4

Analysis in $L^2(\mathbb{R}^d)$

In this chapter we describe basic properties of operators acting on $L^2(\mathbb{R}^d)$. After a preliminary Sect. 4.1, we will study the *Weyl commutation relations* and prove the famous *Stone–von Neumann uniqueness theorem*. Then we define the so-called *x, D-quantization*, with position to the left and the momentum to the right. We will compare it to the *D, x-quantization*, which uses the reverse order of position and momentum. The *Weyl–Wigner quantization*, in some sense superior to the x, D- and D, x-quantizations, will be introduced in Chap. 8, which can be viewed as a continuation of the present chapter.

4.1 Distributions and the Fourier transformation

Throughout this section, $\mathcal{X}$ is a real vector space of dimension d with a Lebesgue measure $\mathrm{d}x$. As in Subsect. 3.6.5, the dual space $\mathcal{X}^{\#}$ is then equipped with a canonical Lebesgue measure, which we denote $\mathrm{d}\xi$. If additionally $\mathcal{X}$ is equipped with a Euclidean structure, we take $\mathrm{d}x$ to be the unique compatible Lebesgue measure (see Subsect. 3.6.5).

4.1.1 Distributions

Let Ω be an open subset of $\mathcal{X}$.

Definition 4.1 *$C_{\mathrm{c}}^{\infty}(\Omega)$ denotes the space of smooth functions compactly supported in Ω. We equip $C_{\mathrm{c}}^{\infty}(\Omega)$ with the usual topology and rename it $\mathcal{D}(\Omega)$. $\mathcal{D}'(\Omega)$ denotes its topological dual. Elements of $\mathcal{D}'(\Omega)$ are called* distributions.

A large class of distributions in $\mathcal{D}'(\Omega)$ is given by functions $f \in L^1_{\mathrm{loc}}(\Omega)$ with the action on $\Phi \in C_{\mathrm{c}}^{\infty}(\Omega)$ given by

$$\langle f|\Phi\rangle := \int f(x)\Phi(x)\mathrm{d}x. \tag{4.1}$$

We will use the integral notation on the r.h.s. of (4.1) also in the case of distributions that do not belong to $L^1_{\mathrm{loc}}(\Omega)$. Here are some examples with $\Omega = \mathcal{X} = \mathbb{R}$:

$$\int \delta(t)\Phi(t)\mathrm{d}t := \Phi(0),$$
$$\int (t \pm \mathrm{i}0)^{\lambda}\Phi(t)\mathrm{d}t := \lim_{\epsilon \searrow 0} \int (t \pm \mathrm{i}\epsilon)^{\lambda}\Phi(t)\mathrm{d}t.$$

https://doi.org/10.1017/9781009290876.005 Published online by Cambridge University Press

4.1.2 Pullback of distributions

Let $\chi : \Omega_1 \to \Omega_2$ be a diffeomorphism between two open sets $\Omega_i \subset \mathbb{R}^d$, $i = 1, 2$.

Definition 4.2 *One defines the* pullback $\chi^\# : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ *by*

$$\int \chi^\# f(x_1)\Phi(x_1)\mathrm{d}x_1 := \int f(x_2)\Phi \circ \chi^{-1}(x_2)|\det \nabla\chi^{-1}(x_2)|\mathrm{d}x_2, \quad \Phi \in \mathcal{D}(\Omega_1).$$

Clearly, if $f \in L^1_{\mathrm{loc}}(\Omega_2)$, then $\chi^\# f(x_1) = f \circ \chi(x_1)$.

The pullback of distributions can be generalized to a large class of transformations between sets of different dimension. Let $\Omega_i \subset \mathbb{R}^{d_i}$, $i = 1, 2$, be two open sets and $\tau : \Omega_1 \to \Omega_2$ a *submersion*, that is, a smooth map whose derivative is everywhere surjective. We can find an open set $\Omega_3 \subset \mathbb{R}^{d_1 - d_2}$ and a diffeomorphism $\chi : \Omega_1 \to \Omega_2 \times \Omega_3$ such that

$$\pi_{\Omega_2} \circ \chi = \tau, \tag{4.2}$$

where π_{Ω_2} is the projection onto Ω_2. We then define the map $\tau^\# : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ as

$$\tau^\# f := \chi^\#(f \otimes 1),$$

where we consider $f \otimes 1$ as an element of $\mathcal{D}'(\Omega_2 \times \Omega_3)$. One can show that $\tau^\#$ is independent on the choice of χ satisfying (4.2).

Definition 4.3 *The map* $\tau^\# : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ *is also called the* pullback of distributions.

In particular, if $f \in L^1_{\mathrm{loc}}(\Omega_2)$, then

$$\int \tau^\# f(x_1)\Phi(x_1)\mathrm{d}x_1 = \int f \circ \tau(x_1)\Phi(x_1)\mathrm{d}x_1. \tag{4.3}$$

We will use the notation of the r.h.s. of (4.3) also for the pullback of distributions that do not belong to $L^1_{\mathrm{loc}}(\Omega)$. For instance,

$$\int \delta(\tau(t))\Phi(t)\mathrm{d}t = \sum_{\tau(s)=0} |\tau'(s)|^{-1}\Phi(s).$$

4.1.3 Schwartz functions and distributions

Definition 4.4 *The space of* Schwartz functions on $\mathcal{X}$ *is defined as*

$$\mathcal{S}(\mathcal{X}) := \left\{\Psi \in C^\infty(\mathcal{X}) \ : \ \int |x^\alpha \nabla_x^\beta \Psi(x)|^2 \mathrm{d}x < \infty, \quad \alpha, \beta \in \mathbb{N}^d\right\}. \tag{4.4}$$

(In the definition we use an identification of $\mathcal{X}$ with $\mathbb{R}^d$. It is clear that $\mathcal{S}(\mathcal{X})$ does not depend on this identification.)

Remark 4.5 *The definition (4.4) is equivalent to*

$$\mathcal{S}(\mathcal{X}) = \left\{\Psi \in C^\infty(\mathcal{X}) \ : \ |x^\alpha \nabla_x^\beta \Psi(x)| \le c_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{N}^d\right\}. \tag{4.5}$$

The definition (4.5) is more common in the literature, even though one can argue that (4.4) is more natural.

Definition 4.6 *On $\mathcal{S}(\mathcal{X})$ we introduce semi-norms*

$$\|\Psi\|_{\alpha,\beta} := \Big(\int |x^\alpha \nabla_x^\beta \Psi(x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}},$$

which make it into a Fréchet space. $\mathcal{S}'(\mathcal{X})$ denotes its topological dual.

Note that we have continuous inclusions

$$\mathcal{D}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X}) \subset L^2(\mathcal{X}) \subset \mathcal{S}'(\mathcal{X}) \subset \mathcal{D}'(\mathcal{X}).$$

4.1.4 Derivatives

Let f be a complex function on $\mathcal{X}$. Recall from the real case of Def. 2.50 (1) that the *derivative of f at $x_0 \in \mathcal{X}$ in the direction $q \in \mathcal{X}$* is defined by

$$q\cdot\nabla_x f(x_0) := \frac{\mathrm{d}}{\mathrm{d}t} f(x_0 + tq)\big|_{t=0}. \tag{4.6}$$

Proposition 4.7 *The derivative of a C^1 function at a point is a complex linear functional on $\mathcal{X}$, that is, $\nabla_x f(x_0) \in \mathbb{C}\mathcal{X}^{\#}$.*

Definition 4.8 *If $f \in C^2(\mathcal{X}, \mathbb{R})$, its* Hessian at $x_0 \in \mathcal{X}$ *is denoted $\nabla_x^{(2)} f(x_0) \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$ and defined by*

$$q_2\cdot\nabla_x^{(2)} f(x_0) q_1 := \frac{\mathrm{d}^2}{\mathrm{d}t_1 \mathrm{d}t_2} f(x_0 + t_1 q_1 + t_2 q_2)\big|_{t_1=t_2=0}, \quad q_1, q_2 \in \mathcal{X}.$$

If $\zeta \in L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X})$, then $\nabla_x\cdot\zeta\nabla_x$ denotes the corresponding differential operator:

$$\nabla_x\cdot\zeta\nabla_x f(x_0) := \mathrm{Tr}\,\zeta\nabla_x^{(2)} f(x).$$

If $\mathcal{X}$ is a Euclidean space with the scalar product denoted by $x_1 \cdot x_2$, then $\nabla_x \cdot \nabla_x = \nabla_x^2 = \Delta_x$ stands for the Laplacian.

