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RESEARCH ARTICLE

A Helfrich functional for compact surfaces in $\mathbb{C}P^2$

Zhongwei Yao[1,2]

[1]School of Mathematics and Statistics, Fujian Normal University, Fuzhou, 350117, P. R. China
[2]School of Mathematics and Physics, Leshan Normal University, Leshan, Sichuan, 614004, P. R. China
Email: 1976796476@qq.com

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Abstract
Let $f: M \to \mathbb{C}P^2$ be an isometric immersion of a compact surface in the complex projective plane $\mathbb{C}P^2$. In this paper, we consider the Helfrich-type functional $\mathcal{H}_{\lambda_1,\lambda_2}(f) = \int_M (|H|^2 + \lambda_1 + \lambda_2 C^2)\mathrm{d}M$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \geqslant 0$, H and C are respectively the mean curvature vector and the Kähler function of M in $\mathbb{C}P^2$. The critical surfaces of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$ are called Helfrich surfaces. We compute the first variation of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$ and classify the homogeneous Helfrich tori in $\mathbb{C}P^2$. Moreover, we study the Helfrich energy of the homogeneous tori and show the lower bound of the Helfrich energy for such tori.

1. Introduction

Helfrich functional dates back to Helfrich's seminal work [7], which proposed the functional model of the elastic lipid bilayer or membrane. Let $f: M \to \mathbb{R}^3$ be a smooth immersed surface in $\mathbb{R}^3$, the Helfrich functional (see [5]) is given by:

$$\mathcal{H}_{\lambda_1,\lambda_2}(f) = \int_M [(H - c_0)^2 - \gamma K]\mathrm{d}M + \lambda \int_M \mathrm{d}M,$$

where H denotes the mean curvature vector of surface, K is the Gaussian curvature, dM is the area measure of the induced metric, $\gamma \in \mathbb{R}$ is a constant bending rigidity, $\lambda \geqslant 0$ is the weight factor of the area functional, and $c_0 \in \mathbb{R}$ is a given spontaneous curvature caused by asymmetry between the two layers of the membrane. The functional with zero spontaneous curvature can be considered as a weighted sum of the Willmore functional and the area, which represent the bending energy and the surface energy, respectively. The critical surfaces of the first variation of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$ are called Helfrich surfaces. In recent years, many important researches have been developed in the study of the functional in geometry. Examples include the existence and regularity of solutions for Helfrich immersion from surfaces into $\mathbb{R}^3$(see, for instance, [3, 5, 6, 13, 16]), the classification ([1, 15]) of Helfrich surfaces in $\mathbb{R}^3$.

It is well know that the Willmore functional is conformal invariant and has been a field of active research since the work of Willmore [17]. And many of the techniques developed have played important roles in geomotry. Despite this, the functional for a immersed surface in complex manifolds is relatively less discussed. As noted in [2], Castro and Urbano proved the Whitney sphere is the only Willmore Lagrangian surface of genus zero in $\mathbb{C}^2$. Hu and Li [9] considered higher-dimensional case, and they proved Whitney spheres are Willmore submanifolds of $\mathbb{C}^n$ if and only if $n = 2$ and constructed examples of Willmore Lagrangian spheres in $\mathbb{C}^n$ for all $n \geq 2$. Immersions from surfaces into the complex projective plane $\mathbb{C}P^2$ are also considered. In [8], Hu and Li calculate the Euler–Lagrangian equation of the Willmore functional for an n-dimensional submanifold in an $(n + p)$-dimensional Riemannian manifold. As a corollary, the authors have given the Euler–Lagrangian equation of the Willmore functional for an immersed surface in complex projective plane $\mathbb{C}P^2$. In [14], Montiel and Urbano studied the Willmore functional for compact surface M in $\mathbb{C}P^2$. In this case, the Willmore functional is given by:

$$W = \int_M (|H|^2 + 1 + 3C^2)\mathrm{d}M,$$

where C denotes the Kähler function of M in $\mathbb{C}P^2$. The authors decomposed W into two global conformal invariants:

$$W^+ = \int_M (|H|^2 + 6C^2)\mathrm{d}M,\ \ W^- = \int_M (|H|^2 + 2)\mathrm{d}M.$$

They proved that $W^- \geqslant 4\pi\mu - 2\int |C|\mathrm{d}M$, where μ denotes the maximum multiplicity of the immersion. The equality holds if and only if $\mu = 1$ and M is either the complex projective line or totally geodesic real projective plane, or $\mu = 2$ and M is the Lagrangian Whitney sphere. Moreover, Montiel and Urbano obtained $W^- \geqslant 8\pi^2/3\sqrt{3}$ for all homogeneous tori in $\mathbb{C}P^2$ and conjectured that the Clifford torus attains the minimum of W^- among all Lagrangian tori in $\mathbb{C}P^2$. In this regard, Ma, Mironov, and Zuo [11] studied a family of Hamiltonian-minimal Lagrangian tori and proved Montiel–Urbano's conjecture is valid. For arbitrary Lagrangian tori, the conjecture remains open.

