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Partially hyperbolic endomorphisms with expanding linear part

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Abstract. In this paper, we study transitivity of partially hyperbolic endomorphisms of the two torus whose action in the first homology group has two integer eigenvalues of moduli greater than one. We prove that if the Jacobian is everywhere greater than the modulus of the largest eigenvalue, then the map is robustly transitive. For this, we introduce Blichfedt's theorem as a tool for extracting dynamical information from the action of a map in homology. We also treat the case of specially partially hyperbolic endomorphisms, for which we obtain a complete dichotomy: either the map is transitive and conjugated to its linear part, or its unstable foliation must contain an annulus which may either be wandering or periodic.

Key words: partial hyperbolicity, robust transitivity, endomorphisms
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1. *Introduction*

Although it may now be long forgotten, dynamicists once believed that diffeomorphisms with gradient-like dynamics (so-called Morse–Smale systems) make up a dense subset among diffeomorphisms on any compact manifold. That should remind us about how striking the existence of robustly transitive diffeomorphisms actually is. Recall that a diffeomorphism f is *transitive* if it has a dense orbit, and *robustly transitive* if there is a C^1 neighbourhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is transitive. The first examples of robustly transitive diffeomorphisms were Anosov diffeomorphisms and, for some time, it was believed that there were no others. However, in the 70s, Shub and Mañé gave examples of robustly transitive diffeomorphisms on $\mathbb{T}^4$ and $\mathbb{T}^3$ that are not Anosov. Both of these examples are homotopic to Anosov (that is 'derived-from-Anosov') and partially hyperbolic.

Partial hyperbolicity is not a necessary condition for robust transitivity, but an even weaker form of hyperbolicity (dominated splitting with uniform contraction/expansion in the extreme bundles, see [DPU99, BDP03]) is. In particular, in dimension three, any robustly transitive diffeomorphism must have a non-trivial dominated splitting with uniform expansion or contraction in the one-dimensional bundle. Until the 90s, there were no known examples of robustly transitive diffeomorphisms which are not homotopic to Anosov. That changed with the publication of [BD96], where a new tool called a blender was introduced, allowing for a whole range of new examples. Yet it still remains an open problem to describe and classify all robustly transitive derived-from-Anosov diffeomorphisms, even on $\mathbb{T}^3$.

In hindsight, it may seem surprising that the research on this topic was born in the context of invertible maps, since the simplest examples of robustly transitive maps are actually uniformly expanding maps. It is therefore natural to ask whether it is possible to describe and classify robustly transitive 'derived-from-expanding' maps, that is, maps which are robustly transitive and homotopic to an expanding map without being themselves expanding. In a sense, it is a more elementary problem to classify derived-from expanding maps on, say, $\mathbb{T}^2$ than the analogous problem for derived-from-Anosov diffeomorphisms on $\mathbb{T}^3$ and we believe that the former is the right starting point for both problems. This is because of the simpler topology present in the derived-from-expanding case. In fact, there is a strong analogy between uniformly expanding maps and Anosov diffeomorphisms which becomes apparent by lifting a uniformly expanding map to its natural extension in the inverse limit space. Similarly, there is a strong analogy between derived-from-expanding maps on $\mathbb{T}^2$ and derived-from-Anosov maps with a dominated splitting and a uniformly contracted one-dimensional bundle.

In spite of their more straightforward topological description, linear expanding maps on $\mathbb{T}^2$ come in a greater variety than linear Anosov maps on $\mathbb{T}^3$. Whereas the latter must have either three real irrational eigenvalues or one irrational and a pair of complex ones, the former allows for a pair of irrational, a pair of complex or a pair of integer eigenvalues. This paper is dedicated to this latter case.

Problem 1.1. Fix a linear expanding map A on $\mathbb{T}^2$ with integer eigenvalues. What are the robustly transitive maps homotopic to A?

Note that every homotopy class contains maps with attractors, which is an obvious obstacle to transitivity, so the robustly transitive maps cannot make up the whole homotopy class. Something extra is needed. In previous works, we have considered this question for maps which are conservative [And16] or for which the non-wandering set is the whole of $\mathbb{T}^2$ [Ran18]. Both conditions serve to make sure the map has no attractors and are in fact sufficient for transitivity. A possible candidate for a weaker condition would be maps which are *volume expanding*. Indeed, a volume expanding map cannot have an attractor whose trapping region is *inessential*, that is, which does not wind around the torus. However, even volume expanding maps may have attractors with *essential* trapping regions.

Example 1.2. Let F be the direct product of two maps $f, g : S^1 \to S^1$, where $f(x) = 3x \bmod 1$ and $g(x)$ a map homotopic to $x \mapsto 2x \bmod 1$, satisfying:

(1) $g(0) = 0$;
(2) $g'(0) < 1$;
(3) $\frac{2}{3} < g'(x) < 3$ for all $x \in S^1$.

Then F has Jacobian larger than 2 everywhere but is clearly not transitive. Indeed, g has an attractor at 0, so F has an attractor with trapping region of the form $S^1 \times (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Once an orbit enters this region, it cannot escape.

Our main finding is that when the map is *partially hyperbolic* and has a *sufficiently large* Jacobian, then it is robustly transitive. Let us be more specific.

In this paper, an *endomorphism* is synonymous with a non-invertible local diffeomorphism. A *partially hyperbolic* endomorphism is a local diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ admitting an unstable cone-field $\mathcal{C}^u : p \mapsto \mathcal{C}^u_p$, where $\mathcal{C}^u_p$ is a closed cone in $T_p\mathbb{T}^2$, and constants $\ell > 0$ and $\lambda > 1$ satisfying:

(i) $\mathcal{C}^u$ is Df^ℓ-invariant, that is,

$$Df^\ell_p \mathcal{C}^u_p \subseteq \operatorname{int} \mathcal{C}^u_{f^\ell(p)} \cup \{0\},$$

where $\operatorname{int}(\mathcal{C}^u_p)$ denotes the interior of $\mathcal{C}^u_p$;

(ii) for every $v \in \mathcal{C}^u_p$, $\|Df^\ell(v)\| \geq \lambda\|v\|$.

The action of an endomorphism in the first homology group is given by a 2×2 matrix with integer entries. We refer to this matrix (and the maps it induces on $\mathbb{R}^2$ and $\mathbb{T}^2$) as the *linear part* of the endomorphism.

THEOREM A. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a partially hyperbolic endomorphism whose linear part A has integer eigenvalues λ_1, λ_2 with $|\lambda_1| \geq |\lambda_2| > 1$. Suppose that*

$$|\det(Df_p)| > |\lambda_1| \quad \textit{for every } p \in \mathbb{T}^2. \tag{1}$$

Then f is transitive.

