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ON THE HOMOLOGICAL DIMENSION OF VALUATED VECTOR SPACES

BY
ERRIN ERB WHITE

Following L. Fuchs [1], we define a valuated vector space to be a vector space V with a valuation from V to a totally ordered set Γ in which every nonempty subset has a supremum. It is assumed that Γ has a maximum element $\infty \neq \sup(\Gamma \backslash \infty)$. A standard model for Γ is a closed initial segment of ordinals with the symbol ∞ adjoined. For $x \in V$, the valuation of x is denoted by $|x|$, and the following properties are satisfied:

(0) $|x| = \infty$ if and only if $x = 0$.
(1) $|cx| = |x|$ if c is a nonzero scalar.
(2) $|x+y| \geq \min(|x|, |y|)$.

A map from a space V to W is a linear transformation that does not decrease values.

Fuchs observes in [1] that the valuated vector spaces, with a fixed scalar field K and a fixed set of values Γ, form a pre-abelian category $\mathbf{V}$. Thus $\mathbf{V}$ has a zero object, kernels and cokernels, products and coproducts. However, in general not every monomorphism in $\mathbf{V}$ is a kernel.

The valuation on a quotient space B/A is defined by

$$|b+A| = \sup\{|b+a| : a \in A\},$$

and A is nice in B if $|b+A| = |b+a|$ for some $a \in A$. Following P. Hill [3], we say that A is separable in B if, for each $b \in B$, there exists a sequence $\{a_n\}_{n<\omega}$ in A such that

$$|b+A| = \sup_{n<\omega}\{|b+a_n|\}.$$

The projective and injective valuated vector spaces were completely determined in [1]. A projective space is the same as a free space. Following [1], we say that $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ is an exact sequence if α is an embedding, $\operatorname{Im} \alpha = \operatorname{Ker} \beta$ and for each $b \in W$, $|b| = \sup\{|a| : \beta(a) = b\}$. By a projective (=free) resolution for a valuated vector space V, we mean an exact sequence

$$\cdots F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} V \to 0$$

such that each F_i is free. It is noted that by the Corollary in [3], if there exists a free resolution $0 \to F_1 \to F_0 \to V \to 0$ of V, then there is such a free resolution of V where F_1 is nice in F_0.

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Some of the results of [1] were used in [2]. One of the most important aspects of [1] is the investigation of projective and injective dimension [4]. Theorem 7 in [1] claims that no valuated vector space has injective dimension exceeding 1, but as F. Richman and E. A. Walker suggested in [5] there is a flaw in the proof of this theorem. The falsity of Theorem 7 also led Fuchs to the erroneous conclusion of Theorem 3 in [1], which asserts, in essence, that every valuated vector space has projective dimension less than or equal to 1. By definition, a non-projective space V has projective dimension 1 if there is an exact sequence

$$0 \to P_1 \to P_0 \to V \to 0$$

where P_0 and P_1 are projective. Since a projective is free, a space V has projective dimension 1 only if V is the quotient of a free space by a free subspace. Richman and Walker [5] were the first to give an example of a valuated vector space that has projective dimension greater than 1. Their example is the product $P = \prod_{\alpha<\omega_1} \langle x_\alpha \rangle$ if the continuum hypothesis is assumed. Otherwise, the product needs to be larger.

In [3], Hill raised the question as to which valuated vector spaces can be embedded in free spaces; he called such a space an SF-space. We shall improve on the example of Richman and Walker by showing that there is an SF-space that has projective dimension 2. Thus subspaces of free spaces can be very nonfree.

THEOREM 1. *There exists a subspace of a free space with projective dimension 2.* ∎

Proof. A valuated vector space which is the quotient of a free space by a free subspace is called a QFF-space in [3]. Since Fuchs [1] characterized the projectives in $\mathbf{V}$ as the free spaces in $\mathbf{V}$, it follows that a valuated vector space is a QFF-space if and only if its projective dimension is either 0 or 1. Thus we need a space that is not a QFF-space.

For each $\alpha < \omega_2$, let $\langle x_\alpha \rangle$ denote the one-dimensional valuated vector space having value α. For notational convenience, the scalar field is the two-element field. Hence scalars do not appear, and $+$'s and $-$'s are the same. Let $S = \sum_{\alpha<\omega_2} \langle x_\alpha \rangle$. Then S is a free valuated vector space of dimension $\aleph_2$. Our example will be a subspace E of S. Let $E = \langle x_0 + x_\alpha \rangle_{\alpha<\omega_2}$. It is known [3], that a free space is separable in every containing space. It is easy to show that E is not separable in S because S/E has an element with value ω_2. Therefore, E is not free (and thus not projective).

We have shown that the *SF*-space E does not have projective dimension 0. We shall now show that its projective dimension is greater than 1. Assume that the projective dimension of E is 1 and let $E = A/B$ where A and B are free.

According to [3, Theorem 5], E is the union of a smooth chain of subspaces E_α satisfying the following conditions for each α:

(1) $\dim(E_\alpha) < \aleph_2$.
(2) E_α is separable in E.

Our intent is to construct a smooth chain of nonseparable subspaces D_α of E, or at least to construct a smooth chain of subspaces D_α with many of them being nonseparable, if not all. For each $\alpha < \omega_2$, let $D_\alpha = \langle x_0 + x_\lambda \rangle_{\lambda < \alpha}$. Then E is the union of the smooth chain of subspaces D_α, and for all $\alpha < \omega_2$, $\dim(D_\alpha) < \aleph_2$. Notice that D_α is separable in E only if α is cofinal with ω. There exist strictly increasing functions f and g from ω_2 into ω_2 such that:

(i) $f(1) = 1$,
(ii) $E_{f(\alpha)} \subseteq D_{g(\alpha)} \subseteq E_{f(\alpha+1)}$, for all $\alpha < \omega_2$,
(iii) $E_{f(\beta)} = D_{g(\beta)}$, for each limit ordinal β.

So in particular, $E_{f(\omega_1)} = D_{g(\omega_1)}$. Since all the E_α's are separable, this implies that $D_{g(\omega_1)}$ is separable in E. However, this is impossible because $g(\omega_1)$ is not cofinal with ω. Therefore E is not the quotient of a free space by a free subspace, and thus its projective dimension is greater than or equal to 2.

Richman and Walker [5, Theorem 16] showed that if $A \in \mathbf{V}_p$, the category of p-local valuated groups, is torsion and if the cardinality of A does not exceed $\aleph_n$, then the projective dimension of A does not exceed $n+1$. Adjusting the count on projective dimension from valuated groups to the category of valuated vector spaces $\mathbf{V}$, we can conclude that the projective dimension of our space E is less than or equal to 2. Therefore E has projective dimension exactly 2, and the theorem is proved. ∎

Any space having projective dimension 2 or greater can be converted to a counterexample to Theorem 7 in [1]. For the particular space E constructed in our Theorem 1, this conversion leads to another subspace of a free space. We omit the proof of the next theorem since all one has to do is to go through Fuchs' argument in [1, p. 31]. However, we remark that it would be of considerable interest to have a direct counterexample to Theorem 7 in [1].

THEOREM 2. *There exist subspaces of free spaces having injective dimension greater than* 1. ∎

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Department of Mathematics
Auburn University
Auburn, Alabama 36830

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