In this paper, we will focus on the Helfrich functional for surfaces in the complex projective plane $\mathbb{C}P^2$ (with holomorphic sectional curvature 4). Let $f: M \to \mathbb{C}P^2$ be an isometric immersion of a compact surface in $\mathbb{C}P^2$. For simplicity, we assume that the spontaneous curvature $c_0 = 0$. The Helfrich functional is defined by:

$$\mathcal{H}_{\lambda_1,\lambda_2}(f) = \int_M (|H|^2 + \lambda_1 + \lambda_2 C^2)\mathrm{d}M, \tag{1.1}$$

where $\lambda_1,\ \lambda_2 \in \mathbb{R}$ and $\lambda_1 \geqslant 0$. When $\lambda_1 = 1, \lambda_2 = 3$, the functional reduces to Willmore functional W. When $\lambda_1 = 0, \lambda_2 = 6$, the functional reduces to W^+ and when $\lambda_1 = 2, \lambda_2 = 0$, reduces to W^-. We first give the Euler–Lagrange equation of Helfrich functional $\mathcal{H}_{\lambda_1,\lambda_2}(f)$. This can also be derived by Hu–Li's result (See [8]).

Let $\{e_A\}_{1\leq A\leq 4}$ be a local orthonormal frame on $\mathbb{C}P^2$ such that when restricts to M, $\{e_1, e_2\}$ is a local orthonormal basis for TM. Then the Kähler function C on M can be given by $C = \langle Je_1, e_2\rangle$. Without loss of generality, we assume that $\{e_A\}$ satisfy

$$\begin{cases} Je_1 = C\,e_2 + \sqrt{1-C^2}\,e_4,\ Je_2 = -C\,e_1 - \sqrt{1-C^2}\,e_3, \\ Je_3 = -C\,e_4 + \sqrt{1-C^2}\,e_2,\ Je_4 = C\,e_3 - \sqrt{1-C^2}\,e_1. \end{cases} \tag{1.2}$$

Then, we have

Theorem 1.1. *Let $f: M \to \mathbb{C}P^2$ be an isometric immersion of a compact surface in the complex projective plane $\mathbb{C}P^2$. Then, M is a Helfrich surface if and only if*

$$\begin{cases} \Delta^\perp H^3 + (5 - 2\lambda_1 - (3 - 2\lambda_2)C^2 - 2|H|^2)H^3 + \sum_{\beta ij} h^3_{ij}h^\beta_{ij}H^\beta - 2\lambda_2\sqrt{1-C^2}C_{,1} = 0, \\ \Delta^\perp H^4 + (5 - 2\lambda_1 - (3 - 2\lambda_2)C^2 - 2|H|^2)H^4 + \sum_{\beta ij} h^4_{ij}h^\beta_{ij}H^\beta - 2\lambda_2\sqrt{1-C^2}C_{,2} = 0, \end{cases}$$

where C denotes the Kähler function of M in $\mathbb{C}P^2$, $C_{,i}$ $(1 \leq i \leq 2)$ denote the first covariant derivatives of C, and H^β $(3 \leq \beta \leq 4)$ are the coefficient of the mean curvature vector H of M.

It follows from the above Euler–Lagrange equation that every minimal surfaces with constant Kähler angle is Helfrich surface. In particular, the complex curve and Lagrangian minimal surface in $\mathbb{C}P^2$ are Helfrich surfaces.

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We, on the other hand, will focus on the homogeneous tori in $\mathbb{C}P^2$. We are going to show the homogeneous Helfrich tori in $\mathbb{C}P^2$ and compute the Helfrich energy for the homogenous tori, thereby determining the energy minimizers within this class of surfaces.

Theorem 1.2. *Let T_{r_1,r_2,r_3} be a homogeneous torus in $\mathbb{C}P^2$. Then, T_{r_1,r_2,r_3} is a Helfrich surface if and only if*

1. *When $0 \leqslant \lambda_1 \leqslant \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$.*
2. *When $\lambda_1 > \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$ or $T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$.*

Theorem 1.3. *Let T_{r_1,r_2,r_3} be a homogeneous torus in $\mathbb{C}P^2$. Then,*

1. *When $0 \leqslant \lambda_1 \leqslant 3$,*
$$\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{4\lambda_1\pi^2}{3\sqrt{3}}$$
with equality holding if and only if $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$.
2. *When $\lambda_1 > 3$,*
$$\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{(4\lambda_1-8)\pi^2}{\sqrt{4\lambda_1-9}}$$
with equality holding if and only if $T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$.

The arrangement of this paper is as follows. In Section 2, we recall the basic theory of surfaces in $\mathbb{C}P^2$. In Section 3, we calculate the Euler–Lagrangian equation of the critical surfaces of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$. Then, in Section 4, we consider the homogeneous tori in $\mathbb{C}P^2$ and give the proof of Theorems 1.2 and 1.3.