4.1.5 Complex derivatives

Let $\mathcal{Z}$ be a complex vector space. Let f be a complex function on $\mathcal{Z}$.

Definition 4.9 *The* holomorphic, *resp.* anti-holomorphic derivative of f at $z_0 \in \mathcal{Z}$ in the direction of $w \in \mathcal{Z}$, resp. $\overline{w} \in \overline{\mathcal{Z}}$ *is defined by*

$$w\cdot\nabla_z f(z_0) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(z_0 + tw) - \mathrm{i}f(z_0 + \mathrm{i}tw)\right)\big|_{t=0},$$
$$\overline{w}\cdot\nabla_{\overline{z}} f(z_0) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(f(z_0 + tw) + \mathrm{i}f(z_0 + \mathrm{i}tw)\right)\big|_{t=0}.$$

Proposition 4.10 *The holomorphic, resp. anti-holomorphic derivative of a C^1 function at a point is a linear, resp. anti-linear functional on $\mathcal{Z}$, that is, $\nabla_z f(z_0) \in \mathcal{Z}^{\#}$, resp. $\nabla_{\overline{z}} f(z_0) \in \overline{\mathcal{Z}}^{\#}$.*

Recall from the complex case of Def. 2.50 (1) that f possesses a (complex) derivative at z_0 in the direction of w if there exists the limit

$$\lim_{u\to 0}\frac{f(z_0+uw)-f(z_0)}{u}, \tag{4.7}$$

where u is a complex parameter.

Definition 4.11 *Assume that $\mathcal{Z}$ is finite-dimensional and let $U\subset\mathcal{Z}$ be an open set. We say that $f:U\to\mathbb{C}$ is* holomorphic *in U if it possesses a complex derivative at each $z_0\in U$.*

Proposition 4.12 *A function $f:U\to\mathbb{C}$ is holomorphic iff $f\in L^1_{\mathrm{loc}}(U)$ and $\nabla_{\overline{z}}f=0$ in U in the distribution sense. Then (4.7) equals $w\cdot\nabla_z f(z_0)$.*

We consider also the realification of $\mathcal{Z}$, denoted $\mathcal{Z}_{\mathbb{R}}$, where the multiplication by i is denoted by j.

Let $\nabla_z^{\mathbb{R}}$ denote the usual (real) derivative on $\mathcal{Z}_{\mathbb{R}}$. We can express the holomorphic and anti-holomorphic derivative in terms of the real derivative:

$$\begin{aligned} w\cdot\nabla_z &= \frac{1}{2}\left(w\cdot\nabla_z^{\mathbb{R}}-\mathrm{i}(\mathrm{j}w)\cdot\nabla_z^{\mathbb{R}}\right),\\ \overline{w}\cdot\nabla_{\overline{z}} &= \frac{1}{2}\left(w\cdot\nabla_z^{\mathbb{R}}+\mathrm{i}(\mathrm{j}w)\cdot\nabla_z^{\mathbb{R}}\right),\\ w\cdot\nabla_z+\overline{w}\cdot\nabla_{\overline{z}} &= w\cdot\nabla_z^{\mathbb{R}}. \end{aligned} \tag{4.8}$$

(On the left w is treated as an element of $\mathcal{Z}$ and on the right w as a real vector in $\mathcal{Z}_{\mathbb{R}}$.)

Note that if we make the identification $\mathcal{Z}_{\mathbb{R}}\ni w\mapsto(w,\overline{w})\in\mathcal{Z}\oplus\overline{\mathcal{Z}}$, as in (1.31), then (4.8) can be written as $\nabla_z+\nabla_{\overline{z}}=\nabla_z^{\mathbb{R}}$.

4.1.6 Position and momentum operators

Definition 4.13 *For $\eta\in\mathcal{X}^{\#}$ and $q\in\mathcal{X}$ we set*

$$(\eta\cdot x\Psi)(x):=\eta\cdot x\Psi(x),\quad \mathrm{Dom}\ \eta\cdot x:=\left\{\Psi\in L^2(\mathcal{X}):\ \int|\eta\cdot x|^2|\Psi(x)|^2\mathrm{d}x<\infty\right\},$$

$$(q\cdot D\Psi)(x):=-\mathrm{i}q\cdot\nabla\Psi(x),\quad \mathrm{Dom}\ q\cdot D:=\left\{\Psi\in L^2(\mathcal{X}):\ \int|q\cdot\nabla_x\Psi(x)|^2\mathrm{d}x<\infty\right\}.$$

$\eta\cdot x$ and $q\cdot D$ are called respectively position and momentum operators *and are self-adjoint operators.*

Remark 4.14 *In the formulas above the symbol x is used with as many as three different meanings:*

(1) *as an element of the space $\mathcal{X}$, e.g. in $\Psi(x)$ or in $\eta\cdot x$ on the right of $:=$;*
(2) *as the name of the "generic variable in $\mathcal{X}$"; e.g. in $\mathrm{d}x$ or ∇_x;*
(3) *as a vector of self-adjoint operators, e.g. in $\eta\cdot x$ on the left of $:=$.*

This ambiguous usage of the same symbol, although sometimes confusing, seems to be difficult to avoid and is often employed. Sometimes one tries to differentiate the third meaning by decorating x in some way, e.g. writing $\hat{x}$.

Proposition 4.15 *The Schwartz space $\mathcal{S}(\mathcal{X})$ is the largest subspace of $L^2(\mathcal{X})$ contained in the domain of position and momentum operators and preserved by all the operators $\eta{\cdot}x$ and $q{\cdot}D$.*

The operator $\eta{\cdot}x$ and $q{\cdot}D$, viewed as operators on $\mathcal{S}(\mathcal{X})$, satisfy the so-called *Heisenberg commutation relations*:

$$[\eta_1{\cdot}x, \eta_2{\cdot}x] = [q_1{\cdot}D, q_2{\cdot}D] = 0, \qquad [\eta{\cdot}x, q{\cdot}D] = \mathrm{i}\eta{\cdot}q\mathbb{1}. \tag{4.9}$$

Definition 4.16 *The algebra of differential operators with polynomial coefficients will be denoted $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$.*

Elements of $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$ act naturally on $\mathcal{S}(\mathcal{X})$. By duality, they also act on $\mathcal{S}'(\mathcal{X})$.

Remark 4.17 *In Subsect. 8.3.1 we will define a more general class of algebras, denoted $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{Y})$, where $\mathcal{Y}$ is a symplectic space.*

Remark 4.18 *The algebra $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$ is sometimes called the* Weyl algebra*. However, we prefer to use this name for a different class of algebras; see Subsect. 8.3.5.*

4.1.7 Fourier transformation

Definition 4.19 *We denote by $C_\infty(\mathcal{X})$ the Banach space of continuous functions on $\mathcal{X}$ tending to 0 at ∞.*

Definition 4.20 *For $f \in L^1(\mathcal{X})$ the* Fourier transform of f*, denoted either $\mathcal{F}f$ or $\hat{f}$, is given by the formula*

$$\hat{f}(\xi) = \int f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}x.$$

It is well known that $\mathcal{F}$ extends to a unique bounded operator from $L^2(\mathcal{X}, \mathrm{d}x)$ to $L^2(\mathcal{X}^{\#}, \mathrm{d}\xi)$, where $\mathrm{d}\xi$ is the dual Lebesgue measure on $\mathcal{X}^{\#}$.

The Riemann–Lebesgue lemma says that if $f \in L^1(\mathcal{X})$, then $\hat{f} \in C_\infty(\mathcal{X}^{\#})$.

$(2\pi)^{-\frac{d}{2}}\mathcal{F}$ is unitary, and we have the Fourier inversion formula

$$f(x) = (2\pi)^{-d}\int \hat{f}(\xi)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

The space $\mathcal{S}(\mathcal{X})$ is mapped by $\mathcal{F}$ continuously onto $\mathcal{S}(\mathcal{X}^{\#})$. $\mathcal{F}$ can be extended to a unique continuous linear map from $\mathcal{S}'(\mathcal{X})$ onto $\mathcal{S}'(\mathcal{X}^{\#})$.