The condition in equation (1) says that the Jacobian of f at every point is larger than the spectral radius of the linear part of f. It can be slightly relaxed by asking that it holds on an iterate of f or, equivalently, that there is some $C > 0$ and $\lambda > \lambda_1$ such that $|\det(Df^n_p)| \geq C\lambda^n$ for every $n \geq 1$ and every $p \in \mathbb{T}^2$. We say that an endomorphism with this property is *strongly volume expanding*.

It should be noted that partial hyperbolicity and the strongly volume expanding condition are both persistent under C^1-perturbations. As a consequence, we have the following corollary.

COROLLARY A. *Suppose that f is a partially hyperbolic endomorphism whose linear part is expanding with integer eigenvalues. If f is strongly volume expanding, then f is C^1 robustly transitive.*

In §5, we give an explicit example of an endomorphism satisfying the hypotheses of Theorem A which is (C^1 robustly) not conjugated to its linear part.

Theorem A is similar in flavour to a theorem by Rodriguez Hertz, Ures and Yang [**RHUY22**] about partially hyperbolic diffeomorphisms on $\mathbb{T}^3$. Using the hypothesis that f

is C^2 and a slightly weaker version of equation (1) (they allow for equality in equation (1) in a set with zero leaf volume along unstable leaves), they conclude that the strong stable and unstable foliations are C^1 robustly minimal, which in particular implies C^1 robust transitivity. Here we require less regularity but a slightly stronger condition on the Jacobian than that of [**RHUY22**]. Notwithstanding the apparent similarities, the approaches taken in the two works are very different. The argument in [**RHUY22**] relies on the existence of positive Lyapunov exponents in the centre direction and makes thorough use of the partially hyperbolic structure. In contrast, the present work applies Blichfedt's theorem to show that the strongly volume expanding condition has a rather far reaching topological consequence: a sufficiently high iterate of any open set must wind around the torus in two directions (Lemma 3.1). This is entirely independent of the map being partially hyperbolic or not and is of independent interest. Partial hyperbolicity is used to guarantee that this property indeed implies transitivity.

1.1. *Specially partially hyperbolic endomorphisms.* Whenever f is a partially hyperbolic endomorphism, we may define the *centre direction* at a point p by

$$E_p^c = \{v \in T_x\mathbb{T}^2 : Df_p^n(v) \notin \mathcal{C}^u(f^n(p)) \text{ for all } n \geq 0\} \cup \{0\}.$$

However, in contrast to the invertible case, there may not be a well-defined unstable direction. More precisely, given a choice of pre-orbit $\hat{p} = (\ldots, p_{-2}, p_{-1}, p_0)$ of p, that is, a sequence of points in $\mathbb{T}^2$ satisfying $p_0 = p$ and $f(p_{i-1}) = p_i$ for every $i \geq 0$, we define the direction

$$\hat{E}^u_{\hat{p}} = \bigcap_{n \geq 0} Df^n(\mathcal{C}^u(p_n)). \tag{2}$$

In general, $\hat{E}^u_{\hat{p}}$ will depend on the particular choice of pre-orbit $\hat{p}$. In the exceptional case where it does not, we say that f is a *specially partially hyperbolic endomorphism* and write $E^u_p = \hat{E}^u_{\hat{p}}$. In this case, E^u_p can easily be shown to be f-invariant and continuous.

For specially partially hyperbolic endomorphisms, we are able to give a full characterization of transitivity both in terms of conjugacy and in terms of absence of periodic or wandering annuli. By an *annulus*, we mean an open subset $\mathbb{A}$ of $\mathbb{T}^2$ homeomorphic to $(-1, 1) \times S^1$. We say that an annulus $\mathbb{A}$ is *periodic* if there is $n \geq 1$ such that $f^n(\mathbb{A}) = \mathbb{A}$; and it is *wandering* if $f^n(\mathbb{A}) \cap \mathbb{A} = \emptyset$ for every $n \geq 1$.

THEOREM B. *Let f be a specially partially hyperbolic endomorphism with linear part A. Suppose that A has integer eigenvalues $|\lambda_1| > |\lambda_2| > 1$. Then the following are equivalent:*

(a) *f is transitive;*

(b) *f is topologically conjugated to A;*

(c) *f admits neither a periodic nor a wandering annulus.*

When they exist, periodic and wandering annuli can always be chosen to be saturated by unstable leaves. We can therefore restate Theorem B as the following corollary.

THEOREM B'. *Let f be a specially partially hyperbolic endomorphism with linear part A having eigenvalues $|\lambda_1| > |\lambda_2| > 1$. Then one of the following holds:*

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(a) *f is transitive and topologically conjugated to A;*
(b) *f is not transitive and there is a periodic or wandering annulus saturated by the unstable foliation.*

Note that in virtue of being a direct product, Example 1.2 is in fact specially partially hyperbolic, so it serves as an example for the non-transitive case in Theorem B (and B'). In that example, the origin is an attractor for g whose basin is a union of intervals. If I is the interval that contains 0, then $\mathbb{T} \times I$ is a periodic (in fact fixed) annulus.

2. *Some preliminaries*

An endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ induces an action $f_\star$ on $\pi_1(\mathbb{T}^2)$. Since $\pi_1(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z}^2$, this action can be represented by a 2×2 integer matrix A. Now, A itself induces an endomorphism on $\mathbb{T}^2$, called a *linear endomorphism*. Each endomorphism is homotopic to one and only one such linear endomorphism, which we refer to as the *linear part* of f. One good reason for this is that if $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f, then

$$\tilde{f}(\tilde{x} + v) = \tilde{f}(\tilde{x}) + Av \tag{3}$$

for every $\tilde{x} \in \mathbb{R}^2$ and every $v \in \mathbb{Z}^2$. In particular, $\tilde{f}$ can be neatly decomposed as $A + (\tilde{f} - A)$, where $\tilde{f} - A$ is $\mathbb{Z}^2$-periodic and hence bounded.

A linear map A on $\mathbb{R}^2$ is called *expanding* when all its eigenvalues have magnitude larger than one. In the case where the linear part A of f is expanding, there is a surjective continuous map $h : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, such that

$$h \circ f = A \circ h. \tag{4}$$

The existence of h was proved by Franks in [**Fra70**] for diffeomorphisms with hyperbolic linear part, but the proof can be easily adapted to endomorphisms with expanding linear part. (We remark that if the linear part is a hyperbolic endomorphism, such a map may not exist. See [**CVa23**].) The map h is called a *semiconjugacy* from f to A. When h is a homeomorphism, we say that it is a *conjugacy* between f and A.