2. Preliminaries

In this section, we will review the moving frame method for surfaces in $\mathbb{C}P^2$ following Chern and Wolfson (for more details, see [4]). In the paper, we will adopt the following ranges of indices:

$$0 \leqslant a, b, c \leqslant 2,\ 1 \leqslant i, j, k \leqslant 2,\ 3 \leqslant \alpha, \beta \leqslant 4,\ 1 \leqslant A, B \leqslant 4.$$

Let $\langle , \rangle$ be the hermitian product in $\mathbb{C}^3$, that is,

$$\langle Z, W\rangle = \sum_{l=1}^{3} Z_l\overline{W}_l$$

for any $Z, W \in \mathbb{C}^3$, where $\overline{W}$ denotes the conjugate of W. Let $\mathbb{C}P^2$ be the complex projective plane with its canonical Fubini–Study metric of constant holomorphic sectional curvature 4. Then,

$$\mathbb{C}P^2 = \{[Z_0] = \Pi(Z_0) | Z_0 = (z_1, z_2, z_3) \in \mathbb{C}^3 - \{0\}, |Z_0| = 1\},$$

where $\Pi : S^5 \to \mathbb{C}P^2$ is the Hopf projection. We denote g by its Fubini–Study metric and J by its complex structure induced by $\mathbb{C}^3$ on $\mathbb{C}P^2$. Then,

$$g = \langle dZ_0, dZ_0\rangle - \langle dZ_0, Z_0\rangle\langle Z_0, dZ_0\rangle. \tag{2.1}$$

The Kähler form Ω on $\mathbb{C}P^2$ is defined by:

$$\Omega(u, v) = g(Ju, v),\ \text{for any } u, v \in \Gamma(T(\mathbb{C}P^2)). \tag{2.2}$$

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Let $\{Z_0, Z_1, Z_2\}$ be a unitary frames in $\mathbb{C}^3$. Then, we have

$$\langle Z_a, Z_b\rangle = \delta_{a\bar{b}},\ dZ_a = \sum_b \psi_{a\bar{b}} Z_b, \tag{2.3}$$

where $\psi_{a\bar{b}} = \overline{\psi_{\bar{a}b}}$ is connection 1-form and satisfies structure equation:

$$d\psi_{a\bar{b}} = \sum_c \psi_{a\bar{c}} \wedge \psi_{c\bar{b}}. \tag{2.4}$$

Moreover, the Fubini–Study metric (2.1) can be written as:

$$g = \sum_i \psi_{0\bar{i}} \overline{\psi_{0\bar{i}}}. \tag{2.5}$$

On the other hand, let $\{\varsigma_i\}$ be a unitary frames in $\mathbb{C}P^2$ with dual frames $\{\omega_i\}$, the structure equation of $\mathbb{C}P^2$ can be written as:

$$d\omega_i = \sum_j \omega_{i\bar{j}} \wedge \omega_j,\ \omega_{i\bar{j}} + \omega_{\bar{j}i} = 0,\ \omega_{\bar{j}i} = \overline{\omega_{j\bar{i}}}, \tag{2.6}$$

with $\{\omega_{i\bar{j}}\}$ being the unitary connection forms with respect to $\{\omega_i\}$. We have then

$$g = \sum_i \omega_i \overline{\omega_i} = \sum_i \omega_i \omega_{\bar{i}}. \tag{2.7}$$

Combining with (2.5) and (2.7) and (2.4) and (2.6), we get

$$\omega_i = \psi_{0\bar{i}}, \tag{2.8}$$

$$\omega_{i\bar{j}} = \delta_{ij}\psi_{0\bar{0}} - \psi_{j\bar{i}}. \tag{2.9}$$

Let $f: M \to \mathbb{C}P^2$ be an isometric immersion of a compact surface M in the complex projective plane $\mathbb{C}P^2$. The Kähler function C on M is defined by:

$$f^*\Omega = C\mathrm{d}M, \tag{2.10}$$

where $\mathrm{d}M$ is the area form on M. The surface f is holomorphic, anti-holomorphic, or Lagrangian, respectively, depending on $C \equiv 1, -1$, or 0. Now, we consider $M \subset \mathbb{C}P^2$ first from the Riemannian geometry version and then from the complex version due to Chern and Wolfson [4].

Let us choose a (new) local orthonormal frame $\{e_A\}$ of $\mathbb{C}P^2$ with its dual $\{\theta_A\}$ such that restricting to M, $\{e_i\}$ is a local orthonormal basis of TM. Then, we have restricted to M

$$\theta_\alpha = 0,\ \theta_{i\alpha} = \sum_j h_{ij}\theta_j.$$

The second fundamental form II and the mean curvature vector H are defined by:

$$II = \sum_{\alpha ij} h^\alpha_{ij}\theta_i \otimes \theta_j \otimes e_\alpha,\ H = \frac{1}{2}\sum_{\alpha i} h^\alpha_{ii} e_\alpha = \sum_\alpha H^\alpha e_\alpha.$$

Let $\overline{\nabla}$ be the Riemannian connection of $\mathbb{C}P^2$, and let ∇ and $\nabla^\perp$ be the induced connection and normal connection of M, respectively. The covariant derivative and Laplacian of H on the normal bundle of M are, respectively, defined as:

$$\sum_i H^\alpha_{,i}\theta_i = dH^\alpha + \sum_\beta H^\beta\theta_{\beta\alpha},$$

$$\sum_j H^\alpha_{,ij}\theta_j = dH^\alpha_{,i} + \sum_j H^\alpha_{,j}\theta_{ji} + \sum_\beta H^\beta_{,i}\theta_{\beta\alpha},$$

$$\triangle^\perp H^\alpha = \sum_i H^\alpha_{,ii}.$$

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Let $\overline{R}_{ABCD}$ be the Riemannian curvature tensor of $\mathbb{C}P^2$, we have then

$$\begin{aligned}\overline{R}_{ABCD} &= g(e_A, e_C)g(e_B, e_D) - g(e_A, e_D)g(e_B, e_C) + g(Je_A, e_C)g(Je_B, e_D) \\ &\quad - g(Je_A, e_D)g(Je_B, e_C) + 2g(Je_A, e_B)g(Je_C, e_D).\end{aligned} \tag{2.11}$$