4.1.8 Gaussian integrals

Let $\nu \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$ be positive definite. Let $\eta \in \mathbb{C}\mathcal{X}^{\#}$. Then

$$(2\pi)^{-\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2}x\cdot\nu x + \eta\cdot x} \mathrm{d}x = (\det \nu)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\eta\cdot\nu^{-1}\eta}. \tag{4.10}$$

Note that the determinant $\det \nu$ is defined w.r.t. the Lebesgue measure $\mathrm{d}x$ (see Subsect. 3.6.6). In particular, if $f(x) = \mathrm{e}^{-\frac{1}{2}x\cdot\nu x}$, then

$$\hat{f}(\xi) = (2\pi)^{\frac{d}{2}} (\det \nu)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}\xi\cdot\nu^{-1}\xi}. \tag{4.11}$$

If $\nu \in L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}^{\#})$ is not necessarily positive definite and $\eta \in \mathcal{X}^{\#}$, then

$$\lim_{R\to\infty} (2\pi)^{-\frac{d}{2}} \int_{|x|<R} \mathrm{e}^{\frac{\mathrm{i}}{2}x\cdot\nu x + \mathrm{i}\eta\cdot x} \mathrm{d}x = |\det \nu|^{-\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{4}\pi \,\mathrm{inert}\, \nu} \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta\cdot\nu^{-1}\eta}. \tag{4.12}$$

In particular, if $g(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}x\cdot\nu x}$, then

$$\hat{g}(\xi) = (2\pi)^{\frac{d}{2}} \mathrm{e}^{\frac{\mathrm{i}}{4}\pi \,\mathrm{inert}\, \nu} (\det \nu)^{-\frac{d}{2}} \mathrm{e}^{-\frac{\mathrm{i}}{2}\xi\cdot\nu^{-1}\xi}.$$

We will sometimes abuse the notation and write $\det(-\mathrm{i}\nu)^{-\frac{1}{2}}$ for $|\det \nu|^{-\frac{1}{2}} \mathrm{e}^{\frac{\mathrm{i}}{4}\pi \,\mathrm{inert}\, \nu}$.

4.1.9 Gaussian integrals for complex variables

Let $\mathcal{Z}$ be a complex space of dimension d. Recall from Subsect. 3.6.9 that the integral of a function $\mathcal{Z} \ni z \mapsto F(z)$ over $\mathcal{Z}$ is interpreted as the integral of the pullback of F by

$$\mathcal{Z} \ni z \mapsto (z, \overline{z}) \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$$

on the space $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, and $\mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z$ is used as the standard volume form.

Let us translate formula (4.10) into the context of complex variables. Let $\beta \in L_{\mathrm{h}}(\mathcal{Z}, \mathcal{Z}^{*})$ be positive definite, and $w_1, w_2 \in \mathcal{Z}^{*}$. Then

$$(2\pi\mathrm{i})^{-d} \int \mathrm{e}^{-\overline{z}\cdot\beta z + w_1\cdot\overline{z} + \overline{w}_2\cdot z} \mathrm{d}\overline{z}\mathrm{d}z = (\det \beta)^{-1} \mathrm{e}^{w_1\cdot\beta^{-1}\overline{w}_2}, \tag{4.13}$$

where $\det \beta$ is computed w.r.t. the volume form $\mathrm{d}z$.

Let us explain the proof of (4.13). As mentioned above, the integral in (4.13) is interpreted as an integral on the real vector space $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$. We choose any scalar product on $\mathcal{Z}$ compatible with $\mathrm{d}z$. Note from Subsect. 3.6.9 that the volume form $\mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z$ is compatible with the Euclidean scalar product on $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$. We identify β with an element of $L(\mathcal{Z})$ using the unitary structure of $\mathcal{Z}$. Then, setting

$$v = (z, \overline{z}), \quad m = 2d, \quad \mathrm{d}v = \mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z,$$

$$\nu := \begin{bmatrix} 0 & \overline{\beta} \\ \beta & 0 \end{bmatrix}, \qquad \xi = (w_1, \overline{w}_2) \in \mathbb{C}\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathbb{C}\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})^{\#},$$

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we see that (4.13) reduces to (4.10). To compute the determinant of ν as an operator on $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$, we use that

$$\det \nu = \det \nu_{\mathbb{C}} = \det \beta \det \overline{\beta} = \det \beta \det \beta^{\#} = \det \beta^2,$$

since $\beta = \beta^*$. Then (4.13) follows from (4.10).

4.1.10 Convolution operators

Definition 4.21 *If $f \in \mathcal{S}'(\mathcal{X})$, $\Psi \in \mathcal{S}(\mathcal{X})$, then their* convolution product $f \star \Psi$ *is defined by*

$$f \star \Psi(x) := \int f(x - x_1)\Psi(x_1)\mathrm{d}x_1.$$

We have

$$\mathcal{F}(f \star \Psi) = (\mathcal{F}f)(\mathcal{F}\Psi).$$

Recall that $D = \frac{1}{\mathrm{i}}\nabla_x$ is a vector of commuting self-adjoint operators. Note that $\mathcal{F}D\mathcal{F}^{-1} = \xi$ where ξ is the operator of multiplication by $\xi \in \mathcal{X}^{\#}$ on $L^2(\mathcal{X}^{\#})$.

Note the identities

$$\begin{aligned} f(D)\Psi(x) &= (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} f(\xi)\Psi(y)\mathrm{d}\xi\mathrm{d}y \\ &= (2\pi)^{-d} \int \hat{f}(y - x)\Psi(y)\mathrm{d}y, \quad f \in \mathcal{S}(\mathcal{X}^{\#}). \end{aligned}$$

If $\nu \in L_{\mathrm{s}}(\mathcal{X}^{\#}, \mathcal{X})$, then

$$\begin{aligned} \mathrm{e}^{-\frac{1}{2}D_x\cdot\nu D_x}\Psi(x) &= \mathrm{e}^{\frac{1}{2}\nabla_x\cdot\nu\nabla_x}\Psi(x) \\ &= (2\pi)^{-\frac{d}{2}}(\det\nu)^{-\frac{1}{2}} \int \mathrm{e}^{-\frac{1}{2}(x-x_1)\cdot\nu^{-1}(x-x_1)}\Psi(x_1)\mathrm{d}x_1. \end{aligned} \tag{4.14}$$

As a consequence, we obtain the following identity for $\Psi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X})$:

$$(2\pi)^{-\frac{d}{2}} \int \Psi(x)\mathrm{e}^{-\frac{1}{2}x\cdot\nu x}\mathrm{d}x = |\det\nu|^{-\frac{1}{2}} \left(\mathrm{e}^{\frac{1}{2}\nabla_x\cdot\nu^{-1}\nabla_x}\Psi\right)(0). \tag{4.15}$$

As an example of (4.14) let us note

$$\mathrm{e}^{-\mathrm{i}tD_x\cdot D_\xi}\Psi(x,\xi) = (2\pi t)^{-d} \int \mathrm{e}^{\frac{\mathrm{i}}{t}(x-x_1)\cdot(\xi-\xi_1)}\Psi(x_1,\xi_1)\mathrm{d}x_1\mathrm{d}\xi_1.$$

Let us write the analog of (4.14) on a complex space $\mathcal{Z}$ of dimension d, for $\beta \in L(\mathcal{Z}, \mathcal{Z}^*)$ and $\beta > 0$:

$$\mathrm{e}^{\nabla_z\cdot\beta\nabla_{\overline{z}}}\Psi(\overline{z}, z) = (2\pi\mathrm{i})^{-d}(\det\beta)^{-1} \int \mathrm{e}^{-(\overline{z}-\overline{z}_1)\cdot\beta^{-1}(z-z_1)}\Psi(\overline{z}_1, z_1)\mathrm{d}\overline{z}_1\mathrm{d}z_1. \tag{4.16}$$

4.1.11 Sesquilinear forms on $\mathcal{S}(\mathcal{X})$

Definition 4.22 *$A \in B\big(L^2(\mathcal{X})\big)$ is called an $\mathcal{S}$-*type operator *if it is given by an integral kernel in $\mathcal{S}(\mathcal{X}\times\mathcal{X})$, that is, there exists $A(\cdot,\cdot)\in\mathcal{S}(\mathcal{X}\times\mathcal{X})$ such that*

$$A\Psi(x) := \int A(x,y)\Psi(y)\mathrm{d}y.$$

The set of $\mathcal{S}$-type operators is denoted $\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X})$.

