathrm{BMOA}} = \sup\{\|f \circ \varphi_\lambda - f(\lambda)\|_{H^2} : \lambda \in \mathbf{D}\},$$

where $\| \; \|_{H^2}$ denotes the usual Hardy space H^2-norm. An analytic function f on $\mathbf{D}$ is called a *Bloch function* if $\|f\|_{\mathcal{B}} < \infty$ and a BMOA function if $\|f\|_{\mathrm{BMOA}} < \infty$. Since the Hardy space H^2-norm is larger than the Bergman space norm $\| \; \|_2$, it is clear that every BMOA function is a Bloch function.

PROPOSITION 3. *Let f and g be analytic functions on $\mathbf{D}$. If $f + \bar{g}$ is bounded on $\mathbf{D}$, then both f and g are in* BMOA.

PROOF. Put $u = f + \bar{g}$. By the mean value property the functions $f - f(0)$ and $\bar{g} - \bar{g}(0)$ are orthogonal on the circle $\{z \in \mathbf{C} : |z| = r\}$ with $0 < r < 1$, so that

$$\int_0^{2\pi} |f(re^{i\theta}) - f(0)|^2 \frac{d\theta}{2\pi} + \int_0^{2\pi} |\bar{g}(re^{i\theta}) - \bar{g}(0)|^2 \frac{d\theta}{2\pi} = \int_0^{2\pi} |u(re^{i\theta}) - u(0)|^2 \frac{d\theta}{2\pi}.$$

We conclude that

$$\|f - f(0)\|_{H^2}^2 + \|g - g(0)\|_{H^2}^2 = \|u - u(0)\|_{H^2}^2.$$

For $\lambda \in \mathbf{D}$ replace f by $f \circ \varphi_\lambda$ and g by $g \circ \varphi_\lambda$ to obtain

$$\|f \circ \varphi_\lambda - f(\lambda)\|_{H^2}^2 + \|g \circ \varphi_\lambda - g(\lambda)\|_{H^2}^2 = \|u \circ \varphi_\lambda - u(\lambda)\|_{H^2}^2 \leq 4\|u\|_\infty^2.$$

Thus both f and g are in BMOA. ∎

Axler and Čučković showed in [4] that if f and g are in H^2, then the radialization $\mathcal{R}(f\bar{g})$ extends to a continuous function on $\overline{\mathbf{D}}$. Using the Möbius-invariance of BMOA their result yields the following lemma.

LEMMA 4. *If $f, g \in$ BMOA, then $\mathcal{R}(f \circ \varphi_z \bar{g} \circ \varphi_z)$ extends to a continuous function on $\overline{\mathbf{D}}$, for all $z \in \mathbf{D}$.*

We will need to use some elementary properties of reproducing kernels. Let $\langle\, , \rangle$ denote the usual inner product in $L^2(\mathbf{D}, dA)$, that is, $\langle f, g\rangle = \int_{\mathbf{D}} f(z)\overline{g(z)}\, dA(z)$ for $f, g \in L^2(\mathbf{D}, dA)$. For $\lambda \in \mathbf{D}$ the functional of evaluation at λ is bounded on L^2_a, and thus there exists a unique function K_λ in L^2_a, called the reproducing kernel at λ, for which $\langle f, K_\lambda\rangle = f(\lambda)$, for all $f \in L^2_a$. It is easily verified that $K_\lambda(z) = (1 - \bar{\lambda} z)^{-2}$. Using the reproducing property we have $\|K_\lambda\|_2^2 = K_\lambda(\lambda) = (1 - |\lambda|^2)^{-2}$. We will write k_λ for the normalized reproducing kernel, that is, $k_\lambda = (1 - |\lambda|^2)K_\lambda$. If $h \in L^2_a$, then it is easily verified that $P(\bar{h}k_\lambda) = \bar{h}(\lambda)k_\lambda$, for each $\lambda \in \mathbf{D}$. In particular, $P(\bar{h}) \equiv \bar{h}(0)$, for every $h \in L^2_a$.

Note that $k_\lambda = -\varphi'_\lambda$. Thus for an integrable function h on $\mathbf{D}$ we have the following change-of-variable formula:

$$\int_{\mathbf{D}} h|k_\lambda|^2\, dA = \int_{\mathbf{D}} h \circ \varphi_\lambda\, dA.$$

In particular, $\tilde{u}(\lambda) = \langle uk_\lambda, k_\lambda\rangle$, for every $u \in L^1(\mathbf{D}, dA)$.

We will also need the following well-known result; for completeness we include a proof.

LEMMA 5. *If f is a continuous function on $\mathcal{M}$, $m \in \mathcal{M}$, and $\{\lambda_\alpha\}$ is a net in $\mathbf{D}$ converging to m in $\mathcal{M}$, then $f \circ \varphi_{\lambda_\alpha} \to f \circ L_m$ uniformly on each compact subset of $\mathbf{D}$.*

PROOF. If f is a continuous function on $\mathcal{M}$, then f can be uniformly approximated by functions of the form $g_1\bar{h}_1 + \cdots + g_n\bar{h}_n$, where $g_1, \ldots, g_n, h_1, \ldots, h_n \in H^\infty$, and n is a positive integer (by the Stone-Weierstrass theorem). Thus it suffices to prove the lemma for the case where $f \in H^\infty$. Without loss of generality we may assume that also $\|f\|_\infty < 1$. By the definition of L_m we have $f \circ \varphi_{\lambda_\alpha} \to f \circ L_m$ pointwise on $\mathbf{D}$, so it suffices to show that the family $\{f \circ \varphi_\lambda : \lambda \in \mathbf{D}\}$ is equicontinuous on each set $r\mathbf{D}$, for $0 < r < 1$. For $z, w \in \mathbf{D}$ we have $|f(z) - f(w)| = |1 - \overline{f(w)}f(z)|\, \big|\varphi_{f(w)}\big(f(z)\big)\big| \leq 2d(z, w)$. Replacing f by $f \circ \varphi_\lambda$ we get $|(f \circ \varphi_\lambda)(z) - (f \circ \varphi_\lambda)(w)| \leq 2d(z, w) \leq \frac{2}{1-r^2}|z - w|$, for $z, w \in r\mathbf{D}$, proving the equicontinuity of $\{f \circ \varphi_\lambda : \lambda \in \mathbf{D}\}$ on $r\mathbf{D}$. ∎

3. Towards the Proof of Theorem 1.

In this section we will prove the implications "(a) ⇒ (c)" and "(c) ⇒ (b)" of Theorem 1.

Suppose f and g are bounded harmonic functions on $\mathbf{D}$. Let f_1, f_2, g_1, and g_2 be analytic functions on $\mathbf{D}$ such that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$.