Now, let us recall the complex version of the geometry of M which is due to Chern and Wolfson [4]. Set $\phi = \theta_1 + i\theta_2$. It defines a complex structure on M. The induced metric on M is of the form:

$$f^*g = \sum_i \omega_i \overline{\omega_i} = \phi\overline{\phi}. \tag{2.12}$$

Then after a normalization of $\{\varsigma_i\}$ if necessary ([4, p. 66]), we can assume that there exist complex-valued smooth functions s, t which satisfy $|s|^2 + |t|^2 = 1$ such that

$$\omega_1 = s\phi,\ \omega_2 = t\overline{\phi}. \tag{2.13}$$

In particular, setting $|s| = \cos\frac{\alpha}{2}$, $|t| = \sin\frac{\alpha}{2}$, one has then $C = \cos\alpha$ with $0 \le \alpha \le \pi$. Now we have, along M,

$$\bar{s}\omega_1 + t\overline{\omega_2} = \phi, \tag{2.14}$$

$$\bar{t}\omega_1 - s\overline{\omega_2} = \theta_3 + i\theta_4 = 0. \tag{2.15}$$

Taking exterior derivative of the first term of (2.15), we get

$$((sd\bar{t} - \bar{t}ds) + s\bar{t}(\omega_{1\bar{1}} + \omega_{2\bar{2}})) \wedge \phi + \omega_{1\bar{2}} \wedge \overline{\phi} = 0. \tag{2.16}$$

Set

$$(sd\bar{t} - \bar{t}ds) + s\bar{t}(\omega_{1\bar{1}} + \omega_{2\bar{2}}) = a\phi + b\overline{\phi}, \tag{2.17}$$

$$\omega_{1\bar{2}} = b\phi + c\overline{\phi}, \tag{2.18}$$

then the complex-valued second fundamental forms can be given by:

$$II^{\mathbb{C}} = a\phi^2 + 2b\phi\overline{\phi} + c\overline{\phi}^2. \tag{2.19}$$

Lemma 2.1. *The coefficients $a,\ b,\ c$ of the complex-valued second fundamental forms $II^{\mathbb{C}}$ satisfy*

$$\begin{aligned}&Re(a+c) + Re(2b) = h^3_{11},\ Im(c-a) = h^3_{12},\ Re(2b) - Re(a+c) = h^3_{22},\\ &Im(a+c) + Im(2b) = h^4_{11},\ Re(a-c) = h^4_{12},\ Im(2b) - Im(a+c) = h^4_{22},\end{aligned} \tag{2.20}$$

where Re *and* Im *denote, respectively, the real and imaginary parts.*

Proof. Taking exterior derivative of the second term of (2.15), we get

$$\begin{aligned}&d(\theta_3 + i\theta_4)\\ &= -\frac{1}{2}\left(\frac{h^3_{11} - h^3_{22} + 2h^4_{12} + i(h^4_{11} - h^4_{22} - 2h^3_{12})}{4}\phi + \frac{h^3_{11} + h^3_{22} + i(h^4_{11} + h^4_{22})}{4}\overline{\phi}\right) \wedge \phi\\ &\quad - \frac{1}{2}\left(\frac{h^3_{11} + h^3_{22} + i(h^4_{11} + h^4_{22})}{4}\phi + \frac{h^3_{11} - h^3_{22} - 2h^4_{12} + i(h^4_{11} - h^4_{22} + 2h^3_{12})}{4}\overline{\phi}\right) \wedge \overline{\phi}.\end{aligned}$$

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Then from (2.15), (2.16), (2.17), and (2.18), we have that

$$a = \frac{h^3_{11} - h^3_{22} + 2h^4_{12} + i(h^4_{11} - h^4_{22} - 2h^3_{12})}{4},$$

$$b = \frac{h^3_{11} + h^3_{22} + i(h^4_{11} + h^4_{22})}{4},$$

$$c = \frac{h^3_{11} - h^3_{22} - 2h^4_{12} + i(h^4_{11} - h^4_{22} + 2h^3_{12})}{4},$$

and (2.20) follows. □

Remark 2.2. *Note that we can re-choose the unitary coframe* $\{\omega_1, \omega_2\}$ *such that*

$$s = \cos\frac{\alpha}{2},\ t = \sin\frac{\alpha}{2}. \tag{2.21}$$

The Kähler angle α *is smooth at the points with* $0 < \alpha < \pi$. *At the points* $\alpha = 0$ *or* π, α *could be only continuous. Moreover, under the assumption of* (2.21), *we can obtain by* (2.13) *that*

