A NOTE ON THE QUOTIENTS OF INDECOMPOSABLE INJECTIVE MODULES

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Let R be a commutative domain, I an ideal of R and write E_I for the injective envelope of R / I. In this note the following theorem will be proved:

THEOREM. *For a prime ideal* P *of a commutative domain* R *the following are equivalent*:

(i) *Every factor module of* E_P *is an indecomposable injective module*;

(ii) *Every non-zero prime ideal* $P' \subseteq P$ *is contained in only one maximal ideal* M *of* R, *and* R_M *is an almost maximal valuation ring*.

In [1] A.K. Tiwary proved that if P is a maximal ideal and R_P is a principal ideal domain (P.I.D.) then every non-zero factor module of E_P is isomorphic to E_P. This result together with its converse will be obtained as a corollary to the above theorem.

Among valuation rings the almost maximal ones are characterized by the property that every factor module of their quotient field is injective. Consequently, in the case of an almost maximal valuation ring, (i) holds for every indecomposable injective. Note that (i) is equi-

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valent to the condition that every quotient of E_P is injective and the submodules of E_P are totally ordered. For if $A, B \subseteq E_P$ and neither $A \subseteq B$ nor $B \subseteq A$ then $E_P / A \cap B$ is decomposable.

By a local ring we mean a ring with a unique maximal ideal and $\subset$ denotes strict inclusion.

LEMMA. *Let* R *be a local domain and* M *its maximal ideal. Then every factor module of* E_M *is an indecomposable injective if and only if* R *is an almost maximal valuation ring*.

Proof. The sufficiency of the condition has already been noted.

Suppose that every quotient of E_M is an indecomposable injective. Then the submodules of E_M are totally ordered. It is known that E_M is a cogenerator i.e., $\mathrm{Hom}_R (A, E_M) = 0$ implies that $A = 0$ for all R-modules A. It follows that the correspondence $I \to 0 :_E I$ gives rise to a lattice anti-isomorphism from the ideals I of R into the submodules of E_M. Thus the ideals of R are totally ordered and R is a valuation ring. To prove that R is almost maximal it will suffice to show that Q / R is injective where Q is the field of quotients of R ([2] Th. 4). Since R is a valuation ring, Q / M is clearly an essential extension of R / M and we may assume that $Q / M \subseteq E_M$. By assumption $E = E_M / (R / M)$ is an indecomposable injective. Hence E is the injective envelope of Q / R and the submodules of E are totally ordered. Assume that $Q / R \subset E$ and let $x \in E, x \notin Q / R$. We have $Q / R \subset Rx \approx R / 0 :_R x$ and $0 :_R x = F \neq 0$. Choose an element a in F such that $R a \neq F$. Then $1 / a + R \in Q / R \subset R x$ and $R : 1 / a = R a \subset F$, a contradiction. Thus Q / R is injective and R is almost maximal.

Proof of Theorem. The following fact will be frequently used. If $P' \subseteq P$ are prime ideals of R then $E_{P'}$, when regarded (in a natural way) as an R_P module, is the injective envelope of $R_P / R_P P'$. An R_P module is injective if and only if it is R-injective.

(i) => (ii). Let M be a maximal ideal containing P. Then the canonical mapping $R / P \to R / M$ induces a non-zero homomorphism $E_P \to E_M$. Hence E_M is isomorphic to a factor module of E_P by the assumption and (i) holds for E_M as well. Moreover, the same holds true if we replace R by R_M. By the Lemma R_M is an almost maximal valuation ring.

Let P' be a non-zero prime ideal contained in P and $a \in P'$, $a \neq 0$. We obtain an isomorphism of the R_M-ideals $R_M M$ and $I = (R_M M)a$. But R_M is a valuation ring and so $E_M \approx E(R_M / I)$ as R_M-modules ([2] Prop. 1). This gives rise to a non-zero mapping between E_M and $E_{P'}$, since $E_{P'}$ as an R_M-module is the injective envelope of $R_M / R_M P'$. Thus $E_{P'}$ is a quotient of E_M and the R-submodules of $E_{P'}$ are totally ordered. Hence $R / P' \subseteq E_{P'}$ is local and M is the only maximal ideal containing P'.

(ii) => (i). It follows from the assumption and the Lemma that E_M, when regarded as an R_M-module has the property that every R_M-factor module of it is an indecomposable injective. We will know that E_M satisfies (i) as an R-module once we show that every R-submodule of E_M is an R_M-submodule as well. In other words, we have to show that for every $x \in E_M$, $x \neq 0$, and $t \in R - M$ there is an $r \in R$ such that $t(rx) = x$ or, equivalently, that M is the only maximal ideal containing $I = 0 :_R x$. Since R_M is a valuation ring, there is a prime ideal P' of R such that $R_M P' = \text{Rad}\,(R_M I)$. Moreover, we can arrange that either

$P \subseteq P' \subseteq M$ or $P' \subseteq P \subseteq M$. Suppose that N is a maximal ideal of R and $I \subseteq N$. If $r \in P'$ then $r^k t \in I$ for some integer $k > o$ and $t \in R - M$. Hence $(r^k t)\, x = 0$. But $0 \colon x = 0 : tx$ since t acts faithfully on E_M. Thus $r^k x = 0$ and $r^k \in I \subseteq N$. It follows that $P' \subseteq N$. By assumption, M is the only maximal ideal containing a prime ideal which is contained in P. If $P' \subseteq P$ then it follows immediately that $N = M$ while in the case $P \subseteq P' \subseteq N$ the assumption is applied on P. Thus $M = N$ in either case and this proves that E_M satisfies (i) as an R-module. Utilizing the fact that R_M is a valuation ring we can show, as in the first part, that E_P is a factor module of E_M. Hence every R-factor module of E_P is an indecomposable injective.

COROLLARY. For a prime ideal P of a commutative domain R the following are equivalent:

(a) Every proper factor module of E_P is isomorphic to E_P;

(b) P is maximal and R_P is a P.I.D.

Proof. (b) => (a) (Tiwary [1]) The assumption in (b) imply that R_P is almost maximal and there is no non-zero prime ideal strictly contained in P. Hence, as in Theorem, the R-structure and the R_P-structure of E_P are the same. But it is known that E_P as an R_P-module satisfies (a).

(a) => (b). If there is a maximal ideal $M \supset P$ then we obtain a non-zero homomorphism $E_P \to E_M$, a contradiction. Hence P is maximal. Clearly (a) => (i). Hence R_P is an almost maximal valuation ring. For R_M-ideals $I, F, E_I \approx E_F$ if and only if $I \approx F$ ([2] Prop. 1). Since we can construct a non-zero homomorphism $E_M \to E_I$ for all non-zero R_M-

ideals I, it follows that the non-zero ideals of R_M are isomorphic. Thus R_M is a P.I.D.

We note that we do not necessarily have to confine ourselves to prime ideals. Let I be an irreducible ideal and P a prime ideal of R in such a way that $rs \in I$, $s \in I$ implies that $r \in P$. Then E_I satisfies (i) if and only if E_P does so.

REFERENCES

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