(a) ⇒ (c): Suppose that T_f and T_g are essentially commuting, that is $T_fT_g - T_gT_f$ is compact. Then:

$$\begin{aligned}\langle T_fT_gk_\lambda, k_\lambda\rangle &= \langle fP(gk_\lambda), k_\lambda\rangle = \big\langle (f_1 + \bar{f}_2)\big(g_1k_\lambda + \bar{g}_2(\lambda)k_\lambda\big), k_\lambda\big\rangle \\ &= \langle f_1, g_1k_\lambda, k_\lambda\rangle + \langle \bar{f}_2g_1k_\lambda, k_\lambda\rangle + \bar{g}_2(\lambda)\langle f_1k_\lambda, k_\lambda\rangle + \bar{g}_2(\lambda)\langle \bar{f}_2k_\lambda, k_\lambda\rangle \\ &= f_1(\lambda)g_1(\lambda) + \langle \bar{f}_2g_1k_\lambda, k_\lambda\rangle + f_1(\lambda)\bar{g}_2(\lambda) + \bar{f}_2(\lambda)\bar{g}_2(\lambda).\end{aligned}$$

Combining the above result with a similar formula for $\langle T_g T_f k_\lambda, k_\lambda\rangle$ (obtained by interchanging the f_j's and g_j's) we conclude

$$\langle (T_f T_g - T_g T_f) k_\lambda, k_\lambda\rangle = \tilde{u}(\lambda) - u(\lambda),$$

where $u = \bar{f}_2 g_1 - f_1 \bar{g}_2$. It is easy to show that $k_\lambda \longrightarrow 0$ weakly on L^2_a as $|\lambda| \longrightarrow 1^-$ (see [2]), so that the compactness of $T_f T_g - T_g T_f$ implies that $\|(T_f T_g - T_g T_f)k_\lambda\|_2 \longrightarrow 0$ as $|\lambda| \longrightarrow 1^-$, and thus

$$\tilde{u}(\lambda) - u(\lambda) \longrightarrow 0 \text{ as } |\lambda| \longrightarrow 1^-.$$

It follows with the help of Lemma 5 that both functions $f \circ L_m$ and $g \circ L_m$ are harmonic on $\mathbf{D}$, so that there exist analytic functions F_1, F_2, G_1, and G_2 on $\mathbf{D}$ for which $f \circ L_m = F_1 + \bar{F}_2$ and $g \circ L_m = G_1 + \bar{G}_2$. Put $v = \bar{F}_2 G_1 - F_1 \bar{G}_2$. We will show that $\tilde{v} = v$. By Lemma 5, $f \circ \varphi_{\lambda_\alpha} - f(\lambda_\alpha) \longrightarrow f \circ L_m - f \circ L_m(0)$ uniformly on compact subsets of $\mathbf{D}$. Because f is bounded, we have $f \circ \varphi_{\lambda_\alpha} - f(\lambda_\alpha) \longrightarrow f \circ L_m - f \circ L_m(0)$ in $L^2(\mathbf{D}, dA)$. Using the boundedness of the Bergman projection P it follows that $f_1 \circ \varphi_{\lambda_\alpha} - f_1(\lambda_\alpha) = P\big(f \circ \varphi_{\lambda_\alpha} - f(\lambda_\alpha)\big) \longrightarrow P\big(f \circ L_m - f \circ L_m(0)\big) = F_1 - F_1(0)$ in L^2_a. Also, $f_2 \circ \varphi_{\lambda_\alpha} - f_2(\lambda_\alpha) = P\big(\bar{f} \circ \varphi_{\lambda_\alpha} - \bar{f}(\lambda_\alpha)\big) \longrightarrow P\big(\bar{f} \circ L_m - \bar{f} \circ L_m(0)\big) = F_2 - F_2(0)$ in L^2_a. Likewise $g_j \circ \varphi_{\lambda_\alpha} - g_j(\lambda_\alpha) \longrightarrow G_j - G_j(0)$ in L^2_a, for $j = 1, 2$. Then:

$$\begin{aligned}
\tilde{v}(0) - v(0) &= \int_{\mathbf{D}} \Big(\big(\bar{F}_2 - \bar{F}_2(0)\big)\big(G_1 - G_1(0)\big) - \big(F_1 - F_1(0)\big)\big(\bar{G}_2 - \bar{G}_2(0)\big)\Big)\, dA \\
&= \lim_\alpha \int_{\mathbf{D}} \Big\{ \big(\bar{f}_2 \circ \varphi_{\lambda_\alpha} - \bar{f}_2(\lambda_\alpha)\big)\big(g_1 \circ \varphi_{\lambda_\alpha} - g_1(\lambda_\alpha)\big) \\
&\qquad - \big(\bar{g}_2 \circ \varphi_{\lambda_\alpha} - \bar{g}_2(\lambda_\alpha)\big)\big(f_1 \circ \varphi_{\lambda_\alpha} - f_1(\lambda_\alpha)\big)\Big\}\, dA \\
&= \lim_\alpha \tilde{u}(\lambda_\alpha) - u(\lambda_\alpha) = 0.
\end{aligned}$$

Fix $z \in \mathbf{D}$. It is an easy consquence of Schwarz's lemma that there is a unimodular constant ζ such that $L_{m(z)}(w) = L_m \circ \varphi_z(\zeta w)$ for $w \in \mathbf{D}$. Then $f \circ L_{m(z)} = \Phi_1 + \bar{\Phi}_2$ and $g \circ L_{m(z)} = \Psi_1 + \bar{\Psi}_2$, with $\Phi_j(w) = F_j \circ \varphi_z(\zeta w)$ and $\Psi_j(w) = G_j \circ \varphi_z(\zeta w)$. By the above paragraph,

$$\int_{\mathbf{D}} \Big(\big(\bar{\Phi}_2 - \bar{\Phi}_2(0)\big)\big(\Psi_1 - \Psi_1(0)\big) - \big(\Phi_1 - \Phi_1(0)\big)\big(\bar{\Psi}_2 - \bar{\Psi}_2(0)\big)\Big)\, dA = 0.$$

By a simple change-of-variable this implies

$$\begin{aligned}
\int_{\mathbf{D}} \Big(&\big(\bar{F}_2 \circ \varphi_z(w) - \bar{F}_2(z)\big)\big(G_1 \circ \varphi_z(w) - G_1(z)\big) \\
&- \big(F_1 \circ \varphi_z(w) - F_1(z)\big)\big(\bar{G}_2 \circ \varphi_z(w) - \bar{G}_2(z)\big)\Big)\, dA(w) = 0,
\end{aligned}$$

that is, $\tilde{v}(z) - v(z) = 0$.