$$\begin{cases} \omega_1 = \cos\frac{\alpha}{2}\theta_1 + \sin\frac{\alpha}{2}\theta_3 + i\left(\cos\frac{\alpha}{2}\theta_2 + \sin\frac{\alpha}{2}\theta_4\right), \\ \omega_2 = \sin\frac{\alpha}{2}\theta_1 - \cos\frac{\alpha}{2}\theta_3 + i\left(-\sin\frac{\alpha}{2}\theta_2 + \cos\frac{\alpha}{2}\theta_4\right), \end{cases}$$

and hence

$$\begin{cases} \varsigma_1 = \frac{1}{2}\left(\cos\frac{\alpha}{2}e_1 + \sin\frac{\alpha}{2}e_3 - i\left(\cos\frac{\alpha}{2}e_2 + \sin\frac{\alpha}{2}e_4\right)\right), \\ \varsigma_2 = \frac{1}{2}\left(\sin\frac{\alpha}{2}e_1 - \cos\frac{\alpha}{2}e_3 - i\left(-\sin\frac{\alpha}{2}e_2 + \cos\frac{\alpha}{2}e_4\right)\right). \end{cases}$$

So, we have

$$J\left(\cos\frac{\alpha}{2}e_1 + \sin\frac{\alpha}{2}e_3\right) = \cos\frac{\alpha}{2}e_2 + \sin\frac{\alpha}{2}e_4,\ J\left(\sin\frac{\alpha}{2}e_1 - \cos\frac{\alpha}{2}e_3\right) = -\sin\frac{\alpha}{2}e_2 + \cos\frac{\alpha}{2}e_4,$$

from which we get

$$\begin{cases} Je_1 = Ce_2 + \sqrt{1-C^2}e_4,\ Je_2 = -Ce_1 - \sqrt{1-C^2}e_3, \\ Je_3 = -Ce_4 + \sqrt{1-C^2}e_2,\ Je_4 = Ce_3 - \sqrt{1-C^2}e_1. \end{cases}$$

This is exactly we assume in (1.2) *of Section* 1.

3. Euler–Lagrange equation of Helfrich functional

Let $f(p,t): M \times (-\epsilon, \epsilon) \to \mathbb{C}P^2$ be a variation of M with $f_0(p) = f(p)$. Here, we denote by $f_t(p) = f(p,t): M \to \mathbb{C}P^2$ for $t \in (-\epsilon, \epsilon)$. Let $\{x_1, x_2, t\}$ be a local coordinate system around the point $(p, 0)$ such that $\{\mathrm{d}f\left(\frac{\partial}{\partial x_1}\right), \mathrm{d}f\left(\frac{\partial}{\partial x_2}\right)\}|_p$ is an orthonormal basis of T_pM. Set $\bar{V} = \mathrm{d}f_t\left(\frac{\partial}{\partial t}\right)$, $X_i = \mathrm{d}f_t\left(\frac{\partial}{\partial x_i}\right)$. Then, we have the induced metric of f_t and its area form as follows:

$$(g_t)_{ij} = g_t(X_i, X_j),\ \mathrm{d}M_t = \sqrt{G_t}\mathrm{d}x_1 \wedge \mathrm{d}x_2 \text{ with } G_t = \det(g_t).$$

Then $(g_0)_{ij}(p) = g_{ij}(p)\delta_{ij}$. Set $\tilde{e}_i = \mathrm{d}f_t|_{t=0}\left(\frac{\partial}{\partial x_i}\right)$ and $V = \mathrm{d}f_t|_{t=0}\left(\frac{\partial}{\partial t}\right) = V^\top + V^\perp$ with $V^\top \in \Gamma(TM)$ and $V^\perp \in \Gamma(T^\perp M)$.

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We first consider $\frac{\partial(\sqrt{G_t})}{\partial t}|_{t=0}$ and $\frac{\partial|H|^2}{\partial t}|_{t=0}$. It is well known that

$$\frac{\partial(\sqrt{G_t})}{\partial t}|_{t=0} = (\mathrm{div}V^\top - 2\langle H, V\rangle)\sqrt{G_t}. \tag{3.1}$$

Now we consider $\frac{\partial|H|^2}{\partial t}|_{t=0}$. It follows the definition of the mean curvature vector that

$$2H_t = \sum_{ij} g_t^{ij} h(X_i, X_j),$$

where $((g_t)^{ij})$ denotes the inverse matrix of $((g_t)_{ij})$. Thus, we have at the point p that

$$\frac{\partial}{\partial t}(2H_t)|_{t=0} = \sum_{ij}\left(\frac{\partial(g_t)^{ij}}{\partial t}|_{t=0} h(\tilde{e}_i, \tilde{e}_j) + \delta_{ij}\frac{\partial h(X_i, X_j)}{\partial t}|_{t=0}\right). \tag{3.2}$$

Differentiating the formula $\sum_j g^{ij} g_{jk} = \delta_{ik}$ and using the fact that $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial x_i}\right] = 0$, we get

$$\begin{aligned}\frac{\partial(g_t)^{ij}}{\partial t}|_{t=0} &= -\frac{\partial(g_t)_{ij}}{\partial t}|_{t=0} = -\overline{\nabla}_V g(\tilde{e}_i, \tilde{e}_j)\\ &= -\overline{\nabla}_{V^\top} g(\tilde{e}_i, \tilde{e}_j) + (\overline{\nabla}_{\tilde{e}_i} V^\perp, \tilde{e}_j) + (\tilde{e}_i, \overline{\nabla}_{\tilde{e}_j} V^\perp))\\ &= -\nabla_{V^\top} g(\tilde{e}_i, \tilde{e}_j) + 2g(h(\tilde{e}_i, \tilde{e}_j), V^\perp).\end{aligned} \tag{3.3}$$