One of the consequences of the existence of the semi-conjugacy is that $\tilde{f}^n$ and $A^n \circ \tilde{h}$ stay uniformly close. Indeed, if $\tilde{h}$ is a lift of h, then $\tilde{h} - \mathrm{id}$ is $\mathbb{Z}^2$-periodic (since h is homotopic to the identity) and hence bounded by some constant, say κ. However, $A^n(\tilde{h}(\tilde{x})) = \tilde{h}(\tilde{f}^n(\tilde{x}))$ so that

$$\|\tilde{f}^n(\tilde{x}) - A^n(\tilde{h}(\tilde{x}))\| < \kappa \tag{5}$$

for every $\tilde{x} \in \mathbb{R}^2$ and every $n \geq 1$.

It is sometimes useful to consider the set-valued function

$$\phi : \mathbb{T}^2 \to \mathcal{K}(\mathbb{T}^2) \tag{6}$$

$$x \mapsto h^{-1}(h(x)) \tag{7}$$

and its lift $\tilde{\phi}(\tilde{x}) = \tilde{h}^{-1}(\tilde{h}(\tilde{x}))$. Here $\mathcal{K}(\mathbb{T}^2)$ denotes the class of compact subsets of $\mathbb{T}^2$. The set $\tilde{\phi}(\tilde{x})$ is the set of points whose forward orbit stays a bounded distance away from the orbit of $\tilde{x}$ under iterations of $\tilde{f}$, that is,

$$\tilde{\phi}(\tilde{x}) = \{\tilde{y} \in \mathbb{R}^2 : \sup_{n \geq 0} \|\tilde{f}^n(\tilde{x}) - \tilde{f}^n(\tilde{y})\| < \infty\}.$$

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PROPOSITION 2.1. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an endomorphism with expanding linear part A and $\tilde{f}$ a lift of f. Then the following hold.*

(a) *There is $r > 0$ such that*

$$\tilde{\phi}(\tilde{x}) = \bigcap_{k\geq 0} \tilde{f}^{-n_k}(B(\tilde{f}^{n_k}(\tilde{x}), r))$$

for each $\tilde{x} \in \mathbb{R}^2$ and each sequence $n_k \to \infty$.

(b) *There exists r_0 and $k \geq 1$ such that $\tilde{f}^k(B(\tilde{x}, r)) \supset \overline{B(\tilde{f}^k(\tilde{x}), r)}$ for every $\tilde{x} \in \mathbb{R}^2$ and $r > r_0$, where $B(\tilde{x}, r)$ is the ball of radius r centred at $\tilde{x}$.*

(c) *For each $\tilde{x} \in \mathbb{R}^2$, $\tilde{\phi}(\tilde{x})$ is a connected set.*

(d) *For each $\tilde{x}$, $\tilde{h}^{-1}(\tilde{x})$ is connected.*

(e) *For each compact connected set $\mathcal{C}$ in $\mathbb{T}^2$, the set $\tilde{h}^{-1}(\mathcal{C})$ is connected.*

Proof. The inclusion '$\supset$' in item (a) holds for every $r > 0$. This follows by noting that iterates of any two points in the set on the right remain a bounded distance from one another. Since the linear part is expanding, this can only happen if they have the same image under $\tilde{h}$.

The inclusion '$\subset$' in item (a) holds for any $r > 2\kappa$, where $\kappa > 0$ is chosen in such a way that $\|\tilde{h} - \mathrm{id}\| \leq \kappa$. To see this, let $\tilde{y} \in \tilde{\phi}(\tilde{x})$. Then $\tilde{h}(\tilde{y}) = \tilde{h}(\tilde{x})$ and, for $n \geq 0$,

$$\tilde{h}(\tilde{f}^n(\tilde{y})) = A^n(\tilde{h}(\tilde{y})) = A^n(\tilde{h}(\tilde{x})) = \tilde{h}(\tilde{f}^n(\tilde{x})).$$

Hence,

$$\|\tilde{f}^n(\tilde{y}) - \tilde{f}^n(\tilde{x})\| \leq \|\tilde{f}^n(\tilde{y}) - \tilde{h}(\tilde{f}^n(\tilde{y}))\| + \|\tilde{h}(\tilde{f}^n(\tilde{x})) - \tilde{f}^n(\tilde{x})\| < r,$$

and we conclude that $\tilde{y} \in \bigcap_{n\geq 0} \tilde{f}^{-n}(B(\tilde{f}^n(\tilde{x}), r))$.

Item (b) holds because of equation (5) and the fact that A is expanding.

To show item (c), fix k and r such that item (b) holds. If necessary, increase r so that item (a) holds as well. Consider the sets $D_n(r) = \tilde{f}^{-n}(B(\tilde{f}^n(\tilde{x}), r))$. From item (a), we have that $\tilde{\phi}(\tilde{x}) = \bigcap_{k\geq 0} D_{kn}$. Now,

$$\tilde{f}^{k(n+1)}(\overline{D_{k(n+1)}}) = \overline{B(\tilde{f}^{k(n+1)}(\tilde{x}), r)} \subset \tilde{f}^k(B(\tilde{f}^{nk}(\tilde{x}), r)) = \tilde{f}^{k(n+1)}(D_{nk}),$$

so that $\overline{D_{k(n+1)}} \subset D_{nk}$. Hence, $\tilde{\phi}(\tilde{x})$ can be written as $\bigcap_{n\geq 0} \overline{D_{nk}}$. In other words, $\tilde{\phi}(\tilde{x})$ is the intersection of a decreasing sequence of compact connected sets, so it is itself connected.

Item (d) is an immediate consequence of item (c).

We prove item (e) by contradiction. First note that $\tilde{h}^{-1}(\mathcal{C})$ is necessarily compact, since $\tilde{h}$ is a bounded distance from the identity. Suppose that $\tilde{h}^{-1}(\mathcal{C})$ is not connected. Then there are disjoint compact sets A and B such that $\tilde{h}^{-1}(\mathcal{C}) = A \cup B$. Hence, $\mathcal{C} = \tilde{h}(A) \cup \tilde{h}(B)$ with both $\tilde{h}(A)$ and $\tilde{h}(B)$ compact. Now, since $\mathcal{C}$ is connected, there exists some point $p \in \tilde{h}(A) \cap \tilde{h}(B)$. However, then $\tilde{h}^{-1}(p)$ can be written as the disjoint union $(\tilde{h}^{-1}(p) \cap A) \cup (\tilde{h}^{-1}(p) \cap B)$, both of which are closed. That is absurd. $\square$

Since $\pi\tilde{h}^{-1} = h^{-1}\pi$, we have the following corollary.