By Proposition 3 the functions F_2 and G_1 are in BMOA, so that by Lemma 4 the function $\mathcal{R}(F_2 \circ \varphi_z \bar{G}_2 \circ \varphi_z)$ extends continuously to $\overline{\mathbf{D}}$. The same is true for $\mathcal{R}(F_1 \circ \varphi_z \bar{G}_2 \circ \varphi_z)$, and thus $\mathcal{R}(v \circ \varphi_z)$ extends continuously to $\overline{\mathbf{D}}$. By Lemma 2 the function v is harmonic on $\mathbf{D}$. Since v is harmonic, we have $\overline{F_2'} G_1' - F_1' \overline{G_2'} = \frac{\partial^2 v}{\partial \bar{z} \partial z} = 0$ on

$\mathbf{D}$. In particular, $\overline{F_2'}(0)G_1'(0) - F_1'(0)\overline{G_2'}(0) = 0$. Using the inequality $|h'(0)| \leq \sqrt{2}\|h\|_2$, valid for every $h \in L_a^2$ (as is easily verified by using power series), it follows from the convergence $f_j \circ \varphi_{\lambda_\alpha} - f_j(\lambda_\alpha) \longrightarrow F_j - F_j(0)$ in L_a^2 that $(|\lambda_\alpha|^2 - 1)f_j'(\lambda_\alpha) \longrightarrow F_j'(0)$. Likewise $(|\lambda_\alpha|^2 - 1)g_j'(\lambda_\alpha) \longrightarrow G_j'(0)$ and we conclude that $(1 - |\lambda_\alpha|^2)^2\{\overline{f_2'}(\lambda_\alpha)g_1'(\lambda_\alpha) - f_1'(\lambda_\alpha)\overline{g_2'}(\lambda_\alpha)\} \longrightarrow \overline{F_2'}(0)G_1'(0) - F_1'(0)\overline{G_2'}(0) = 0$. Thus (c) holds.

(c) $\Rightarrow$ (b): Suppose that (c) holds. Take $m \in \mathcal{M} \setminus \mathbf{D}$. Let F_1, F_2, G_1, and G_2 be analytic functions on $\mathbf{D}$ such that $f \circ L_m = F_1 + \bar{F}_2$ and $g \circ L_m = G_1 + \bar{G}_2$. We claim that $\overline{F_2'}G_1' = F_1'\overline{G_2'}$ on $\mathbf{D}$. First, picking a net $\{\lambda_\alpha\}$ in $\mathbf{D}$ converging to m in $\mathcal{M}$, it follows as in the previous paragraph that

$$\overline{F_2'}(0)G_1'(0) - F_1'(0)\overline{G_2'}(0) = \lim_\alpha (1 - |\lambda_\alpha|^2)^2\{\overline{f_2'}(\lambda_\alpha)g_1'(\lambda_\alpha) - f_1'(\lambda_\alpha)\overline{g_2'}(\lambda_\alpha)\} = 0.$$

For fixed $z \in \mathbf{D}$ let ζ be a unimodular constant such that $L_{m(z)}(w) = L_m \circ \varphi_z(\zeta w)$, for all $w \in \mathbf{D}$. Then $f \circ L_{m(z)} = \Phi_1 + \bar{\Phi}_2$ and $g \circ L_{m(z)} = \Psi_1 + \bar{\Psi}_2$, with $\Phi_j(w) = F_j \circ \varphi_z(\zeta w)$ and $\Psi_j(w) = G_j \circ \varphi_z(\zeta w)$. By the previous paragraph $\overline{\Phi_2'}(0)\Psi_1'(0) = \Phi_1'(0)\overline{\Psi_2'}(0)$. An easy computation shows that $\Phi_j'(0) = \zeta(|z|^2 - 1)F_j'(z)$ and $\Psi_j'(0) = \zeta(|z|^2 - 1)G_j'(z)$ for $j = 1, 2$. Thus $\overline{F_2'}(z)G_1'(z) = F_1'(z)\overline{G_2'}(z)$, and our claim is verified.

To show that one of statements, (i), (ii), or (iii) in (b) holds we argue as in [4]. If G_1' is identically zero on $\mathbf{D}$, then it follows that either F_1' identically zero on $\mathbf{D}$ (so that F_1 is constant and thus both $\bar{f} \circ L_m$ and $\bar{g} \circ L_m$ are analytic on $\mathbf{D}$) or G_2' is identically zero on $\mathbf{D}$ (in which case $g \circ L_m$ is constant on $\mathbf{D}$ and (iii) holds). Similarly, if G_2' is identically zero on $\mathbf{D}$, then either (i) or (iii) holds. If neither G_1' nor G_2' is identically zero on $\mathbf{D}$, then on the region $\{z \in \mathbf{D} : G_1'(z) \neq 0 \text{ and } G_2'(z) \neq 0\}$ we have $\overline{F_2'/G_2'} = F_1'/G_1'$. Since the complex conjugate of an analytic function on a region is only analytic if the function is constant, we conclude that for some constant c we must have $F_1' = cG_1'$ and $F_2' = \bar{c}G_2'$, and therefore both $F_1 - cG_1$ and $F_2 - \bar{c}G_2$ are constant on $\mathbf{D}$. It follows that $f \circ L_m - c(g \circ L_m)$ is constant on $\mathbf{D}$, and thus (iii) holds.

4. **Hankel operators.** In this section we will introduce Hankel operators, and after showing how they relate to the commutator of Toeplitz operators we will prove a sufficient condition for compactness of a certain operator involving Hankel operators.

For $f \in L^\infty(\mathbf{D}, dA)$, the Hankel operator with symbol f, denoted by H_f, is the operator from L_a^2 into $(L_a^2)^\perp$, the orthogonal complement of L_a^2 in $L^2(\mathbf{D}, dA)$, defined by $H_f h = fh - P(fh)$, $h \in L_a^2$. The following proposition relates the commutator of two Toeplitz operators to these Hankel operators.