Also, we have

$$\begin{aligned}\delta_{ij}\frac{\partial h(X_i, X_j)}{\partial t}|_{t=0} &= \overline{\nabla}_V(\overline{\nabla}_{\tilde{e}_i}\tilde{e}_i - \nabla_{\tilde{e}_i}\tilde{e}_i)\\ &= \overline{\nabla}_{V^\top}\overline{\nabla}_{\tilde{e}_i}\tilde{e}_i - \overline{\nabla}_{V^\top}\nabla_{\tilde{e}_i}\tilde{e}_i\\ &\quad + \overline{\nabla}_{\tilde{e}_i}\overline{\nabla}_{\tilde{e}_i} V^\perp + \overline{R}(V^\perp, \tilde{e}_i)\tilde{e}_i - \overline{\nabla}_{V^\perp}\nabla_{\tilde{e}_i}\tilde{e}_i.\end{aligned} \tag{3.4}$$

Since $\frac{\partial|H|^2}{\partial t} = g\left(H, 2\frac{\partial H}{\partial t}\right)$, we only need to know the normal part of (3.4). So, by a direct computation we have

$$\left(\sum_i (\overline{\nabla}_{V^\top}\overline{\nabla}_{\tilde{e}_i}\tilde{e}_i - \overline{\nabla}_{V^\top}\nabla_{\tilde{e}_i}\tilde{e}_i)\right)^\perp = \left(\sum_i \overline{\nabla}_{V^\top} h(\tilde{e}_i, \tilde{e}_i)\right)^\perp = 2\nabla^\perp_{V^\top} H, \tag{3.5}$$

$$\begin{aligned}&\left(\sum_i (\overline{\nabla}_{\tilde{e}_i}\overline{\nabla}_{\tilde{e}_i} V^\perp + \overline{R}(V^\perp, \tilde{e}_i)\tilde{e}_i - \overline{\nabla}_{V^\perp}\nabla_{\tilde{e}_i}\tilde{e}_i)\right)^\perp\\ &= \sum_i (-h(\tilde{e}_i, A_{V^\perp}(\tilde{e}_i)) + \nabla^\perp_{\tilde{e}_i}\nabla^\perp_{\tilde{e}_i} V^\perp + (\overline{R}(V^\perp, \tilde{e}_i)\tilde{e}_i)^\perp - \nabla^\perp_{\nabla_{\tilde{e}_i}\tilde{e}_i} V^\perp)\\ &= \triangle^\perp V^\perp + \sum_i (-g(h(\tilde{e}_i, \tilde{e}_j), V^\perp) h(\tilde{e}_i, \tilde{e}_j) + (\overline{R}(V^\perp, \tilde{e}_i)\tilde{e}_i)^\perp).\end{aligned} \tag{3.6}$$

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Here, we used the fact that $(\overline{\nabla}_{V^\perp}\nabla_{\tilde{e}_i}\tilde{e}_i)^\perp = \nabla^\perp_{\nabla_{\tilde{e}_i}\tilde{e}_i}V^\perp$. Finally, substituting (3.3) and (3.4) into (3.2) and using (3.5), (3.6), and (2.11), we obtain at the point p that

$$\begin{aligned}\frac{\partial |H|^2}{\partial t}|_{t=0} &= g\left(H, 2\frac{\partial H}{\partial t}\right)\\ &= \sum_\alpha \Big(H^\alpha \triangle^\perp V^\alpha + V^\alpha(5-3C^2)H^\alpha + \sum_{ij\beta} V^\alpha H^\beta h^\alpha_{ij}h^\beta_{ij}\\ &\quad - \sum_{ijk} H^\alpha h^\alpha_{ij} V^k \nabla_{\tilde{e}_k} g_{ij}\Big) + \sum_i V^i \nabla_{\tilde{e}_i}|H|^2.\\ &= \sum_\alpha \left(H^\alpha \triangle^\perp V^\alpha + V^\alpha(5-3C^2)H^\alpha + \sum_{ij\beta} V^\alpha H^\beta h^\alpha_{ij}h^\beta_{ij}\right) + \sum_i V^i \nabla_{\tilde{e}_i}|H|^2. \end{aligned} \tag{3.7}$$

Here, we used

$$\sum_{ijk} V^k H^\alpha h^\alpha_{ij}\nabla_{\tilde{e}_k} g_{ij} = \sum_{ik}(-V^k\nabla_{\tilde{e}_k}(H^\alpha h^\alpha_{ii}) - H^\alpha h^\alpha_{ii}\mathrm{div}V^\top + \mathrm{div}(H^\alpha h^\alpha_{ii}V^\top)) = 0.$$