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COROLLARY 2.2. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be an endomorphism with expanding linear part A. Then the following hold.*

(a) *For each $p \in \mathbb{T}^2$, the set $h^{-1}(p)$ is a connected set.*
(b) *For each closed connected set $\mathcal{C}$ in $\mathbb{T}^2$, the set $h^{-1}(\mathcal{C})$ is connected.*
(c) *For each $p \in \mathbb{T}^2$, $f(\phi(p)) = \phi(f(p))$.*

2.1. *Dynamical coherence.* A partially hyperbolic endomorphism on $\mathbb{T}^2$ is said to be *dynamically coherent* if there exists an invariant C^0 foliation with C^1 leaves tangent to E^c. When it exists, such a foliation is called a *centre foliation* of f and its leaves are called *centre leaves*. If f and g are two dynamically coherent partially hyperbolic endomorphisms, we say that f and g are *leaf conjugate* if there exists a homeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ mapping centre leaves of f to centre leaves of g. A *periodic centre annulus* is an annulus $\mathbb{A} \subset \mathbb{T}^2$ such that $f^n(\mathbb{A}) = \mathbb{A}$ for some $n \geq 1$ whose boundary consists of either one or two C^1 circles tangent to the centre direction.

THEOREM 2.3. (Hall and Hammerlindl [**HH22**]) *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a partially hyperbolic endomorphism which does not admit a periodic centre annulus. Then f is dynamically coherent and leaf conjugate to A.*

Remark 2.4. In general, a partially hyperbolic endomorphism is not necessarily dynamically coherent, even when having an expanding linear part. An example was given in [**HH23**] with a linear part as in equation (8).

2.2. *Changing coordinates.* This work concerns specifically endomorphisms whose linear part A has *integer* eigenvalues. It is convenient to suppose that one of the eigenspaces is the vertical direction, that is, that A is represented by a lower triangular matrix of the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ \mu & \lambda_2 \end{pmatrix}, \tag{8}$$

where $|\lambda_1| \geq |\lambda_2| > 1$ are the (integer) eigenvalues of A and μ is some integer. There is no loss of generality in doing that.

LEMMA 2.5. *Let A be a 2 by 2 matrix with integer entries and two integer eigenvalues λ_1, λ_2. Then there exists $P \in SL(2, \mathbb{Z})$ such that $P^{-1}AP$ is of the form in equation (8) for some $\mu \in \mathbb{Z}$.*

Proof. Since A has integer eigenvalues, there exists $v \in \mathbb{Z}^2$ such that $Av = \lambda_2 v$. Without loss of generality, we may suppose that the components v_1, v_2 of v are coprime. Let p, q be such that $pv_1 + qv_2 = 1$ and take

$$P = \begin{pmatrix} q & v_1 \\ -p & v_2 \end{pmatrix}.$$

Then $P^{-1}AP$ is of the form in equation (8). □

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3. *Proof of Theorem A*

Before turning to the specific setting of Theorem A, let us take a look at how the strongly volume expanding property serves as a mechanism to produce homology in two linearly independent directions for large iterates of an open set.

Recall that an open set $U \subset \mathbb{T}^2$ is called *essential* if it contains a loop γ such that its homotopy class $[\gamma]$ is non-zero in $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$. Similarly, we define U to be *doubly essential* if it contains loops γ and σ such that $[\gamma]$ and $[\sigma]$ are linearly independent.

It is straightforward to see that if f is volume expanding, then a sufficiently large iterate of any open set is essential. The main idea behind Theorem A is that strong volume expansion leads to high iterates of any open set being doubly essential.

LEMMA 3.1. *Let f be a strongly volume expanding endomorphism on $\mathbb{T}^2$. Then, given any open set $U \subset \mathbb{T}^2$, there exists $N \geq 0$ such that $f^n(U)$ is doubly essential for every $n \geq N$.*

The proof of Lemma 3.1 is a direct consequence of the following lemma.

LEMMA 3.2. *Let f be a strongly volume expanding endomorphism on $\mathbb{T}^2$ and $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ a lift of f. Then, given any open set $\tilde{U} \subset \mathbb{R}^2$, there exists $N \geq 0$ such that for every $n \geq N$, there exist points $\tilde{p}_1, \tilde{q}_1, \tilde{p}_2, \tilde{q}_2$ in $\tilde{f}^n(\tilde{U})$ such that $\tilde{p}_1 - \tilde{q}_1$ is a non-zero multiple of e_1 and $\tilde{p}_2 - \tilde{q}_2$ is a non-zero multiple of e_2.*

The proof of Lemma 3.2 is based on a classical theorem about the geometry of numbers.

THEOREM 3.3. (Blichfeldt's theorem [Bli14]) *Let $B \subseteq \mathbb{R}^2$ be a Lebesgue measurable set such that* $\mathrm{Leb}(B) > k$ *for some positive integer k. Then there exist $x_0, \ldots, x_k$ in B such that $x_i - x_0 \in \mathbb{Z}^n$ for every $i = 1, \ldots, k$.*

Proof of Lemma 3.2. Fix $C > 0$ and $\lambda > \lambda_1$ such that

$$|\det(Df_p^n)| \geq C\lambda^n$$

for every $p \in \mathbb{T}^2$ and every $n \geq 1$. Fix also $\epsilon > 0$ so that $\lambda_1 + \epsilon < \lambda$. Let B be a (non-empty) open connected subset of $\tilde{U}$ contained in a ball of radius less than one. By Gelfand's formula,

$$\|A^n\| < (\lambda_1 + \epsilon)^n$$

for n greater than some n_0. By equation 5, we have that $\tilde{h}(B)$ is contained in a ball of radius $1 + \kappa$ so that for $n > n_0$, $\tilde{f}^n(B)$ is contained in a ball of diameter less than $L_n = 2(1 + \kappa)(\lambda_1 + \epsilon)^n + 2\kappa$. Choose $N > n_0$ so that $L_N < C\lambda^N \mathrm{Leb}(B)$.

Now suppose that $n \geq N$ and let ℓ be the integer part of L_n. Then

$$\mathrm{Leb}(\tilde{f}^n(B)) > \ell,$$

so by Blichfeldt's theorem, there is $\tilde{x} \in \mathbb{R}^2$ such that $\tilde{x} + \tilde{f}^n(B)$ intersects $\mathbb{Z}^2$ in at least $\ell + 1$ points. Recall that L_n is an upper bound for the diameter of $\tilde{f}^n(B)$ so, upon possibly adding an element of $\mathbb{Z}^2$ to $\tilde{x}$, we may assume that

$$(\tilde{x} + \tilde{f}^n(B)) \cap \mathbb{Z}^2 \subset \{1, \ldots, \ell\}^2.$$

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In other words, the intersection of $\tilde{x} + \tilde{f}^n(B)$ with $\mathbb{Z}^2$ consists of at least $\ell + 1$ points and is contained in $\{1, \ldots, \ell\}^2$. By the pigeon hole principle, there must be a line $\{1, \ldots, \ell\} \times \{i\}$ containing two points $\tilde{x}_1, \tilde{y}_1$ of the intersection. Similarly, there is a column $\{j\} \times \{1, \ldots, \ell\}$ containing two points $\tilde{x}_2, \tilde{y}_2$ of the intersection. The proof follows by taking $\tilde{p}_i = \tilde{x}_i - \tilde{x}$ and $\tilde{q}_i = \tilde{y}_i - \tilde{x}$ for $i = 1, 2$. □

LEMMA 3.4. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a partially hyperbolic endomorphism. If f is strongly volume expanding, then f is dynamically coherent and leaf conjugated to its linear part.*

Proof. By Theorem 2.3, it suffices to show that f does not admit a periodic centre annulus. Lemma 3.1 implies that any open set must become doubly essential after a sufficient number of iterations. However, no iterate of a periodic centre annulus is doubly essential. □

Remark 3.5. It is proved in [**HH22**] that the absence of a periodic centre annulus implies that the eigenvalues λ_1 and λ_2 of A are distinct real numbers.