PROPOSITION 6. *Let $f, g \in L^\infty(\mathbf{D}, dA)$. Then*

$$T_f T_g - T_g T_f = H_{\bar{g}}^* H_f - H_{\bar{f}}^* H_g.$$

PROOF. Let $h \in L_a^2$. Then

$$\begin{aligned}\langle T_f T_g h, h\rangle &= \langle fP(gh), h\rangle = \langle P(gh), \bar{f}h\rangle \\ &= \langle gh, \bar{f}h\rangle - \langle H_g h, \bar{f}h\rangle = \langle fgh, h\rangle - \langle H_g h, H_{\bar{f}} h\rangle.\end{aligned}$$

https://doi.org/10.4153/CJM-1993-059-2 Published online by Cambridge University Press

Similarly $\langle T_gT_fh, h\rangle = \langle fgh, h\rangle - \langle H_fh, H_gh\rangle$, and it follows that

$$\langle (T_fT_g - T_gT_f)h, h\rangle = \langle H_fh, H_gh\rangle - \langle H_gh, H_fh\rangle = \langle (H_g^*H_f - H_f^*H_g)h, h\rangle,$$

so that $T_fT_g - T_gT_f = H_g^*H_f - H_f^*H_g$ ∎

To complete the proof of Theorem 1 it remains to show that (b) $\Rightarrow$ (a) In order to prove this implication we will have to extend our definition of Hankel operators

For $f \in L^2(\mathbf{D}, dA)$, the operator H_f is defined by $H_fh = (I - P)(fh)$, for h in H^∞ In [2] Axler proved that for $f \in L_a^2$, the densely defined operator $H_{\bar f}$ is bounded if and only if f is a Bloch function Axler also obtained a characterization for compactness of the operator $H_{\bar f}$ In [7] the author characterized the $f \in L^\infty(\mathbf{D}, dA)$ for which the Hankel operator H_f is compact In the proof of the implication "(b) $\Rightarrow$ (a)" of Theorem 1 we will need the following sufficient condition for compactness of a difference of products of Hankel operators with adjoints of Hankel operators

THEOREM 7 *If f_1, f_2, g_1, and g_2 are Bloch functions on $\mathbf{D}$ and*

$$\int_{\mathbf{D}} \Big| \big(\bar f_2 \circ \varphi_\lambda - \bar f_2(\lambda)\big)\big(g_1 \circ \varphi_\lambda - g_1(\lambda)\big) - \big(\bar g_2 \circ \varphi_\lambda - \bar g_2(\lambda)\big)\big(f_1 \circ \varphi_\lambda - f_1(\lambda)\big)\Big|\, dA \longrightarrow 0, \text{ as } |\lambda| \to 1 ,$$

then $H_{\bar g_1}^* H_{\bar f_2} - H_{\bar f_1}^* H_{\bar g_2}$ is compact

In the proof of the above theorem we will need two lemmas The following lemma is well-known For an elementary proof we refer the reader to [2]

LEMMA 8 *Let $M = \sup_{\lambda\in\mathbf{D}} \int_{\mathbf{D}} \frac{1}{(1-|w|^2)^{3/5}|1-\lambda \bar w|^{6/5}}\, dA(w)$ Then $M < \infty$*

We will need another lemma, which is a one-sided Schur Test

LEMMA 9 *Let F be measurable on $\mathbf{D} \times \mathbf{D}$ If γ is a positive constant such that*

$$\int_{\mathbf{D}} \frac{|F(z,\lambda)|^2}{|1-\lambda\bar z|^2} \frac{1}{(1-|z|^2)^{1/2}}\, dA(z) \le \gamma \frac{1}{(1-|\lambda|^2)^{1/2}}, \quad \textit{for all } \lambda \in \mathbf{D},$$

then the operator S defined on L_a^2 by

$$Sh(\lambda) = \int_{\mathbf{D}} \frac{F(z,\lambda)}{(1-\lambda\bar z)^2} h(z)\, dA(z), \quad h \in L_a^2,$$

is bounded with norm $\|S\|^2 \le \gamma M$

PROOF Let $h \in L_a^2$ Then by Cauchy-Schwarz's Inequality

$$\begin{aligned}
|Sh(\lambda)|^2 &\le \left(\int_{\mathbf{D}} \frac{|F(z,\lambda)|}{|1-\lambda\bar z|} \frac{1}{(1-|z|^2)^{1/4}} \frac{(1-|z|^2)^{1/4}}{|1-\lambda\bar z|} |h(z)|\, dA(z) \right)^2 \\
&\le \int_{\mathbf{D}} \frac{|F(z,\lambda)|^2}{|1-\lambda\bar z|^2} \frac{1}{(1-|z|^2)^{1/2}}\, dA(z) \times \int_{\mathbf{D}} \frac{(1-|z|^2)^{1/2}}{|1-\lambda\bar z|^2} |h(z)|^2\, dA(z) \\
&\le \gamma \frac{1}{(1-|\lambda|^2)^{1/2}} \times \int_{\mathbf{D}} \frac{(1-|z|^2)^{1/2}}{|1-\lambda\bar z|^2} |h(z)|^2\, dA(z)
\end{aligned}$$

Using Fubini's Theorem

$$\int_{\mathbf{D}} |Sh(\lambda)|^2 \, dA(\lambda) \leq \gamma \int_{\mathbf{D}} (1-|z|^2)^{1/2} |h(z)|^2 \int_{\mathbf{D}} \frac{1}{|1-\lambda\bar{z}|^2(1-|\lambda|^2)^{1/2}} \, dA(\lambda)\, dA(z)$$

Now, it is easy to show that

$$\int_{\mathbf{D}} \frac{1}{|1-\lambda\bar{z}|^2(1-|\lambda|^2)^{1/2}} \, dA(\lambda) \leq M \frac{1}{(1-|z|^2)^{1/2}},$$

and the proof is completed ∎

We are now ready to prove Theorem 7

PROOF OF THEOREM 7 Putting

$$R(z,\lambda) = \big(\bar{f}_2(z) - \bar{f}_2(\lambda)\big)\big(g_1(z) - g_1(\lambda)\big) - \big(\bar{g}_2(z) - \bar{g}_2(\lambda)\big)\big(f_1(z) - f_1(\lambda)\big),$$

we have

$$\int_{\mathbf{D}} \Big| R\big(\varphi_\lambda(w), \lambda\big)\Big| \, dA(w) \longrightarrow 0, \text{ as } |\lambda| \longrightarrow 1$$

Let $h \in H^\infty$ and $\lambda \in \mathbf{D}$ Then

$$H^*_{g_1} H_{f_2} h(\lambda) = \langle H^*_{g_1} H_{f_2} h, K_\lambda \rangle = \langle H_{f_2} h, H_{g_1} K_\lambda \rangle = \Big\langle \bar{f}_2 h, \big(\bar{g}_1 - \bar{g}_1(\lambda)\big) K_\lambda \Big\rangle$$

Using the reproducing property of K_λ we also have $\Big\langle h, \big(\bar{g}_1 - \bar{g}_1(\lambda)\big) K_\lambda \Big\rangle = \Big\langle h\big(g_1 - g_1(\lambda)\big), K_\lambda \Big\rangle = 0$ Thus

$$H^*_{g_1} H_{f_2} h(\lambda) = \Big\langle \big(\bar{f}_2 - \bar{f}_2(\lambda)\big) h, \big(\bar{g}_1 - \bar{g}_1(\lambda)\big) K_\lambda \Big\rangle$$