Definition 4.23 *Continuous linear functionals on $\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X})$ are called* $\mathcal{S}'$-type forms. *Their space is denoted by $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$.*

Clearly, elements of $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$ are represented by distributions in $\mathcal{S}'(\mathcal{X}\oplus\mathcal{X})$. We have the obvious pairing for $B\in\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$ and $A\in\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X})$:

$$B(A) = \int\int B(x,y)A(x,y)\mathrm{d}x\mathrm{d}y.$$

Let

$$\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X}) \ni A \mapsto B(A)\in\mathbb{C} \tag{4.17}$$

be an $\mathcal{S}'$-type form. Clearly, for any $\Psi_1,\Psi_2\in\mathcal{S}(\mathcal{X})$, the operator $|\Psi_2)(\Psi_1|$ belongs to $\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X})$. Thus we obtain a sesquilinear form

$$\mathcal{S}(\mathcal{X})\times\mathcal{S}(\mathcal{X})\ni(\Psi_1,\Psi_2)\mapsto B\big(|\Psi_2)(\Psi_1|\big)\in\mathbb{C}. \tag{4.18}$$

We can interpret (4.18) as the action of $B\Psi_2$ on $\overline{\Psi_1}$, where B is a continuous linear map from $\mathcal{S}(\mathcal{X})$ to $\mathcal{S}'(\mathcal{X})$. Thus (4.18) can be written as $(\Psi_1|B\Psi_2)$. We call it the "operator notation for (4.18)", and we will use it henceforth.

We can write

$$\mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#}\oplus\mathcal{X})\subset B\big(L^2(\mathcal{X})\big)\subset\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X}).$$

Theorem 4.24 (The Schwartz kernel theorem) *B is a continuous linear transformation from $\mathcal{S}(\mathcal{X})$ to $\mathcal{S}'(\mathcal{X})$ iff B belongs to $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$, that is, iff there exists a distribution $B(\cdot,\cdot)\in\mathcal{S}'(\mathcal{X}\oplus\mathcal{X})$ such that*

$$(\Psi_1|B\Psi_2) = \int\overline{\Psi_1(x_1)}B(x_1,x_2)\Psi_2(x_2)\mathrm{d}x_1\mathrm{d}x_2,\quad \Psi_1,\Psi_2\in\mathcal{S}(\mathcal{X}).$$

Definition 4.25 *The distribution $B(\cdot,\cdot)\in\mathcal{S}'(\mathcal{X}\oplus\mathcal{X})$ is called the* distributional kernel of the transformation B.

Definition 4.26 *We define the* adjoint form B^* *by $(\Psi_1|B^*\Psi_2) = \overline{(\Psi_2|B\Psi_1)}$. If B_1 or B_2^* are continuous operators on $\mathcal{S}(\mathcal{X})$, then we can define $B_2\circ B_1$ as an element of $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$ by*

$$(\Psi_1|B_2\circ B_1\Psi_2) := (\Psi_1|B_2(B_1\Psi)),\ \text{or}\ (\Psi_1|B_2\circ B_1\Psi_2) := (B_2^*\Psi|B_1\Psi).$$

In particular this is possible if B_1 or $B_2\in\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#}\oplus\mathcal{X})$.

4.1.12 Hilbert–Schmidt and trace-class operators on $L^2(\mathcal{X})$

Note that $B \in B^2(L^2(\mathcal{X}))$ iff the distributional kernel of B belongs to $L^2(\mathcal{X} \oplus \mathcal{X})$. Moreover, if $B_1, B_2 \in B^2(L^2(\mathcal{X}))$, then

$$\operatorname{Tr} B_1^* B_2 = \int \overline{B_1(x_2, x_1)} B_2(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

Consider a trace-class operator $B \in B^1(L^2(\mathcal{X}))$. On the formal level we have the formula

$$\operatorname{Tr} B = \int B(x, x) \mathrm{d}x.$$

The following theorem gives some of many possible rigorous versions of the above identity:

Theorem 4.27 (1) *If $B \in \mathrm{CCR}^{\mathcal{S}}(\mathcal{X}^{\#} \oplus \mathcal{X})$, then*

$$\operatorname{Tr} B = \int B(x, x) \mathrm{d}x.$$

(2) *Fix an arbitrary Euclidean structure on $\mathcal{X}$. If $B \in B^1(L^2(\mathcal{X}))$ then*

$$\operatorname{Tr} B = \lim_{\epsilon \searrow 0} (2\pi/\epsilon)^{\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2\epsilon}(x_1 - x_2)^2} B(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

Proof (1) is left to the reader. To prove (2) we set $P_\epsilon := \mathrm{e}^{-\frac{\epsilon}{2} D^2}$. Note that $0 \leq P_\epsilon \leq \mathbb{1}$ and $\mathrm{w} - \lim_{\epsilon \to 0} P_\epsilon = \mathbb{1}$. By Subsect. 2.2.6, we know that $\operatorname{Tr} B = \lim_{\epsilon \to 0} \operatorname{Tr}(P_\epsilon B) = \lim_{\epsilon \to 0} \operatorname{Tr}(P_{\epsilon/2} B P_{\epsilon/2})$. By (4.14), P_ϵ has the kernel

$$(2\pi\epsilon)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2\epsilon}(x_x - x_2)^2},$$

and $P_{\epsilon/2} B P_{\epsilon/2}$ has kernel $B \star T_\epsilon$, where $T_\epsilon(x_1, x_2) = (\pi\epsilon)^{-d} \mathrm{e}^{-\frac{1}{\epsilon}(x_1^2 + x_2^2)}$. Now $B \star T_\epsilon \in \mathcal{S}(\mathcal{X} \oplus \mathcal{X})$, and by (1) we get

$$\operatorname{Tr}(P_{\epsilon/2} B P_{\epsilon/2}) = (\pi\epsilon)^{-d} \int \mathrm{e}^{-\frac{1}{\epsilon}(x - x_1)^2 - \frac{1}{\epsilon}(x - x_2)^2} B(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x.$$

Next we use (4.14) and the fact that $\mathrm{e}^{-\frac{\epsilon}{4} D^2} \mathrm{e}^{-\frac{\epsilon}{4} D^2} = \mathrm{e}^{-\frac{\epsilon}{2} D^2}$ to perform the integral in x, which yields

$$\operatorname{Tr}(P_{\epsilon/2} B P_{\epsilon/2}) = (2\pi\epsilon)^{-\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2\epsilon}(x_1 - x_2)^2} B(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2. \qquad \square$$

4.2 Weyl operators

As in the previous section, $\mathcal{X}$ is a finite-dimensional real vector space with the Lebesgue measure $\mathrm{d}x$.

The *Heisenberg commutation relations* (4.9) involve two unbounded operators: position and momentum. This makes them problematic as rigorous statements.

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In the early period of quantum mechanics Weyl noticed that for many purposes it is preferable to replace the Heisenberg commutation relations by relations involving the unitary groups generated by the position and momentum, since then called the *Weyl commutation relations*. These relations involve only bounded operators, hence their meaning is clear. On the formal level they are equivalent to the Heisenberg relations.

Linear combinations of the position and the momentum are self-adjoint. Their exponentials are often called *Weyl operators*. They are very useful in quantum mechanics.

One of the central results of mathematical foundations of quantum mechanics is the *Stone–von Neumann theorem*, which says that the properties of the position and momentum, up to a unitary equivalence, are essentially determined by the Weyl relations.

4.2.1 Definition of Weyl operators

Let us consider the one-parameter unitary groups on $L^2(X)$

$$\begin{aligned} \mathcal{X}^\# \ni \eta &\mapsto \mathrm{e}^{\mathrm{i}\eta\cdot x} \in U\big(L^2(\mathcal{X})\big), \\ \mathcal{X} \ni q &\mapsto \mathrm{e}^{\mathrm{i}q\cdot D} \in U\big(L^2(\mathcal{X})\big) \end{aligned}$$

generated by the position and the momentum operators.