In the proof of Theorem A, it will be convenient to reduce the argument to the case in which f is a skew-product. This can always be done—at least at the cost of sacrificing differentiability. Indeed, by Lemma 3.4, f is leaf conjugated to its linear part A. Let us denote the leaf conjugacy by ψ. Then the map $g = \psi \circ f \circ \psi^{-1}$ preserves the foliation of $\mathbb{T}^2$ into vertical circles (the centre leaves of the map A), and is therefore a skew product. It is clear that the map $h_g = h \circ \psi^{-1}$ is a semi-conjugacy from g to A.

Remark 3.6. Although it is not stated explicitly in [**HH22**], it can be read from the proofs that the leaf conjugacy $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to the identity and $g = \psi f \psi^{-1}$ is of the form $g(x, y) = (\lambda_1 x, \tau_x(y))$, where $\tau_x : S^1 \to S^1$ is a continuous family of differentiable maps of degree λ_2. Since ψ and h are homotopic to the identity, so is h_g.

Proof of Theorem A. Let $U \subset \mathbb{T}^2$ be a (non-empty) open set. We shall show that there is some n such that $f^n(U) = \mathbb{T}^2$. We denote $\pi^{-1}(U)$ by $\tilde{U}$. Since $\tilde{\psi}(\tilde{U})$ is open, it contains an open rectangle $R = (x_1, x_2) \times (y_1, y_2)$. By Lemma 3.2, there exists k such that $\tilde{f}^k(\tilde{\psi}^{-1}(R))$ contains points that differ by a non-zero multiple of e_2. However, then the same is true for $\tilde{g}^k(R)$ (see Remark 3.6). We are assuming A to be of the form in equation (8) so that $\tilde{g}^k(R)$ is a union of vertical lines. This means that $\tilde{g}^k(R)$ must contain a vertical line whose length is larger than one. Since $\tilde{g}^n(R)$ is open, $\pi(\tilde{g}^n(R))$ contains a vertical strip, that is, a set of the form $I \times S^1$ for some open interval $I = (a, b)$. Iterating this strip ℓ times by g, where $|\lambda_1|^\ell (b - a) > 1$, we get the whole torus $\mathbb{T}^2$. The proof follows by taking $n = k + \ell$. □

Remark 3.7. The proof of Theorem A shows that given any open $U \subset \mathbb{T}^2$, there exists n such that $f^n(U) = \mathbb{T}^2$. This property, sometimes so called *topological exactness* or *locally eventually onto*, is much stronger than transitivity. In fact, it is straightforward to see that it implies topological mixing. Hence, Theorem A and Corollary 1 remain valid if we replace 'transitive' with 'mixing'.

4. *Proof of Theorem B*

In what follows, we shall fix a specially partially hyperbolic endomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ and $\lambda_1, \lambda_2 \in \mathbb{Z}$ with $|\lambda_1| > |\lambda_2| > 1$ as the eigenvalues of A. Since the unstable direction (defined by equation (2)) is independent of the past, f has a non-trivial invariant splitting

$$T_p\mathbb{T}^2 = E^c \oplus E^u, \tag{9}$$

such that for all $p \in \mathbb{T}^2$ and all unit vectors $v \in E^c_p$ and $w \in E^u_p$,

$$\|Df(v)\| < \|Df(w)\| \quad \text{and} \quad \|Df(w)\| > 1.$$

Such an endomorphism always has a foliation tangent to the unstable bundle E^u. Indeed, this follows by applying the classical arguments of Hirsh, Pugh and Shub to the lift and then projecting to the torus (or whatever be the manifold under consideration). Let us denote by $\mathcal{F}^u$ the foliation tangent to E^u and call it the *unstable foliation*.

Although every specially hyperbolic endomorphism has an unstable foliation, it does not necessarily have a central one. Indeed, in [**HSW19**], there is an example of a *dynamically incoherent* specially partially hyperbolic endomorphism (whose linear part is not expanding). However, when the linear part is expanding, the next result follows as a direct consequence of [**HH22**, Theorem E].

PROPOSITION 4.1. *A specially partially hyperbolic endomorphism with expanding linear part does not admit a periodic centre annulus.*

By Theorem 2.3, f is dynamically coherent and leaf conjugate to A. We fix $\mathcal{F}^c$ as the centre foliation. Let E^u_A and E^c_A be the eigenspaces corresponding to λ_1 and λ_2, respectively. We denote by $\tilde{\mathcal{A}}^u$ and $\tilde{\mathcal{A}}^c$ the foliations of $\mathbb{R}^2$ by lines parallel to these spaces, and by $\mathcal{A}^u$ and $\mathcal{A}^c$ the foliations they induce on $\mathbb{T}^2$.

We denote by π^u the projection to E^u_A whose whose kernel is E^c_A and π^c is the projection to E^c_A whose kernel is E^u_A. We say that a foliation $\mathcal{F}$ in $\mathbb{R}^2$ is at a bounded distance from $\mathcal{A}^c$ (respectively $\mathcal{A}^u$) if there is some $M > 0$ such that the length of $\pi^u(\mathcal{L})$ (respectively $\pi^c(\mathcal{L})$) is smaller than M for every $\mathcal{L} \in \mathcal{F}$.

Since the eigenvalues of A are integers, $\mathcal{A}^u$ and $\mathcal{A}^c$ consist of circles. In particular, we also have that all the leaves of the centre foliation $\mathcal{F}^c$ of f are also circles and, moreover, the leaves of $\widetilde{\mathcal{F}}^c$ are at a bounded distance from the lines of $\tilde{\mathcal{A}}^c$.