Next, we consider $\frac{\partial (C_t)^2}{\partial t}|_{t=0}$. First for an oriented orthonormal basis $\{e_1, e_2\}$ of TM, set

$$X_V = \Omega(V, e_2)e_1 + \Omega(e_1, V)e_2.$$

It is direct to check that X_V is independent of the choice $\{e_1, e_2\}$, and hence it defines a smooth vector field on M with

$$\mathrm{div}X_V = \nabla_{e_1}\Omega(V, e_2) + \nabla_{e_2}\Omega(e_1, V). \tag{3.8}$$

The definition of the Kähler function means that

$$C_t = \frac{\Omega(X_1, X_2)}{\sqrt{G_t}}.$$

So we get, at the point p,

$$\frac{\partial (C_t)}{\partial t}|_{t=0} = \frac{\partial \Omega(X_1, X_2)}{\partial t}|_{t=0} - C\frac{\partial(\sqrt{G_t})}{\partial t}|_{t=0}. \tag{3.9}$$

Since $[\tilde{e}_1, \tilde{e}_2]|_p = 0$ and $[V, \tilde{e}_i]|_p = 0$ for $1 \le i \le 2$, we have at the point p that

$$\begin{aligned}\frac{\partial \Omega(X_1, X_2)}{\partial t}|_{t=0} &= \Omega(\overline{\nabla}_{\tilde{e}_1}V, \tilde{e}_2) + \Omega(\tilde{e}_1, \overline{\nabla}_{\tilde{e}_2}V)\\ &= \overline{\nabla}_{\tilde{e}_1}\Omega(V, \tilde{e}_2) - \Omega(V, \overline{\nabla}_{\tilde{e}_1}\tilde{e}_2) + \overline{\nabla}_{\tilde{e}_2}\Omega(\tilde{e}_1, V) - \Omega(\overline{\nabla}_{\tilde{e}_2}\tilde{e}_1, V)\\ &= \nabla_{\tilde{e}_1}\Omega(V, \tilde{e}_2) + \nabla_{\tilde{e}_2}\Omega(\tilde{e}_1, V) = \mathrm{div}(X_V).\end{aligned} \tag{3.10}$$

Substituting (3.1) and (3.10) into (3.9), we obtain at the point p that

$$\frac{\partial C^2}{\partial t}|_{t=0} = \Big(2C\,\mathrm{div}X_V - 2C^2(\mathrm{div}V^\top - 2\langle H, V\rangle)\Big). \tag{3.11}$$

Noting that the right sides of (3.7) and (3.11) are independent of the coordinates and hence valid at any point of M. Thus, from (3.1), (3.7), and (3.11), we get

$$\begin{aligned}\frac{d}{dt}|_{t=0}(\mathcal{H}_{\lambda_1,\lambda_2}(f)) &= \int_M \frac{\partial}{\partial t}(|H|^2+\lambda_1+\lambda_2 C^2)|_{t=0}\mathrm{d}M + \int_M (|H|^2+\lambda_1+\lambda_2 C^2)\frac{\partial(\mathrm{d}M)}{\partial t}|_{t=0} \\ &= \int_M \Bigg(\sum_\alpha \Big(H^\alpha \triangle^\perp V^\alpha + [(5-2\lambda_1-(3-2\lambda_2)C^2-2|H|^2)H^\alpha + \sum_{\beta ij} h^\alpha_{ij}h^\beta_{ij}H^\beta]V^\alpha\Big) \\ &\quad + \sum_i V^i\nabla_{\tilde{e}_i}|H|^2 + 2\lambda_2 C\,\mathrm{div}X_V + (\lambda_1+\lambda_2C^2+|H|^2)\mathrm{div}V_\top\Bigg)\mathrm{d}M.\end{aligned} \tag{3.12}$$

Furthermore, it follows from the divergence theorem that

$$\int_M H^\alpha \triangle^\perp V^\alpha \mathrm{d}M = \int_M V^\alpha \triangle^\perp H^\alpha \mathrm{d}M. \tag{3.13}$$

$$\int_M \sum_i V^i\nabla_{\tilde{e}_i}|H|^2\mathrm{d}M = -\int_M |H|^2\mathrm{div}(V^\top)\mathrm{d}M. \tag{3.14}$$

$$\int_M 2C\,\mathrm{div}X_V = -2\int_M \langle\nabla C, X_V\rangle\mathrm{d}M = -2\int_M \Big(\nabla_{e_1}C\cdot\Omega(V,e_2)+\nabla_{e_2}C\cdot\Omega(e_1,V)\Big)\mathrm{d}M. \tag{3.15}$$

$$\int_M C^2\mathrm{div}V^\top\mathrm{d}M = -2\int_M \Big(\nabla_{e_1}C\cdot\Omega(V^\top,e_2)+\nabla_{e_2}C\cdot\Omega(e_1,V^\top)\Big)\mathrm{d}M. \tag{3.16}$$