Theorem 4.28 *Let $\eta \in \mathcal{X}^\#$, $q \in \mathcal{X}$. We have the so-called* Weyl commutation relations*,*

$$\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D} = \mathrm{e}^{-\mathrm{i}\eta\cdot q}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{e}^{\mathrm{i}\eta\cdot x}. \tag{4.19}$$

The operator $\eta\cdot x + q\cdot D$ is essentially self-adjoint on $\mathcal{S}(\mathcal{X})$. For $\Psi \in L^2(\mathcal{X})$ we have

$$\mathrm{e}^{\mathrm{i}(\eta\cdot x+q\cdot D)}\Psi(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot q+\mathrm{i}\eta\cdot x}\Psi(x+q). \tag{4.20}$$

Moreover, the following identities are true:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}(\eta\cdot x+q\cdot D)} &= \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot q}\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\eta\cdot q}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{e}^{\mathrm{i}\eta\cdot x} \\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{e}^{\frac{\mathrm{i}}{2}\eta\cdot x} = \mathrm{e}^{\frac{\mathrm{i}}{2}q\cdot D}\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\frac{\mathrm{i}}{2}q\cdot D}. \end{aligned} \tag{4.21}$$

Proof Clearly, we have

$$\mathrm{e}^{\mathrm{i}q\cdot D}\Psi(x) = \Psi(x+q).$$

This easily implies (4.19).

Define

$$U(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}t^2\eta\cdot q}\mathrm{e}^{\mathrm{i}t\eta\cdot x}\mathrm{e}^{\mathrm{i}tq\cdot D},$$

or

$$U(t)\Psi(x) := \mathrm{e}^{\frac{\mathrm{i}}{2}t^2\eta\cdot q+\mathrm{i}t\eta\cdot x}\Psi(x+tq).$$

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We compute

$$\partial_t U(t)\Psi = \mathrm{i}(\eta x + qD)U(t)\Psi, \quad \Psi \in \mathcal{S}(\mathcal{X}).$$

Clearly, if $\Psi \in \mathcal{S}(\mathcal{X})$, then $U(t)\Psi \in \mathcal{S}(\mathcal{X})$ for all t. Therefore, by Nelson's invariant domain theorem, Thm. 2.74 (2),

$$U(t) = \mathrm{e}^{\mathrm{i}t(\eta\cdot x + q\cdot D)}$$

and $\mathcal{S}(\mathcal{X})$ is a core of $\eta\cdot x + q\cdot D$. This implies (4.20).

The identities (4.21) follow from (4.20). □

Theorem 4.29 *If $B \in B\big(L^2(\mathcal{X})\big)$ commutes with all operators in*

$$\big\{\mathrm{e}^{\mathrm{i}\eta\cdot x}, \quad \mathrm{e}^{\mathrm{i}q\cdot D} \ : \ \eta \in \mathcal{X}^\#, \ q \in \mathcal{X}\big\}, \tag{4.22}$$

then B is proportional to identity. In other words, the set (4.22) is irreducible in $B\big(L^2(\mathcal{X})\big)$.

Proof $L^\infty(\mathcal{X})$, identified with multiplication operators in $L^2(\mathcal{X})$, is a maximal Abelian algebra in $B\big(L^2(\mathcal{X})\big)$. By the Fourier transformation, linear combinations of operators of the form $\mathrm{e}^{\mathrm{i}\eta\cdot x}$ are #-weakly dense in $L^\infty(\mathcal{X})$. Hence if B commutes with all operators $\mathrm{e}^{\mathrm{i}\eta\cdot x}$, it has to be of the form $f(x)$ with $f \in L^\infty(\mathcal{X})$.

We have $\mathrm{e}^{\mathrm{i}q\cdot D} f(x)\mathrm{e}^{-\mathrm{i}q\cdot D} = f(x+q)$. Hence if $f(x)$ commutes with $\mathrm{e}^{\mathrm{i}q\cdot D}$, then $f(x+q) = f(x)$. If this is the case for all $q \in \mathcal{X}$, f has to be constant. □

Theorem 4.30 *Let $\Psi \in L^2(\mathcal{X})$. Then $\Psi \in \mathcal{S}(\mathcal{X})$ iff*

$$\mathcal{X}^\# \oplus \mathcal{X} \ni (\eta, q) \mapsto (\Psi|\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}\Psi) \tag{4.23}$$

belongs to $\mathcal{S}(\mathcal{X}^\# \oplus \mathcal{X})$.

Proof (4.23) is a partial Fourier transform of the function $\mathcal{X} \oplus \mathcal{X} \ni (x, q) \mapsto \overline{\Psi}(x)\Psi(x+q)$. Thus (4.23) belongs to $\mathcal{S}(\mathcal{X}^\# \oplus \mathcal{X})$ iff $\overline{\Psi}(x)\Psi(x+q)$ belongs to $\mathcal{S}(\mathcal{X} \oplus \mathcal{X})$, which is equivalent to $\Psi \in \mathcal{S}(\mathcal{X})$. □

4.2.2 Quantum Fourier transform

Operators can be represented as an integral of $\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}$. This fact resembles the Fourier transformation; therefore we call it the *quantum Fourier transformation.*

The following proposition will be used in our analysis of the x, D and Weyl quantizations:

Proposition 4.31 (1) *Let $w \in L^1(\mathcal{X}^\# \oplus \mathcal{X})$. Then the operator*

$$(2\pi)^{-d} \int w(\eta, q)\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}\mathrm{d}\eta\mathrm{d}q \tag{4.24}$$

belongs to $B_\infty(L^2(\mathcal{X}))$ and is bounded by $(2\pi)^{-d}\|w\|_1$.

(2) *Let $B \in B^1(L^2(\mathcal{X}))$. Then the function*

$$w(\eta, q) := \mathrm{Tr}\, B\mathrm{e}^{-\mathrm{i}q\cdot D}\mathrm{e}^{-\mathrm{i}\eta\cdot x} \tag{4.25}$$

belongs to $C_\infty(\mathcal{X}^\# \oplus \mathcal{X})$ and is bounded by $\mathrm{Tr}|B|$.

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(3) *If $B \in B^1(L^2(\mathcal{X}))$ and w is defined by (4.25), then*

$$B = (2\pi)^{-d} \int w(\eta, q) \mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D} \mathrm{d}\eta \mathrm{d}q, \tag{4.26}$$

as a quadratic form identity on $\mathcal{S}(\mathcal{X})$.

(4) *If, moreover, $w \in L^1(\mathcal{X}^\# \oplus \mathcal{X})$, then (4.26) is an operator identity on $L^2(\mathcal{X})$.*

Remark 4.32 *Note that (4.26) follows from the following formal identity:*

$$\mathrm{Tr}\, \mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D} = (2\pi)^d \delta(\eta)\delta(q).$$

Proof (1) Let $w_n \in \mathcal{S}(\mathcal{X}^\# \oplus \mathcal{X})$ be a sequence such that $w_n \to w$ in $L^1(\mathcal{X}^\# \oplus \mathcal{X})$ and

$$B_n = (2\pi)^{-d} \int w_n(\eta, q) \mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D} \mathrm{d}\eta \mathrm{d}q.$$

Then the integral kernel of B_n belongs to $\mathcal{S}(\mathcal{X})$, hence B_n is Hilbert–Schmidt. Besides, $B_n \to B$ in $B\big(L^2(\mathcal{X})\big)$; therefore B is compact as the norm limit of compact operators.

(2) The map $\mathcal{X}^\# \oplus \mathcal{X} \ni (\eta, q) \mapsto \mathrm{e}^{-\mathrm{i}q\cdot D} \mathrm{e}^{-\mathrm{i}\eta\cdot x} \in B\big(L^2(\mathcal{X})\big)$ is continuous for the weak topology and $\mathrm{e}^{-\mathrm{i}q\cdot D} \mathrm{e}^{-\mathrm{i}\eta\cdot x}$ tends weakly to 0 when $(\eta, q) \to \infty$. This easily implies that $w \in C_\infty(\mathcal{X}^\# \oplus \mathcal{X})$.