Combining this with a similar formula for $H^*_{f_1} H_{g_2} h(\lambda)$ we obtain

$$(H^*_{g_1} H_f - H^*_{f_1} H_g)h(\lambda) = \int_{\mathbf{D}} R(z,\lambda) h(z) K_\lambda(z) \, dA(z), \text{ for } h \in L^2_a$$

For $0 < r < 1$ define $S_r \; L^2_a \longrightarrow L^2(\mathbf{D}, dA)$ by

$$S_r h(\lambda) = \chi_{r\mathbf{D}}(\lambda) \int_{\mathbf{D}} R(z,\lambda) h(z) K_\lambda(z) \, dA(z), \text{ for } h \in L^2_a$$

We claim that S_r is Hilbert-Schmidt To verify this claim we need to show that the kernel of S_r is square-integrable over $\mathbf{D} \times \mathbf{D}$ By a change-of-variables it suffices to show that the integrals $\int_{\mathbf{D}} \Big| R\big(\varphi_\lambda(w), \lambda\big)\Big|^2 dA(w)$ are uniformly bounded in $\lambda \in \mathbf{D}$ By Cauchy-Schwarz's inequality

$$\Big(\int_{\mathbf{D}} \Big| R\big(\varphi_\lambda(w), \lambda\big)\Big|^2 dA(w) \Big)^{1/2} \leq \|f_2 \circ \varphi_\lambda - f_2(\lambda)\|_4 \, \|g_1 \circ \varphi_\lambda - g_1(\lambda)\|_4 + \|f_1 \circ \varphi_\lambda - f_1(\lambda)\|_4 \, \|g_2 \circ \varphi_\lambda - g_2(\lambda)\|_4,$$

which is uniformly bounded in $\lambda \in \mathbf{D}$, because f_1, f_2, g_1, and g_2 are Bloch functions on $\mathbf{D}$ Now,

$$(H^*_{g_1} H_{f_2} - H^*_{f_1} H_{g_2} - S_r) h(\lambda) = \int_{\mathbf{D}} F(z,\lambda) h(z) K_\lambda(z) \, dA(z), \text{ for } h \in L^2_a,$$

where $F(z,\lambda) = \chi_{\mathbf{D}\setminus r\mathbf{D}}(\lambda)R(z,\lambda)$ By a change-of-variables

$$\int_{\mathbf{D}} \frac{|F(z,\lambda)|^2}{|1-\bar{\lambda}z|^2(1-|z|^2)^{1/2}}\,dA(z)$$
$$= \frac{1}{(1-|\lambda|^2)^{1/2}}\chi_{\mathbf{D}\setminus r\mathbf{D}}(\lambda)\int_{\mathbf{D}} \frac{\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^2}{|1-\bar{\lambda}w|(1-|w|^2)^{1/2}}\,dA(w)$$

An application of Holder's inequality yields

$$\int_{\mathbf{D}} \frac{\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^2}{|1-\bar{\lambda}w|(1-|w|^2)^{1/2}}\,dA(w)$$
$$\leq M\Big(\int_{\mathbf{D}}\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^{12}dA(w)\Big)^{1/6}$$
$$\leq M\Big(\int_{\mathbf{D}}\left|R\big(\varphi_\lambda(w),\lambda\big)\right|dA(w)\Big)^{1/12}\Big(\int_{\mathbf{D}}\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^{23}dA(w)\Big)^{1/12}$$

Using that f_1, f_2, g_1 and g_2 are Bloch functions on $\mathbf{D}$, it follows as above that the integrals $\int_{\mathbf{D}}\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^{23}dA(w)$ are uniformly bounded in $\lambda \in \mathbf{D}$ Thus there exists a finite constant C such that

$$\int_{\mathbf{D}} \frac{\left|R\big(\varphi_\lambda(w),\lambda\big)\right|^2}{|1-\bar{\lambda}w|(1-|w|^2)^{1/2}}\,dA(w) \leq C\Big(\int_{\mathbf{D}}\left|R\big(\varphi_\lambda(w),\lambda\big)\right|dA(w)\Big)^{1/12},$$

for all $\lambda \in \mathbf{D}$ With the help of Lemma 9 we get

$$\|H_{g_1}^*H_f - H_{f_1}^*H_{g_2} - S_r\|^2 \leq CM \sup_{\lambda\in\mathbf{D}\setminus r\mathbf{D}}\Big(\int_{\mathbf{D}}\left|\big(\varphi_\lambda(w),\lambda\big)\right|dA(w)\Big)^{1/12},$$

from which we see that $S_r \longrightarrow H_{g_1}^*H_{f_2} - H_{f_1}^*H_{g_2}$ in operator norm as $r \longrightarrow 1$ Since each S_r is compact we conclude that $H_{g_1}^*H_{f_2} - H_{f_1}^*H_g$ is compact ∎

5 **Completion of the Proof of Theorem 1.** It remains to show that (b) implies (a) Suppose f and g are bounded harmonic functions on $\mathbf{D}$ for which statement (b) in Theorem 1 holds Let f_1, f_2, g_1, and g_2 be analytic functions on $\mathbf{D}$ such that $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$ We claim that

$$\int_{\mathbf{D}}\left|\big(\bar{f}_2\circ\varphi_\lambda - \bar{f}_2(\lambda)\big)\big(g_1\circ\varphi_\lambda - g_1(\lambda)\big) - \big(\bar{g}_2\circ\varphi_\lambda - \bar{g}_2(\lambda)\big)\big(f_1\circ\varphi_\lambda - f_1(\lambda)\big)\right|dA$$
$$\longrightarrow 0, \text{ as } |\lambda| \longrightarrow 1 ,$$

so that by Theorem 7, $H_{g_1}^*H_{f_2} - H_{f_1}^*H_{g_2}$ is compact Using that a Hankel operator with symbol in L_a^2 is the zero operator, we have $H_g^*H_f - H_f^*H_g = H_{g_1}^*H_{f_2} - H_{f_1}^*H_g$, and by Proposition 6 statement (a) follows

To prove our claim let $\{\lambda_n\}$ be a sequence in $\mathbf{D}$ tending to $\partial\mathbf{D}$ Let $\{\lambda_\alpha\}$ be a subnet in $\mathbf{D}$ such that $\lambda_\alpha \longrightarrow m$ in $\mathcal{M}$ Because $\lim_\alpha|\lambda_\alpha| = 1$ we have $m \in \mathcal{M}\setminus\mathbf{D}$ Let F_1, F_2,