As explained in [**Pot12**, §4.A], $\widetilde{\mathcal{F}}^u$ is at a bounded distance from some (unique) linear foliation $\tilde{\mathcal{A}}$ on $\mathbb{R}^2$. We claim that $\tilde{\mathcal{A}}$ is A-invariant. Indeed, let F be a leaf of $\widetilde{\mathcal{F}}^u$ and L be a leaf of $\tilde{\mathcal{A}}$. Then F and L are a bounded distance from each other. Since $\tilde{f}$ is a bounded distance from A, $\tilde{f}(F)$ must be a bounded distance from $A(L)$. However, $\tilde{f}(F)$ belongs to $\widetilde{\mathcal{F}}^u$ and is therefore a bounded distance from F itself. It follows that $A(L)$ is a bounded distance from L. In other words, $A(L)$ must be parallel to L proving that $\tilde{\mathcal{A}}$ is A-invariant. In our setting, there are only two linear A-invariant foliations, namely $\tilde{\mathcal{A}}^c$ and $\tilde{\mathcal{A}}^u$. We shall take a closer look at $\widetilde{\mathcal{F}}^u$ to see that indeed $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^u$. Similarly, we will show that $\widetilde{\mathcal{F}}^c$ is at a bounded distance from $\tilde{\mathcal{A}}^c$.

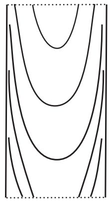

FIGURE 1. Reeb component.

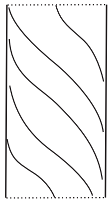

FIGURE 2. Tannulus.

Two important concepts for understanding foliations on $\mathbb{T}^2$ are Reeb components and Tannuli. A *Reeb component* of a foliation $\mathcal{F}$ on $\mathbb{T}^2$ is an annulus $\mathbb{A}$ such that the restriction of $\mathcal{F}$ to the closure of $\mathbb{A}$ is homeomorphic to one of the following:

(1) the foliation on $[-1, 1] \times \mathbb{S}^1$ induced by the foliation on $[-1, 1] \times \mathbb{R}$ given by the lines $\{-1\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$, along with the graphs of the functions $x \mapsto \exp(1/(1 - x^2)) + y$ with $y \in \mathbb{R}$;

(2) the foliation on $\mathbb{T}^2$ induced by the foliation on $S^1 \times \mathbb{R}$ obtained by identifying $\{-1\} \times \mathbb{R}$ with $\{1\} \times \mathbb{R}$ in case (1).

A *Tannulus component* (or simply *tannulus*) is defined analogously, replacing the functions $x \mapsto \exp(1/(1 - x^2)) + y$ with $x \mapsto \tan(\pi x/2) + y$. See Figures 1 and 2.

By the classification of foliations on $\mathbb{T}^2$ (see [**HH86**, Proposition 4.3.2]), if a foliation does not admit Reeb components, then it is a suspension of a circle homeomorphism. Such a foliation may or may not contain a tannulus component.

Remark 4.2. A foliation on $\mathbb{T}^2$ may have infinitely many tannuli, but it can have at most finitely many Reeb components. See [**HH86**].

A main ingredient is the following very general topological lemma.

LEMMA 4.3. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a self-cover. If there exists an annulus $\mathbb{A}$ and $n \geq 1$ such that $\mathbb{A} = f^{-n}(\mathbb{A})$, then the linear part of f has an eigenvalue* ± 1.

Since we are assuming that f has expanding linear part, Lemma 4.3 implies that there cannot be a *backwards* invariant annulus.

The proof of Lemma 4.3 follows by the arguments used in [**And16**, Ran17]. In short, if $\mathbb{A}$ is a periodic annulus with $f^{-n}(\mathbb{A}) = \mathbb{A}$, then the restriction of f^n to $\mathbb{A}$ is a self-cover of degree $\lambda_1^n \cdot \lambda_2^n$. At the same time, if $i : \mathbb{A} \to \mathbb{T}^2$ is the inclusion map, then $i_\star$ sends the fundamental group of $\mathbb{A}$ to a subgroup of $\mathbb{Z}^2$ of the form $G = \{kv : k \in \mathbb{Z}\} \subset \mathbb{Z}^2$, where

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$v \in \mathbb{Z}^2$ is an eigenvalue of the linear part of f. The action f on G produces a subgroup whose index is on the one hand equal to $\lambda_1^n \cdot \lambda_2^n$, and on the other equal to λ_i^n, where λ_i is the eigenvalue associated to v. Hence, the other eigenvalue must be ± 1.

Next, it is showed that $\mathcal{F}^u$ is necessarily a suspension.

LEMMA 4.4. *The unstable foliation $\mathcal{F}^u$ has no Reeb component.*

Proof. Suppose by contradiction that $\mathcal{F}^u$ contains a Reeb component $\mathbb{A} \subseteq \mathbb{T}^2$. Then, by [**HSW19**, Lemma 2.2], there is an integer $n > 0$ such that $f^{-n}(\mathbb{A}) = \mathbb{A}$. However, that is impossible according to Lemma 4.3, since we are assuming that f has expanding linear part. □

As we mentioned above, it follows from the classification of foliations on $\mathbb{T}^2$ that $\mathcal{F}^u$ is a suspension. Moreover, $\widetilde{\mathcal{F}}^u$ has rational slope since its leaves are a bounded distance from an eigenspace of A. Thus by the classification of foliations on $\mathbb{T}^2$, either $\mathcal{F}^u$ has a tannulus or all the leaves of $\mathcal{F}^u$ are circles.

LEMMA 4.5. *Let $\mathcal{F}$ be a foliation of $\mathbb{T}^2$ in which every leaf is a circle. Then every leaf of $\mathcal{F}$ represents the same non-zero element v in $\mathbb{Z}^2$ (the fundamental group of $\mathbb{T}^2$). Suppose, moreover, that γ is a closed C^1 curve transverse to $\mathcal{F}$. Then $[\gamma]$ is not a multiple of v.*

Proof. Let $\mathcal{L}$ be a leaf of $\mathcal{F}$ and write $v = [\mathcal{L}]$. That v is non-zero can be deduced from the Poincaré–Benedixon theorem (a foliation of $\mathbb{R}^2$ cannot have a compact leaf). If $\mathcal{L}'$ is another leaf, then $[\mathcal{L}']$ must be equal to v, for otherwise, $\mathcal{L}$ and $\mathcal{L}'$ would intersect. Fix some lift $\tilde{\gamma} : [0, 1] \to \mathbb{R}^2$ of γ and extend it periodically to $\tilde{\Gamma} : \mathbb{R} \to \mathbb{R}^2$. We claim that $\mathcal{L}$ intersects (the image of) $\tilde{\Gamma}$. Indeed, this also follows from the Poincaré–Benedixon theorem since if it were not true, then the vector field tangent to $\widetilde{\mathcal{F}}$ would exhibit a singularity.