(3) Let us fix $\Psi \in \mathcal{S}(\mathcal{X})$. It is enough to show that

$$(\Psi|B\Psi) = (2\pi)^{-d} \int w(\eta, q)(\Psi|\mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D}\Psi) \mathrm{d}\eta \mathrm{d}q. \tag{4.27}$$

For B of finite rank, (4.27) follows by a direct computation. Let us extend it to B of trace class.

From (2) we know that the map

$$B^1(L^2(\mathcal{X})) \ni B \mapsto w \in C_\infty(\mathcal{X}^\# \oplus \mathcal{X})$$

is continuous. Clearly, $(\eta, q) \mapsto (\Psi|\mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D}\Psi)$ belongs to $\mathcal{S}(\mathcal{X}^\# \oplus \mathcal{X})$. The maps

$$\begin{aligned} B^1(L^2(\mathcal{X})) \ni B &\mapsto (\Psi|B\Psi), \\ C_\infty(\mathcal{X}^\# \oplus \mathcal{X}) \ni w &\mapsto (2\pi)^{-d} \int w(\eta, q)(\Psi|\mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D}\Psi) \mathrm{d}\eta \mathrm{d}q \end{aligned}$$

are continuous. Hence we can extend (4.27) to an arbitrary $B \in B^1(L^2(\mathcal{X}))$ by density.

(4) Clearly, if $w \in L^1(\mathcal{X}^\# \oplus \mathcal{X})$, the r.h.s. of (4.26) is a norm convergent integral. □

Proposition 4.33 *Let us equip $\mathcal{X}$ with a Euclidean structure. Let $P_0 := |\Phi_0)(\Phi_0|$, where $\Phi_0 \in L^2(\mathcal{X})$ is given by*

$$\Phi_0(x) := \pi^{-\frac{d}{4}} \mathrm{e}^{-\frac{1}{2}x^2}.$$

Then

(1) $P_0 = P_0^* = P_0^2$,
(2) $\pi^{\frac{d}{2}} P_0 \mathrm{e}^{x^2} f(x) P_0 = P_0 \int f(x)\mathrm{d}x$, *for* $f \in L^1(\mathcal{X})$,
(3) $P_0 = (2\pi)^{-d} \int \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 + \mathrm{i}\frac{1}{2}q\cdot\eta} \mathrm{e}^{\mathrm{i}\eta\cdot x} \mathrm{e}^{\mathrm{i}q\cdot D} \mathrm{d}\eta \mathrm{d}q$,
(4) $\mathrm{Span}^{\mathrm{cl}}\{\mathrm{e}^{\mathrm{i}\eta\cdot x}\mathrm{e}^{\mathrm{i}q\cdot D}\Phi_0,\ \eta \in \mathcal{X}^\#, q \in \mathcal{X}\} = L^2(\mathcal{X})$.

Proof (1) is immediate, since $\|\Phi_0\| = 1$. To prove (3), we note that $\mathrm{e}^{-\mathrm{i}\eta\cdot x} P_0 \mathrm{e}^{-\mathrm{i}q\cdot D}$ has the kernel $\pi^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{e}^{-\mathrm{i}\eta x} \mathrm{e}^{-\frac{1}{2}(y+q)^2}$, which belongs to $\mathcal{S}(\mathcal{X} \oplus \mathcal{X})$. Hence, by Thm. 4.27,

$$\mathrm{Tr}(P_0 \mathrm{e}^{-\mathrm{i}\eta\cdot x} \mathrm{e}^{-\mathrm{i}q\cdot D}) = \pi^{-\frac{d}{2}} \int \mathrm{e}^{-\frac{1}{2}x^2} \mathrm{e}^{-\mathrm{i}\eta\cdot x} \mathrm{e}^{-\frac{1}{2}(x+q)^2} \mathrm{d}x = \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 + \frac{i}{2}\eta\cdot q}.$$

Then we apply Prop. 4.31. (2) and (4) are left to the reader. □

4.2.3 Stone–von Neumann theorem

Theorem 4.34 (Stone–von Neumann theorem) *Suppose that $\mathcal{X}$ is a finite-dimensional vector space and we are given a pair of strongly continuous unitary representations of the Abelian groups $\mathcal{X}^\#$ and $\mathcal{X}$ on a Hilbert space $\mathcal{H}$,*

$$\mathcal{X}^\# \ni \eta \mapsto V(\eta) \in U(\mathcal{H}),$$
$$\mathcal{X} \ni q \mapsto T(q) \in U(\mathcal{H}),$$

satisfying the Weyl commutation relations

$$V(\eta)T(q) = \mathrm{e}^{-\mathrm{i}\eta\cdot q} T(q) V(\eta).$$

Then there exists a Hilbert space $\mathcal{K}$ and a unitary operator

$$U : L^2(\mathcal{X}) \otimes \mathcal{K} \to \mathcal{H}$$

such that

$$V(\eta)U = U\mathrm{e}^{\mathrm{i}\eta\cdot x} \otimes 1\!\!1_{\mathcal{K}},$$

$$T(q)U = U\mathrm{e}^{\mathrm{i}q\cdot D} \otimes 1\!\!1_{\mathcal{K}}.$$

Proof *Step 1.* Clearly, the groups $V(\eta)$ and $T(q)$ can be written as

$$V(\eta) = \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}}, \qquad T(q) = \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}},$$

for some vectors of self-adjoint operators on $\mathcal{H}$, $\tilde{x}$ and $\tilde{D}$. We can define

$$\begin{aligned} P_0 &:= (2\pi)^{-d} \int \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 + \frac{\mathrm{i}}{2}\eta\cdot q} \mathrm{e}^{\mathrm{i}\eta\tilde{x}} \mathrm{e}^{\mathrm{i}q\tilde{D}} \mathrm{d}\eta\mathrm{d}q \\ &= (2\pi)^{-d} \int \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 - \frac{\mathrm{i}}{2}\eta\cdot q} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \mathrm{d}\eta\mathrm{d}q, \end{aligned} \tag{4.28}$$

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and $\mathcal{K} := \operatorname{Ran} P_0$. The definition of P_0 is suggested by Prop. 4.33. The identities of Prop. 4.33 are true for P_0 defined in (4.28), since they only rely on the Weyl commutation relations. Hence we get

$$\begin{aligned} P_0 = P_0^* &= P_0^2, \\ \pi^{\frac{d}{2}} P_0 \mathrm{e}^{\tilde{x}^2} f(\tilde{x}) P_0 &= P_0 \int f(x)\mathrm{d}x, \quad f \in L^1(\mathcal{X}). \end{aligned} \tag{4.29}$$

Step 2. Let

$$U\Phi \otimes \Psi := \pi^{\frac{d}{4}} \mathrm{e}^{\frac{1}{2}\tilde{x}^2} \Phi(\tilde{x})\Psi, \text{ for } \Phi \in \mathcal{S}(\mathcal{X}),\ \Psi \in \mathcal{K}.$$

(Note that, by (4.29), $f \in L^2(\mathcal{X})$ implies $\mathrm{e}^{\frac{1}{2}\tilde{x}^2} f(\tilde{x}) P_0 \in B(\mathcal{H})$.) We have

$$\begin{aligned} (U\Phi_1 \otimes \Psi_1 | U\Phi_2 \otimes \Psi_2) &= \pi^{\frac{d}{2}} (\Psi_1 | \mathrm{e}^{\tilde{x}^2} \overline{\Phi_1(\tilde{x})} \Phi_2(\tilde{x}) \Psi_2) \\ &= \pi^{\frac{d}{2}} (\Psi_1 | P_0 \mathrm{e}^{\tilde{x}^2} \overline{\Phi_1(\tilde{x})} \Phi_2(\tilde{x}) P_0 \Psi_2) \\ &= (\Psi_1 | \Psi_2) \int \overline{\Phi_1(x)} \Phi_2(x) \mathrm{d}x, \end{aligned}$$

by (4.29). Hence U uniquely extends to an isometry from $L^2(\mathcal{X}) \otimes \mathcal{K}$ into $\mathcal{H}$.
Step 3. We prove that U intertwines the Weyl commutation relations. To this end, using (4.29), we first obtain

$$\mathrm{e}^{\mathrm{i}q\tilde{D}} P_0 = \mathrm{e}^{-q\tilde{x} - \frac{1}{2}q^2} P_0. \tag{4.30}$$