G_1, and G_2 be analytic functions on $\mathbf{D}$ such that $f \circ L_m = F_1 + \bar{F}_2$ and $g \circ L_m = G_1 + \bar{G}_2$. As in the proof of the implication "(a) $\Rightarrow$ (c)" we have

$$\lim_{\alpha} \int_{\mathbf{D}} \Big| \big(\bar{f}_2 \circ \varphi_{\lambda_\alpha} - \bar{f}_2(\lambda_\alpha)\big)\big(g_1 \circ \varphi_{\lambda_\alpha} - g_1(\lambda_\alpha)\big) - \big(\bar{g}_2 \circ \varphi_{\lambda_\alpha} - \bar{g}_2(\lambda_\alpha)\big)\big(f_1 \circ \varphi_{\lambda_\alpha} - f_1(\lambda_\alpha)\big) \Big| \, dA$$
$$= \int_{\mathbf{D}} \Big| \big(\bar{F}_2 - \bar{F}_2(0)\big)\big(G_1 - G_1(0)\big) - \big(F_1 - F_1(0)\big)\big(\bar{G}_2 - \bar{G}_2(0)\big) \Big| \, dA.$$

We will be done if we show that $\big(\bar{F}_2 - \bar{F}_2(0)\big)\big(G_1 - G_1(0)\big) - \big(F_1 - F_1(0)\big)\big(\bar{G}_2 - \bar{G}_2(0)\big) = 0$ on $\mathbf{D}$.

If f and g are both analytic on $\mathcal{P}(m)$, then $f \circ L_m$ and $g \circ L_m$ are both analytic on $\mathbf{D}$, and consequently $\bar{F}_2$ and $\bar{G}_2$ are both constant on $\mathbf{D}$, so that the statement follows. Similarly if both $\bar{f}$ and $\bar{g}$ are both analytic on $\mathcal{P}(m)$. If there are constants a, b, not both zero, such that $af + bg$ is constant on $\mathcal{P}(m)$, then $a(f \circ L_m) + b(g \circ L_m)$ is constant on $\mathbf{D}$, and without loss of generality we may assume that for a constant c the function $f \circ L_m - c(g \circ L_m)$ is constant on $\mathbf{D}$. Differentiating with respect to $\bar{z}$ we obtain $\overline{F_2'} = c\overline{G_2'}$, so that $F_2 - \bar{c}G_2$ is constant on $\mathbf{D}$. It follows that $\big(\bar{F}_2 - \bar{F}_2(0)\big)\big(G_1 - G_1(0)\big) = c\big(\bar{G}_2 - \bar{G}_2(0)\big)\big(G_1 - G_1(0)\big)$. Likewise, differentiation with respect to z yields that $F_1 - cG_1$ is constant on $\mathbf{D}$, hence $\big(F_1 - F_1(0)\big)\big(\bar{G}_2 - \bar{G}_2(0)\big) = c\big(\bar{G}_2 - \bar{G}_2(0)\big)\big(G_1 - G_1(0)\big)$ on $\mathbf{D}$. This proves our claim, and completes the proof of Theorem 1. ∎

Recall that an operator S is called *essentially normal* if $SS^* - S^*S$ is compact. If $f \in L^\infty(\mathbf{D}, dA)$, then it is easy to check that $T_f^* = T_{\bar{f}}$, and we obtain the following corollary of Theorem 1.

COROLLARY 10. *Let f be a bounded harmonic function on $\mathbf{D}$. Then the following statements are equivalent:*

(a) T_f is essentially normal;

(b) f maps each part of $\mathcal{M}$ except $\mathbf{D}$ into a line in $\mathbf{C}$;

(c) $\lim_{|\lambda| \to 1^-} (1 - |\lambda|^2)^2 \{ |\frac{\partial f}{\partial \bar{z}}(\lambda)|^2 - |\frac{\partial f}{\partial z}(\lambda)|^2 \} = 0.$

PROOF. Note that if $\mathcal{P}$ is a part of $\mathcal{M}$ except $\mathbf{D}$, then $f(\mathcal{P})$ is part of a line in $\mathbf{C}$ if and only if there are $a, b \in \mathbf{C}$, not both 0, such that $af + b\bar{f}$ is constant on $\mathcal{P}$. ∎

6. **More on Hankel operators.** In this section we will give several descriptions for compactness of the difference of products of certain Hankel operators and their adjoints.

For $\lambda \in \mathbf{D}$ and $0 < r < 1$ we will write $D(\lambda, r)$ for the pseudo-hyperbolic disk $\{z \in \mathbf{D} : d(z, \lambda) < r\}$. The pseudohyperbolic disk $D(\lambda, r)$ is in fact a euclidean disk whose normalized area is $|D(\lambda, r)| = (1 - |\lambda|^2)^2 r^2 / (1 - r^2|\lambda|^2)^2$ (see [5], p. 3). We have the following theorem for compactness of a difference of products of Hankel operators and their adjoints.

THEOREM 11. *Let f_1, f_2, g_1, and g_2 be bounded analytic functions on $\mathbf{D}$, and let $0 < r < 1$. Then the following statements are equivalent:*

(a) $H_{\bar{g}_1}^* H_{\bar{f}_2} - H_{\bar{f}_1}^* H_{\bar{g}_2}$ is compact;

https://doi.org/10.4153/CJM-1993-059-2 Published online by Cambridge University Press

(b) $\lim_{|\lambda|\to 1} \int_{\mathbf{D}} \left|\left(\bar{f}_2 \circ \varphi_\lambda - \bar{f}_2(\lambda)\right)\left(g_1 \circ \varphi_\lambda - g_1(\lambda)\right) - \left(\bar{g}_2 \circ \varphi_\lambda - \bar{g}_2(\lambda)\right) \left(f_1 \circ \varphi_\lambda - f_1(\lambda)\right)\right| dA = 0$;

(c) $\lim_{|\lambda|\to 1} \int_{r\mathbf{D}} \left|\left(\bar{f}_2 \circ \varphi_\lambda - \bar{f}_2(\lambda)\right)\left(g_1 \circ \varphi_\lambda - g_1(\lambda)\right) - \left(\bar{g}_2 \circ \varphi_\lambda - \bar{g}_2(\lambda)\right) \left(f_1 \circ \varphi_\lambda - f_1(\lambda)\right)\right| dA = 0$;

(d) $\lim_{|\lambda|\to 1} \frac{1}{|D(\lambda,r)|} \int_{D(\lambda,r)} \left|\left(\bar{f}_2-\bar{f}_2(\lambda)\right)\left(g_1-g_1(\lambda)\right)-\left(\bar{g}_2-\bar{g}_2(\lambda)\right)\left(f_1-f_1(\lambda)\right)\right| dA = 0$;

(e) $\lim_{|\lambda|\to 1^-} \int_{D(\lambda,r)} |\overline{f_2'}g_1' - f_1'\overline{g_2'}|\, dA = 0$;

(f) $\lim_{|\lambda|\to 1} (1-|\lambda|^2)^2\{\overline{f_2'}(\lambda)g_1'(\lambda) - f_1'(\lambda)\overline{g_2'}(\lambda)\} = 0$.