We now observe that $\tilde{\Gamma}(t + k) = \tilde{\Gamma}(t) + k[\gamma]$ for every $k \in \mathbb{Z}^2$ so that the image of $\tilde{\Gamma}$ is invariant under translation by $[\gamma]$. Similarly, $\mathcal{L}$ is invariant by translation of v. Hence, $[\gamma]$ cannot be a multiple of v, for if it were, then $\mathcal{L}$ and $\tilde{\Gamma}$ would have infinitely many intersections. □

LEMMA 4.6. *The lifts $\widetilde{\mathcal{F}}^c$ and $\widetilde{\mathcal{F}}^u$ are a bounded distance from $\tilde{A}^c$ and $\tilde{A}^u$, respectively.*

Proof. Recall that every leaf of $\widetilde{\mathcal{F}}^u$ is a bounded distance from a translation of an eigenspace of A. Since $\mathcal{F}^u$ has a tannulus or all its leaves are circles, it is known that in both cases, there is a circle as a leaf. Then, as such a circle of $\mathcal{F}^u$ is transverse to $\mathcal{F}^c$, we can conclude by Lemma 4.5 that this eigenspace cannot be E_A^c. So it has to be E_A^u. □

A consequence of Lemma 4.6 is that the restriction of π^c (respectively π^u) to $\widetilde{\mathcal{F}}^c(\tilde{p})$ (respectively $\widetilde{\mathcal{F}}^u(\tilde{p})$) is onto, so $\widetilde{\mathcal{F}}^c(\tilde{p})$ and $\widetilde{\mathcal{F}}^u(\tilde{p})$ intersect each other. By the Poincaré–Bendixson theorem, we conclude that they intersect each other exactly once. In other words, $\widetilde{\mathcal{F}}^c$ and $\widetilde{\mathcal{F}}^u$ have global product structure and are *quasi-isometric*. That is,

$$\text{there exists } a, b > 0 \quad \text{such that } d_{\widetilde{\mathcal{F}}^*}(\tilde{p}, \tilde{q}) \leq a\|\tilde{p} - \tilde{q}\| + b, \tag{10}$$

where $d_{\widetilde{\mathcal{F}}^*}(\tilde{p}, \tilde{q})$ denotes the distance between $\tilde{p}$ and $\tilde{q}$ along a leaf of $\widetilde{\mathcal{F}}^*$ for $* = c, u$.

LEMMA 4.7. *The map $\tilde{h}$ sends leaves of $\widetilde{\mathcal{F}}^c$ onto leaves of $\tilde{\mathcal{A}}^c$ and leaves of $\widetilde{\mathcal{F}}^u$ onto leaves of $\tilde{\mathcal{A}}^u$.*

Proof. Since $\widetilde{\mathcal{F}}^c$ is at a bounded distance from $\tilde{\mathcal{A}}^c$, there is a constant $R > 0$ such that for every $\tilde{p} \in \mathbb{R}^2$, we can find a line $\mathcal{L} \in \tilde{\mathcal{A}}^c$ such that the leaf $\widetilde{\mathcal{F}}^c(\tilde{p})$ is contained in the R-neighbourhood of $\mathcal{L}$, which is an R-vertical strip. By equation (5), we have that $\|A^n \circ \tilde{h} - \tilde{f}^n\| < \kappa$ for each integer n and, thus, $A^n(\tilde{h}(\widetilde{\mathcal{F}}^c(\tilde{p})))$ is contained in an $(R+\kappa)$-vertical strip.

Now, suppose that $\tilde{q} \in \widetilde{\mathcal{F}}^c(\tilde{p})$ and that $\tilde{h}$ sends $\tilde{p}$ and $\tilde{q}$ to (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2 = E^u_A \oplus E^c_A$, respectively, with $x_1 \neq y_1$. (Recall that we are assuming A to be of the form in equation (8), so that $\mathcal{A}^c$ consists of vertical lines.) Then

$$|\pi^u(A^n(x_1, x_2)) - \pi^u(A^n(y_1, y_2))| = |\lambda_1|^n |x_1 - y_1|$$

gets arbitrarily large as n grows, contradicting that $A^n(\tilde{h}(\widetilde{\mathcal{F}}^c(\tilde{p})))$ is contained in an $(R+\kappa)$-vertical strip. That proves that $\tilde{h}$ sends leaves of $\widetilde{\mathcal{F}}^c$ to lines in $\tilde{\mathcal{A}}^c$. The case of $\widetilde{\mathcal{F}}^u$ is identical. □

LEMMA 4.8. *The map $\tilde{h}$ sends distinct leaves of $\widetilde{\mathcal{F}}^c$ to distinct lines of $\tilde{\mathcal{A}}^c$.*

Proof. We argue by contradiction. Suppose there are distinct leaves, say F_1 and F_2, of $\widetilde{\mathcal{F}}^c$ which are sent to the same line by $\tilde{h}$. Then for every $\tilde{q}_1 \in F_1$ and every $\tilde{q}_2 \in F_2$, we have $\pi^u(\tilde{h}(\tilde{q}_1)) = \pi^u(\tilde{h}(\tilde{q}_2))$ and so $\|\pi^u(\tilde{f}^n(\tilde{q}_1)) - \pi^u(\tilde{f}^n(\tilde{q}_2))\|$ is bounded for $n \geq 0$. By the global product structure, we can choose $\tilde{q}_1$ and $\tilde{q}_2$ in the same leaf of $\widetilde{\mathcal{F}}^u$. Since $\widetilde{\mathcal{F}}^u$ is at a bounded distance from $\tilde{\mathcal{A}}^u$, we have that $\|\pi^c(\tilde{f}^n(\tilde{q}_1)) - \pi^c(\tilde{f}^n(\tilde{q}_1))\|$ is also bounded for $n \geq 0$. Hence, $\|\tilde{f}^n(\tilde{q}_1) - \tilde{f}^n(\tilde{q}_2)\|$ is bounded for $n \geq 0$. However, that is impossible since $\tilde{q}_1$ and $\tilde{q}_2$ are in the same unstable leaf which is quasi-isometric. □

A consequence of Lemma 4.8 is that $\tilde{\phi}(\tilde{p})$ is contained in $\widetilde{\mathcal{F}}^c(\tilde{p})$ for every $\tilde{p} \in \mathbb{R}^2$. Proposition 2.1 then implies that $\tilde{\phi}(\tilde{p})$ must be either a point or a compact line segment in $\widetilde{\mathcal{F}}^c(\tilde{p})$.