Thus, for $\Psi \in \mathcal{K}$,

$$\mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \Psi = \mathrm{e}^{-q\cdot\tilde{x} - \frac{1}{2}q^2} \Psi.$$

Hence

$$\begin{aligned} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} U\Phi \otimes \Psi &= \pi^{\frac{d}{4}} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \mathrm{e}^{\frac{1}{2}\tilde{x}^2} \Phi(\tilde{x})\Psi \\ &= \pi^{\frac{d}{4}} \mathrm{e}^{\frac{1}{2}(\tilde{x}+q)^2} \Phi(\tilde{x}+q) \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \Psi \\ &= \pi^{\frac{d}{4}} \mathrm{e}^{\frac{1}{2}\tilde{x}^2} \Phi(\tilde{x}+q)\Psi = U\ \mathrm{e}^{\mathrm{i}q\cdot D} \Psi \otimes \Phi. \end{aligned}$$

It is easier to check that U intertwines the position operators:

$$\mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} U\Phi \otimes \Psi = \pi^{\frac{d}{4}} \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \mathrm{e}^{\frac{1}{2}\tilde{x}^2} \Psi(\tilde{x})\Phi = U \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \Phi \otimes \Psi.$$

Step 4. Finally, let us show that U is surjective. Clearly, if $\Psi \in \mathcal{K}$, then $U\Phi_0 \otimes \Psi = \Psi$, where we recall that $\Phi_0 = \pi^{-\frac{d}{4}} \mathrm{e}^{-\frac{1}{2}x^2}$. Hence $\mathcal{K} \subset \operatorname{Ran} U$. Thus, using Prop. 4.33 (3) and the intertwining property of U, it is enough to show that the span of

$$\left\{ \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \Psi \ :\ \eta \in \mathcal{X}^{\#},\ q \in \mathcal{X},\ \Psi \in \mathcal{K} \right\} \tag{4.31}$$

is dense in $\mathcal{H}$.

Let $\Xi \in \mathcal{H}$ and $f(\eta, q) := (\Xi | \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} \Xi)$. Assume that Ξ is orthogonal to (4.31). Then

$$\begin{aligned} 0 &= (\Xi | \mathrm{e}^{\mathrm{i}\eta\cdot\tilde{x}} \mathrm{e}^{\mathrm{i}q\cdot\tilde{D}} P_0 \mathrm{e}^{-\mathrm{i}\eta\cdot\tilde{x}} \mathrm{e}^{-\mathrm{i}q\cdot\tilde{D}} \Xi) \\ &= (2\pi)^{-d} \int \mathrm{d}q_1 \mathrm{d}\eta_1 f(\eta_1, q_1) \mathrm{e}^{-\frac{1}{4}\eta^2 - \frac{1}{4}q^2 - \frac{\mathrm{i}}{2}\eta_1\cdot q_1 + \mathrm{i}(q\cdot\eta_1 - \eta\cdot q_1) - \mathrm{i}q\cdot\eta}. \end{aligned}$$

By the properties of the Fourier transformation, $f(\eta, q) = 0$ a.e. (almost everywhere). But $(\eta, q) \mapsto f(\eta, q)$ is a continuous function and $f(0,0) = \|\Xi\|^2$. So $\Xi = 0$. $\square$

4.3 x, D-quantization

As in both previous sections, $\mathcal{X}$ is a finite-dimensional real vector space with the Lebesgue measure $\mathrm{d}x$.

Looking at operators on $L^2(\mathcal{X})$ as a *quantization of classical symbols*, that is, of functions on the *classical phase space* $\mathcal{X} \otimes \mathcal{X}^{\#}$, has a long tradition in quantum physics. In mathematics the usefulness of this point of view seems to have been discovered much later. Apparently, among pure mathematicians this started with a paper of Kohn–Nirenberg (1965). The *calculus of pseudo-differential operators* introduced in that paper proved to be very successful in the study of partial differential equations and originated a branch of mathematics called *microlocal analysis*.

In this section we discuss the two most naive kinds of quantizations, commonly used in the context of partial differential equations – the x, D, and D, x-*quantizations*. Other kinds of quantization, in particular the *Weyl quantization*, will be discussed later in Chap. 8.

We will start with a discussion of quantization of polynomial symbols, where certain properties have elementary algebraic proofs. (Actually, these proofs generalize to the case where the symbols depend polynomially only on, say, momenta.) The definition of the x, D- and D, x-quantizations has a natural generalization to a much larger class of symbols, that of tempered distributions, which we will consider in the following subsection.

4.3.1 Quantization of polynomial symbols

Recall that $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$ denotes the algebra of operators on $\mathcal{S}(\mathcal{X})$ generated by x and D.

Clearly, if $f \in \mathbb{C}\mathrm{Pol_s}(\mathcal{X})$, then $f(x)$ is well defined as an operator on $\mathcal{S}(\mathcal{X})$. Such operators form a commutative sub-algebra in $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$.

Likewise, if $g \in \mathbb{C}\mathrm{Pol_s}(\mathcal{X}^{\#})$, then $g(D)$ is well defined as an operator on $\mathcal{S}(\mathcal{X})$. Such operators form another commutative algebra in $\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X})$.

Definition 4.35 *We define the* x, D-quantization*, resp. the* D, x-quantization *as the maps*

$$\mathbb{C}\mathrm{Pol_s}(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{x,D}(b) \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X}),$$
$$\mathbb{C}\mathrm{Pol_s}(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{D,x}(b) \in \mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X}),$$

as follows: if $b(x,\xi) = f(x)g(\xi)$, $f \in \mathbb{C}\mathrm{Pol_s}(\mathcal{X})$, $g \in \mathbb{C}\mathrm{Pol_s}(\mathcal{X}^{\#})$, *we set*

$$\mathrm{Op}^{x,D}(b) := f(x)g(D), \tag{4.32}$$
$$\mathrm{Op}^{D,x}(b) := g(D)f(x). \tag{4.33}$$

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We extend the definition to $\mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#})$ *by linearity.*

We will treat the ordering x, D as the standard one. Instead of $\mathrm{Op}^{x,D}(b)$ one often uses the notation $b(x, D)$.

Remark 4.36 *The x, D-quantization is sometimes called the* Kohn–Nirenberg quantization.

Definition 4.37 *The maps inverse to (4.32) and (4.33) are denoted*

$$\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X}) \ni B \mapsto \mathrm{s}_B^{x,D} \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#}), \tag{4.34}$$
$$\mathrm{CCR}^{\mathrm{pol}}(\mathcal{X}^{\#} \oplus \mathcal{X}) \ni B \mapsto \mathrm{s}_B^{D,x} \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#}), \tag{4.35}$$

*and the polynomials $s_B^{x,D}$ and $s_B^{D,x}$ are called the x, D- and D, x-*symbols of the operator B.

Theorem 4.38 (1) *If* $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#})$, *then*

$$\mathrm{Op}^{x,D}(b)^* = \mathrm{Op}^{D,x}(\bar{b}). \tag{4.36}$$

(2) *If* $b_-, b_+ \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#})$, *and* $\mathrm{Op}^{D,x}(b_-) = \mathrm{Op}^{x,D}(b_+)$, *then*

$$\begin{aligned} b_+(x,\xi) &= \mathrm{e}^{\mathrm{i}D_x \cdot D_\xi} b_-(x,\xi) \\ &= (2\pi)^{-d} \int \mathrm{e}^{-\mathrm{i}(x-x_1)\cdot(\xi-\xi_1)} b_-(x_1,\xi_1) \mathrm{d}x_1 \mathrm{d}\xi_1. \end{aligned} \tag{4.37}$$

(3) *If* $b_1, b_2 \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#})$ *then* $\mathrm{Op}^{x,D}(b_1)\mathrm{Op}^{x,D}(b_2) = \mathrm{Op}^{x,D}(b)$, *for*

$$\begin{aligned} b(x,\xi) &= \mathrm{e}^{\mathrm{i}D_{\xi_1} \cdot D_{x_2}} b_1(x_1,\xi_1) b_2(x_2,\xi_2) \Big|_{\substack{x_1 = x_2 = x, \\ \xi_1 = \xi_2 = \xi}} \\ &= (2\pi)^{-d} \int \mathrm{e}^{-\mathrm{i}(x-x_1)\cdot(\xi-\xi_1)} b_1(x,\xi_1) b_2(x_1,\xi) \mathrm{d}x_1 \mathrm{d}\xi_1. \end{aligned} \tag{4.38}$$