PROOF. (a) $\Rightarrow$ (f): Let $f = f_1 + \bar{f}_2$ and $g = g_1 + \bar{g}_2$. Then, using Proposition 6, we have $T_fT_g - T_gT_f = H^*_{\bar{g}_1}H_{\bar{f}_2} - H^*_{\bar{f}_1}H_{\bar{g}_2}$, so T_f and T_g are essentially commuting, and by Theorem 1 statement (f) holds.

(f) $\Rightarrow$ (e): This implication follows from the inequality

$$\begin{aligned}
\int_{D(\lambda,r)} |\overline{f_2'}g_1' - f_1'\overline{g_2'}|\, dA
&\leq \sup_{z\in D(\lambda,r)} (1-|z|^2)^2|\overline{f_2'}(z)g_1'(z) - f_1'(z)\overline{g_2'}(z)| \int_{D(\lambda,r)} (1-|z|^2)^{-2}\, dA(z)\\
&= \frac{r^2}{1-r^2} \sup_{z\in D(\lambda,r)} (1-|z|^2)^2|\overline{f_2'}(z)g_1'(z) - f_1'(z)\overline{g_2'}(z)|,
\end{aligned}$$

and the fact that $\sup\{(1-|z|^2) : z\in D(\lambda,r)\} \to 0$ as $|\lambda|\to 1^-$.

(e) $\Rightarrow$ (c): Let $\{\lambda_\alpha\}$ be a net converging to $m \in \mathcal{M}\setminus\mathbf{D}$. Let $F_j = f_j\circ L_m$ and $G_j = g_j \circ L_m$, for $j = 1,2$. By Lemma 5, $f_j\circ\varphi_{\lambda_\alpha} \to F_j$ uniformly on compact subsets of $\mathbf{D}$, and thus $(f_j\circ\varphi_{\lambda_\alpha})' \to F_j'$ uniformly on $r\mathbf{D}$. It follows that

$$\begin{aligned}
\int_{r\mathbf{D}} |\overline{F_2'}(z)G_1'(z) - F_1'(z)\overline{G_2'}(z)|\, dA(z)
&= \lim_\alpha \int_{r\mathbf{D}} |\overline{(f_2\circ\varphi_{\lambda_\alpha})'(z)}(g_1\circ\varphi_{\lambda_\alpha})'(z) - (f_1\circ\varphi_{\lambda_\alpha})'(z)\overline{(g_2\circ\varphi_{\lambda_\alpha})'(z)}|\, dA(z)\\
&= \lim_\alpha \int_{D(\lambda_\alpha,r)} |\overline{f_2'}(w)g_1'(w) - f_1'(w)\overline{g_2'}(w)|\, dA(w) = 0.
\end{aligned}$$

Hence $\overline{F_2'}(z)G_1'(z) - F_1'(z)\overline{G_2'}(z) = 0$ on $r\mathbf{D}$, and as in the proof of the implication (c) $\Rightarrow$ (b) in Theorem 1, it follows that

$$\left(\bar{F}_2(z) - \bar{F}_2(0)\right)\left(G_1(z) - G_1(0)\right) = \left(F_1(z) - F_1(0)\right)\left(\bar{G}_2(z) - \bar{G}_2(0)\right) = 0$$

on $r\mathbf{D}$. Hence

$$\begin{aligned}
\lim_\alpha \int_{r\mathbf{D}} \Big|&\left(\bar{f}_2\circ\varphi_{\lambda_\alpha} - \bar{f}_2(\lambda_\alpha)\right)\left(g_1\circ\varphi_{\lambda_\alpha} - g_1(\lambda_\alpha)\right)\\
&- \left(\bar{g}_2\circ\varphi_{\lambda_\alpha} - \bar{g}_2(\lambda_\alpha)\right)\left(f_1\circ\varphi_{\lambda_\alpha} - f_1(\lambda_\alpha)\right)\Big|\, dA\\
&= \int_{r\mathbf{D}} \Big|\left(\bar{F}_2(z) - \bar{F}_2(0)\right)\left(G_1(z) - G_1(0)\right)\\
&\quad - \left(F_1(z) - F_1(0)\right)\left(\bar{G}_2(z) - \bar{G}_2(0)\right)\Big|\, dA = 0.
\end{aligned}$$

(c) $\Leftrightarrow$ (d): A change-of-variables yields

$$\int_{D(\lambda,r)} \left|\left(\bar{f}_2 - \bar{f}_2(\lambda)\right)\left(g_1 - g_1(\lambda)\right) - \left(\bar{g}_2 - \bar{g}_2(\lambda)\right)\left(f_1 - f_1(\lambda)\right)\right| dA$$
$$= \int_{r\mathbf{D}} \Big|\left(\bar{f}_2 \circ \varphi_\lambda(z) - \bar{f}_2(\lambda)\right)\left(g_1 \circ \varphi_\lambda(z) - g_1(\lambda)\right)$$
$$- \left(\bar{g}_2 \circ \varphi_\lambda(z) - \bar{g}_2(\lambda)\right)\left(f_1 \circ \varphi_\lambda(z) - f_1(\lambda)\right)\Big| \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}z|^4}\, dA(z),$$

and it is easily seen that

$$\frac{(1+r)^2}{r^2(1-r)^2} \leq \frac{1}{|D(\lambda,r)|}\frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda}z|^4} \leq \frac{(1-r)^2}{r^2(1+r)^2},$$

whenever $z \in r\mathbf{D}$.

(c) $\Rightarrow$ (b): Again let $\{\lambda_\alpha\}$ be a net converging to $m \in \mathcal{M} \setminus \mathbf{D}$, and let $F_j = f_j \circ L_m$ and $G_j = g_j \circ L_m$, for $j = 1, 2$. As in the proof of the implication "(e) $\Rightarrow$ (c)" it follows that $\left(\bar{F}_2(z) - \bar{F}_2(0)\right)\left(G_1(z) - G_1(0)\right) - \left(F_1(z) - F_1(0)\right)\left(\bar{G}_2(z) - \bar{G}_2(0)\right) = 0$ on $r\mathbf{D}$. We claim that there is in fact equality on all of $\mathbf{D}$. This is obvious if either G_1 or G_2 is constant on $\mathbf{D}$. If neither G_1 nor G_2 is constant on $\mathbf{D}$, then $\left(\bar{F}_2 - \bar{F}_2(0)\right)/\left(\bar{G}_2 - \bar{G}_2(0)\right) = \left(F_1 - F_1(0)\right)/\left(G_1 - G_1(0)\right)$ on the region $\{z \in r\mathbf{D} : G_1(z) \neq G_1(0) \text{ and } G_2(z) \neq G_2(0)\}$, thus there is a constant such that $F_1 - F_1(0) = c\left(G_1 - G_1(0)\right)$ and $\bar{F}_2 - \bar{F}_2(0) = c\left(\bar{G}_2 - \bar{G}_2(0)\right)$ on $r\mathbf{D}$, and hence on $\mathbf{D}$. This proves the claim. Now it is easy to see that