LEMMA 4.9. *Suppose that $\mathcal{F}^u$ has no tannulus. If $\phi(p) \neq \{p\}$, then the interior of $h^{-1}(\mathcal{A}^u(h(p)))$ is an annulus which is either wandering or periodic for f.*

Proof. Since $\mathcal{F}^u$ has no tannulus, the leaves of $\mathcal{F}^u$ are circles so we may consider fibres of a trivial bundle $\pi : \mathbb{T}^2 \to S^1$ whose fibres are the leaves of $\mathcal{F}^u$. The set $\phi(p)$ is a transversal segment to the fibres and h sends $\mathcal{F}^u(x)$ to $\mathcal{A}^u(h(p))$ for every $x \in \phi(p)$. Hence, $h^{-1}(\mathcal{A}^u(h(p))$ is equal to $\pi^{-1}(\pi(\phi(p)))$. □

Proof of Theorem B. The implication (b) $\Longrightarrow$ (a) is obvious. To see why (a) $\Longrightarrow$ (c), first note that a transitive map may not have a wandering open set of any kind. Suppose that f has a periodic annulus $\mathbb{A} = f^n(\mathbb{A})$ for some $n \geq 1$. Then, by transitivity of f, we must have $f^{-n}(\mathbb{A}) = \mathbb{A}$. Indeed, if it were not so, $f^{-n}(\mathbb{A})$ would consist of a union of several annuli, some of which would be wandering. However, Lemma 4.3 says that it is impossible to have a backwards invariant annulus when the linear part is expanding.

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It remains to show the implication (c) $\Longrightarrow$ (b). Note that h is a conjugacy between f and A if and only if $\phi(p) = \{p\}$ for every $p \in \mathbb{T}^2$. (A continuous bijection on a compact space is a homeomorphism.) Thus, by Lemma 4.9, it suffices to show that if f does not admit a wandering or periodic annulus, then $\mathcal{F}^u$ does not admit a tannulus. Suppose it does admit a tannulus $\mathbb{A}$. Then $f^n(\mathbb{A})$ would be a tannulus for every $n \geq 0$. Moreover, $\mathbb{A}$ and $f^n(\mathbb{A})$ must either coincide or be disjoint. Hence, $\mathbb{A}$ must be either wandering or periodic. □

5. *An example*

Here we present a non-trivial example of an endomorphism satisfying the hypotheses of Theorem A. More precisely, we construct a C^∞ local diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying:

(1) the linear part of f is $A = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 2 \end{smallmatrix}\right)$;
(2) $\det Df(x, y) > 5$ for every $(x, y) \in \mathbb{T}^2$;
(3) f is partially hyperbolic; and
(4) f has a hyperbolic fixed point with stable index 1 and is therefore C^1 persistently not conjugated to A.

By Theorem A, f is robustly transitive. The example is a skew-product, but all properties are robust, so the construction leads implicitly to examples which are not skew-products. They are, however, topologically conjugated to skew-products. However, that is unavoidable according to [**HH22**] (see Theorem 2.3).

Here is the construction. Let $\alpha : \mathbb{T} \to \mathbb{T}$ and $\beta : \mathbb{T}^2 \to \mathbb{T}$ be given by

$$\alpha(x) = 5x + \frac{\sin(2\pi x)}{2\pi}, \tag{11}$$

$$\beta(x, y) = 2y - (1 + \epsilon)\cos^2(\pi x)\frac{\sin(2\pi y)}{2\pi} \tag{12}$$

and take $f(x, y) = (\alpha(x), \beta(x, y))$. Clearly, f is a well-defined C^∞ map on $\mathbb{T}^2$ homotopic to A. That it is a local diffeomorphism will follow as soon as we have proved item (2) above. The derivative of f at (0, 0) is given by

$$\begin{pmatrix} 6 & 0 \\ 0 & 1 - \epsilon \end{pmatrix},$$

which is hyperbolic with stable index 1 for every $\epsilon > 0$. This property persists under C^1 perturbations and guarantees that neither f nor its neighbours are conjugated to A. To see why item (2) holds, note that the Jacobian

$$J(x, y) = |\det Df(x, y)| = (5 + \cos(2\pi x))(2 - (1 + \epsilon)\cos^2(\pi x)\cos(2\pi y))$$

is C^∞ on $\mathbb{T}^2$ and that

$$\partial_y J = 2\pi(1 + \epsilon)(5 + \cos(2\pi x))\cos^2(\pi x)\sin(2\pi y)$$

vanishes only on $x = \frac{1}{2}$, $y = \frac{1}{2}$ and $y = 0$. It therefore suffices to check that J is greater than 5 along these three curves.

- On $x = \frac{1}{2}$, we have $J(\frac{1}{2}, y) \equiv 8$.
- On $y = \frac{1}{2}$, we have $J(x, \frac{1}{2}) = (5 + \cos(2\pi x))(2 + (1 + \epsilon)\cos^2(\pi x)) \geq 8$.

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- On $y = 0$, we have

$$\begin{aligned} J(x,0) &= (5+\cos(2\pi x))(2-(1+\epsilon)\cos^2(\pi x)) \\ &= 6+2\sin^2(\pi x)(2-\sin^2(\pi x)) - \epsilon\cos^2(\pi x)(5+\cos(2\pi x)) \\ &\geq 6-6\epsilon, \end{aligned}$$

which is greater that 5 for every $\epsilon < \frac{1}{6}$. That proves item (2).

Finally, let us verify that f is partially hyperbolic. For that, fix some $p \in \mathbb{T}^2$ and let $(u_1, u_2) = Df_p(1, 1)$, $(w_1, w_2) = Df_p(1, -1)$. We claim that

$$u_1 = w_1 \geq 4, \tag{13}$$

$$1-\epsilon \leq u_2 \leq 3+\epsilon, \tag{14}$$

and

$$-3-\epsilon \leq w_2 \leq 1+\epsilon. \tag{15}$$

Once that is shown, it follows that the cone

$$S = \{(v_1, v_2) \in \mathbb{R}^2 \setminus \{(0,0)\} : |v_1| \geq |v_2|\}$$

is strictly Df_p-invariant at every $p \in \mathbb{T}^2$ as long as $\epsilon < 1$. The estimate in equation (13) also shows that vectors in S are expanded by Df_p by a factor of at least $2\sqrt{2}$. This is because

$$\max_{0\leq t\leq 1} \|t(1,1)+(1-t)(1,-1)\| = \sqrt{2},$$

while

$$\min_{0\leq t\leq 1} \|tDf_p(1,1)+(1-t)Df_p(1,-1)\| \geq 4$$

for every p, and every $v \in S$ is a multiple of a vector of this type.

It remains to prove equations (13), (14) and (15). For that, let us write $p = (x, y)$. Then inequality in equation (13) is immediate, as

$$u_1 = w_1 = \partial_x\alpha(x) = 5+\cos(2\pi x). \tag{16}$$

The inequalities in equation (14) follow by rewriting u_2 as

$$\begin{aligned} u_2 &= \partial_x\beta(x,y)+\partial_y\beta(x,y) \\ &= (\epsilon+1)\sin(\pi x)\sin(2\pi y)\cos(\pi x) \\ &\quad + 2-(\epsilon+1)\cos^2(\pi x)\cos(2\pi y) \\ &= 2-(\cos(2\pi y)+\cos(2\pi(x+y)))/2 \\ &\quad - \epsilon\cos(\pi x)\cos(\pi(x+2y)). \end{aligned}$$

One can rewrite w_2 in a similar fashion to obtain equation (15).

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