The operator $\mathrm{e}^{\mathrm{i}D_x \cdot D_\xi}$ *in (4.37) and the similar operator in (4.38) are understood as the sums of differential operators. In the case of this theorem, the sum is finite, because we deal with polynomial symbols.*

The integral formulas in (4.37) and (4.38) should be understood in the sense of oscillatory integrals.

https://doi.org/10.1017/9781009290876.005 Published online by Cambridge University Press

Proof To prove (4.37) it is sufficient to consider monomials. By a simple combinatorial argument,

$$\begin{aligned}
&(\eta_1\cdot x)\cdots(\eta_n\cdot x)(q_1\cdot D)\cdots(q_m\cdot D)\\
&=\sum_{k=0}^{\min(n,m)}\sum_{i_1<\cdots<i_k}\sum_{\text{distinct }j_1,\dots,j_k}(\eta_{i_1}\cdot q_{j_1})\cdots(\eta_{i_k}\cdot q_{j_k})\\
&\quad\times\prod_{i\in\{1,\dots,m\}\setminus\{j_1,\dots,j_k\}}(q_i\cdot D)\prod_{i\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}}(\eta_i\cdot x)\\
&=\mathrm{Op}^{D,x}\Big(\sum_{k=0}^{\min(n,m)}\frac{1}{k!}(-\mathrm{i}\nabla_x\cdot\nabla_\xi)^k(q_1\cdot\xi)\cdots(q_m\cdot\xi)(\eta_1\cdot x)\cdots(\eta_n\cdot x)\Big).
\end{aligned}$$

(4.38) follows easily from (4.37). In fact, it is enough to assume that $b_i(x,\xi)=f_i(x)g_i(\xi)$. Set $a(x,\xi)=f_2(x)g_1(\xi)$. Then

$$\begin{aligned}
\mathrm{Op}^{x,D}(b_1)\mathrm{Op}^{x,D}(b_2)&=f_1(x)\mathrm{Op}^{D,x}(a)g_2(D)\\
&=f_1(x)\mathrm{Op}^{x,D}(\tilde{b})g_2(D)=b(x,D),
\end{aligned}$$

where

$$\tilde{b}(x,\xi)=\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\nabla_\xi}a(x,\xi),\qquad b(x,\xi)=f_1(x)\tilde{b}(x,\xi)g_2(\xi).$$

□

Formulas (4.37) and (4.38) follow also (in a much larger generality) from integral formulas considered in the next subsection.

The following formula is a version of *Wick's theorem*. It follows from (4.38). We will see similar theorems later on for other quantizations.

Theorem 4.39 *Let* $b_1,\dots,b_n,b\in\mathbb{C}\mathrm{Pol_s}(\mathcal{X}\oplus\mathcal{X}^{\#})$ *and*

$$b(x,D)=b_1(x,D)\cdots b_n(x,D).$$

Then

$$b(x,\xi)=\exp\Big(\mathrm{i}\sum_{i<j}D_{\xi_i}\cdot D_{x_j}\Big)b_1(x_1,\xi_1)\cdots b_n(x_n,\xi_n)\Big|_{\substack{x=x_1=\cdots=x_n,\\ \xi=\xi_1=\cdots=\xi_n.}}$$

4.3.2 Quantization of distributional symbols

Recall that $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#}\oplus\mathcal{X})$ denotes the family of operators (or, actually, quadratic forms on $\mathcal{S}(\mathcal{X})$) whose distributional kernels belong to $\mathcal{S}'(\mathcal{X}\times\mathcal{X})$.

Definition 4.40 *If $b \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#})$, then we define $\mathrm{Op}^{x,D}(b)$ and $\mathrm{Op}^{D,x}(b)$ as the elements of $\mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#} \oplus \mathcal{X})$ whose distributional kernels are*

$$\mathrm{Op}^{x,D}(b)(x_1, x_2) = (2\pi)^{-d} \int_{\mathcal{X}^{\#}} b(x_1, \xi)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\cdot\xi}\mathrm{d}\xi,$$
$$\mathrm{Op}^{D,x}(b)(x_1, x_2) = (2\pi)^{-d} \int_{\mathcal{X}^{\#}} b(x_2, \xi)\mathrm{e}^{\mathrm{i}(x_1 - x_2)\cdot\xi}\mathrm{d}\xi. \tag{4.39}$$

Theorem 4.41 (1) *For $b \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#}) \subset \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#})$, the above definition coincides with (4.32) and (4.33).*

(2) *The maps*

$$\mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{x,D}(b) \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#} \oplus \mathcal{X}),$$
$$\mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#}) \ni b \mapsto \mathrm{Op}^{D,x}(b) \in \mathrm{CCR}^{\mathcal{S}'}(\mathcal{X}^{\#} \oplus \mathcal{X})$$

are bijective. Denote their inverses (symbols) as in (4.34) and (4.35). Then for $B \in \mathrm{Op}(\mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#}))$ we have

$$\mathrm{s}_B^{x,D}(x, \xi) = \int_{\mathcal{X}} B(x, x - y)\mathrm{e}^{-\mathrm{i}\xi\cdot y}\mathrm{d}y,$$
$$\mathrm{s}_B^{D,x}(x, \xi) = \int_{\mathcal{X}} B(x + y, x)\mathrm{e}^{-\mathrm{i}\xi\cdot y}\mathrm{d}y. \tag{4.40}$$

(3) *The formulas (4.36) and (4.37) are true.*

(4) *The formula (4.38) is true, for instance, if either $b_1 \in \mathcal{S}'(\mathcal{X} \oplus \mathcal{X}^{\#})$ and $b_2 \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}(\mathcal{X} \oplus \mathcal{X}^{\#})$, or the other way around.*

(5) *(4.38) is also true if the Fourier transforms of b_1 and b_2 belong to $L^1(\mathcal{X}^{\#} \oplus \mathcal{X})$.*

(6) *We have $b(x, D) \in B^2(L^2(\mathcal{X}))$ iff $b \in L^2(\mathcal{X} \oplus \mathcal{X}^{\#})$. Moreover,*

$$\mathrm{Tr}\, b(x, D)^* a(x, D) = (2\pi)^{-d} \int_{\mathcal{X}\oplus\mathcal{X}^{\#}} \overline{b(x, \xi)}a(x, \xi)\mathrm{d}x\mathrm{d}\xi, \quad a, b \in L^2(\mathcal{X} \oplus \mathcal{X}^{\#}).$$

Proof (2) follows from (4.39) by the inversion of the Fourier transform. (4.37) follows by combining the first formula of (4.39) with the second formula of (4.40). □

Example 4.42 *Fix a Euclidean structure in $\mathcal{X}$. Let P_0 be the orthogonal projection onto the normalized vector $\Phi_0 = \pi^{-\frac{d}{4}}\mathrm{e}^{-\frac{1}{2}x^2}$ (as in Prop. 4.33). The integral kernel of P_0 is*

$$P_0(x, y) = \pi^{-\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}y^2}.$$

Its x, D- and D, x-symbols are

$$\mathrm{s}_{P_0}^{x,D}(x, \xi) = 2^{\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}\xi^2 - \mathrm{i}x\cdot\xi},$$
$$\mathrm{s}_{P_0}^{D,x}(x, \xi) = 2^{\frac{d}{2}}\mathrm{e}^{-\frac{1}{2}x^2 - \frac{1}{2}\xi^2 + \mathrm{i}x\cdot\xi}.$$

4.4 Notes

An exposition of the theory of distributions can be found e.g. in Schwartz (1966) and Gelfand–Vilenkin (1964).

The Stone–von Neumann theorem was announced by Stone in 1930, but the first published proof was given by von Neumann (1931). Proofs can be found in Emch (1972) and Bratteli–Robinson (1996).

The $x, D-$ and $D, x-$ quantization goes back to a paper by Kohn–Nirenberg (1965).