$$\lim_\alpha \int_{\mathbf{D}} \Big|\left(\bar{f}_2 \circ \varphi_{\lambda_\alpha} - \bar{f}_2(\lambda_\alpha)\right)\left(g_1 \circ \varphi_{\lambda_\alpha} - g_1(\lambda_\alpha)\right)$$
$$- \left(\bar{g}_2 \circ \varphi_{\lambda_\alpha} - \bar{g}_2(\lambda_\alpha)\right)\left(f_1 \circ \varphi_{\lambda_\alpha} - f_1(\lambda_\alpha)\right)\Big| dA$$
$$= \int_{\mathbf{D}} \Big|\left(\bar{F}_2(z) - \bar{F}_2(0)\right)\left(G_1(z) - G_1(0)\right)$$
$$- \left(F_1(z) - F_1(0)\right)\left(\bar{G}_2(z) - \bar{G}_2(0)\right)\Big| dA = 0,$$

proving (b).

(b) $\Rightarrow$ (a). By Theorem 7. ∎

In particular we have the following result.

COROLLARY 12. *Let f and g be bounded analytic functions on $\mathbf{D}$. Then the following statements are equivalent:*

(a) $H_{\bar{g}}^* H_{\bar{f}} - H_{\bar{f}}^* H_{\bar{g}}$ *is compact;*

(b) $\lim_{|\lambda|\to 1^-} \int_{\mathbf{D}} \left|\left(\bar{f}\circ\varphi_\lambda - \bar{f}(\lambda)\right)\left(g\circ\varphi_\lambda - g(\lambda)\right) - \left(\bar{g}\circ\varphi_\lambda - \bar{g}(\lambda)\right)\left(f\circ\varphi_\lambda - f(\lambda)\right)\right| dA = 0;$

(c) $\lim_{|\lambda|\to 1} \int_{r\mathbf{D}} \left|\left(\bar{f}\circ\varphi_\lambda - \bar{f}(\lambda)\right)\left(g\circ\varphi_\lambda - g(\lambda)\right) - \left(\bar{g}\circ\varphi_\lambda - \bar{g}(\lambda)\right)\left(f\circ\varphi_\lambda - f(\lambda)\right)\right| dA = 0;$

(d) $\lim_{|\lambda|\to 1^-} \frac{1}{|D(\lambda,r)|} \int_{D(\lambda,r)} \left|\left(\bar{f} - \bar{f}(\lambda)\right)\left(g - g(\lambda)\right) - \left(\bar{g} - \bar{g}(\lambda)\right)\left(f - f(\lambda)\right)\right| dA = 0;$

(e) $\lim_{|\lambda|\to 1^-} \int_{D(\lambda,r)} |\overline{f'}g' - f'\overline{g'}|\, dA = 0;$

(f) $\lim_{|\lambda|\to 1^-} (1-|\lambda|^2)^2 \{\overline{f'}(\lambda)g'(\lambda) - f'(\lambda)\overline{g'}(\lambda)\} = 0;$

PROOF. Let $f_1 = f_2 = f$ and $g_1 = g_2 = g$ in Theorem 11. ∎

7. **Open questions.** In this section we discuss some questions suggested by the results in the paper.

1. Do Theorem 11 and Corollary 12 hold for Bloch functions instead of H^∞ functions? The problem seems to be the condition on the radializations in Lemma 2. The above question has an affirmative answer if the requirement on the radializations can be dropped from Lemma 2, that is, if for integrable (sufficiently nice) functions u on $\mathbf{D}$ the condition $\tilde{u} = u$ is equivalent to the harmonicity of u. Whether this is indeed the case is an open problem.

2. Is there an analogue for the equivalence of (a) and (b) in Theorem 1 for Toeplitz operators on the Hardy space? Let C denote the algebra of complex-valued continuous functions on the circle $\partial\mathbf{D}$. A subset of the maximal ideal space of $L^\infty(\partial\mathbf{D})$ is called a *support set* for $H^\infty + C$ if it is the closed support of the representing measure for a functional in the maximal ideal space of $H^\infty + C$. The combined results of [3] and [8] characterize the bounded measurable functions f and g on $\partial\mathbf{D}$ for which the Toeplitz operators T_f and T_g have compact semi-commutator $T_fT_g - T_{fg}$; an answer is that for each support set S for $H^\infty + C$ either $\bar{f}$ or g is analytic on S. In [9] Dechao Zheng obtained the analogous result for semi-commutators of Toeplitz with bounded harmonic symbols on the Bergman space L^2_a; he showed that for such f and g the semi-commutator $T_fT_g - T_{fg}$ is compact if and only if for each Gleason part $\mathcal{P}$ except $\mathbf{D}$ either $\bar{f}$ or g is analytic on $\mathcal{P}$. Note that the maximal ideal space of $H^\infty + C$ can be identified with the corona, that is, $\mathcal{M} \setminus \mathbf{D}$. The Gleason parts other than $\mathbf{D}$ seem to play the same role in the Bergman space setting as the support sets for $H^\infty + C$ in the Hardy space setting. These results and our Theorem 1 give support to the conjecture that for f and g in $L^\infty(\partial\mathbf{D})$ the Toeplitz operators T_f and T_g are essentially commuting operators on the Hardy space H^2 if and only if for each support set S for $H^\infty + C$: (i) both f and g are analytic on S, or (ii) both $\bar{f}$ and $\bar{g}$ are analytic on S, or (iii) there are constants a and b, not both 0, such that $af + bg$ is constant on S.

Recall that subset S of the maximal ideal space of $L^\infty(\partial\mathbf{D})$ is called an *anti-symmetric set* for $H^\infty + C$ if every function in $H^\infty + C$ which is real-valued on S is necessarily constant on S. An anti-symmetric set for $H^\infty + C$ is called a *maximal anti-symmetric set* if it is not properly contained in another anti-symmetric set for $H^\infty + C$. It is easily verified that each support set for $H^\infty + C$ is a set of anti-symmetry for $H^\infty + C$, and thus is contained in a maximal set of anti-symmetry for $H^\infty + C$. It is a result of Sheldon Axler ([1], Corollary 7.3) that the Toeplitz operators T_f and T_g are essentially commuting if for each maximal anti-symmetric set S for $H^\infty + C$ one of conditions (i), (ii), and (iii) holds. It is however unknown whether a maximal anti-symmetric set for $H^\infty + C$ is in fact a support set for $H^\infty + C$.

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