Substituting (3.13)–(3.16) into (3.12) and noting

$$\begin{aligned}\nabla_{e_1}C\cdot\Omega(V^\perp,e_2)+\nabla_{e_2}C\cdot\Omega(e_1,V^\perp) &= -\nabla_{e_1}C\cdot\langle Je_2,V^\perp\rangle+\nabla_{e_2}C\cdot\langle Je_1,V^\perp\rangle \\ &= \sqrt{1-C^2}\langle(\nabla_{e_1}C\,e_3+\nabla_{e_2}C\,e_4),V\rangle,\end{aligned} \tag{3.17}$$

we obtain that

$$\begin{aligned}\frac{d}{dt}|_{t=0}(\mathcal{H}_{\lambda_1,\lambda_2}(f)) &= \int_M \Bigg(\sum_\alpha V^\alpha[\triangle^\perp H^\alpha + (5-2\lambda_1-(3-2\lambda_2)C^2-2|H|^2)H^\alpha \\ &\quad + \sum_{\beta ij} h^\alpha_{ij}h^\beta_{ij}H^\beta] - 2\lambda_2\sqrt{1-C^2}(C_{,1}V^3+C_{,2}V^4)\Bigg)\mathrm{d}M,\end{aligned}$$

where $C_{,i}$ $(1\le i\le 2)$ denote the first covariant derivatives of C. This implies that the Euler–Lagrange equation of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$ is

$$\begin{cases}\triangle^\perp H^3 + (5-2\lambda_1-(3-2\lambda_2)C^2-2|H|^2)H^3+\sum_{\beta ij}H^\beta h^\beta_{ij}h^3_{ij} - 2\lambda_2\sqrt{1-C^2}C_{,1}=0,\\ \triangle^\perp H^4 + (5-2\lambda_1-(3-2\lambda_2)C^2-2|H|^2)H^4+\sum_{\beta ij}H^\beta h^\beta_{ij}h^4_{ij} - 2\lambda_2\sqrt{1-C^2}C_{,2}=0.\end{cases} \tag{3.18}$$

This gives the proof of Theorem 1.1.

Remark 3.1. *When $\lambda_2=0$, the function reduces to $\mathcal{H}_{\lambda_1,\lambda_2}(f)=\int_M(|H|^2+\lambda_1)\mathrm{d}M$. In this situation, if M is minimal, we obtain from (3.18) that M is Helfrich surface.*

Remark 3.2. *When $C=$ constant, that is, M has constant Kähler angle. If M is minimal, then M is Helfrich surface.*

Combing this we have

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Corollary 3.3. *The complex curves and Lagrangian minimal surfaces in complex projective plane* $\mathbb{C}P^2$ *are Helfrich surfaces.*

Corollary 3.4. *Let* $f : M \to \mathbb{C}P^2$ *be an isometric immersion of a compact surface in* $\mathbb{C}P^2$*. If M is a Helfrich surface for any* λ_1, λ_2*. Then M is minimal.*

4. Homogeneous tori in $\mathbb{C}P^2$

In this section, we consider the homogeneous tori in $\mathbb{C}P^2$.

4.1. The geometry of homogeneous tori

The definition of the homogeneous torus in $\mathbb{C}P^2$ is given by the image of the Hopf projection:

$$T_{r_1,r_2,r_3} = \{\Pi(Z_0) \in \mathbb{C}P^2 | Z_0 = (z_1, z_2, z_3), |z_l| = r_l, l = 1, 2, 3\},$$

for positive numbers r_1, r_2, r_3 that satisfy $r_1^2 + r_2^2 + r_3^2 = 1$. In this case, we also call $T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$ the Clifford torus. We note that Ma, Mironov, and Zuo in [11] gave a basis of the period module for the homogeneous tori. Here, we discuss the conformal structure of the homogeneous tori for completeness.

Taking into account the definition of T_{r_1,r_2,r_3}, we assume that the homogeneous coordinate of T_{r_1,r_2,r_3} is

$$Z_0 = (r_1, r_2e^{i\varphi}, r_3e^{i\psi}), \text{ i.e., } dZ_0 = (0, ir_2e^{i\varphi}, ir_3e^{i\psi}),$$

where $\varphi,\ \psi \in \mathbb{R}$. Then we have from (2.1) that

Lemma 4.1. *The induced metric of* T_{r_1,r_2,r_3} *in* $\mathbb{C}P^2$ *is*

$$g = r_2^2r_3^2\left(\frac{1-r_2^2}{r_3^2}\mathrm{d}\varphi^2 + \frac{1-r_3^2}{r_2^2}\mathrm{d}\psi^2 - 2\mathrm{d}\varphi\mathrm{d}\psi\right). \tag{4.1}$$

Setting

$$\begin{cases} u = \frac{\sqrt{1-r_2^2}}{r_3}\varphi - \frac{r_3}{\sqrt{1-r_2^2}}\psi, \\ v = \frac{r_1}{r_2\sqrt{1-r_2^2}}\psi, \end{cases} \quad \text{i.e.} \quad \begin{cases} \varphi = \frac{r_3}{\sqrt{1-r_2^2}}u + \frac{r_2^2r_3}{r_1\sqrt{1-r_2^2}}v, \\ \psi = \frac{r_2\sqrt{1-r_2^2}}{r_1}\psi. \end{cases} \tag{4.2}$$

Then

$$g = r_2^2r_3^2(du^2 + dv^2) = r_2^2r_3^2|\mathrm{d}z|^2,$$

and hence $z = u + iv$ gives a complex coordinate of T_{r_1,r_2,r_3}. Also, from (4.2) we get a basis (ω_1, ω_2) of the period module for T_{r_1,r_2,r_3} with

$$\omega_1 = \frac{2\pi\sqrt{1-r_2^2}}{r_3}, \ \omega_2 = \frac{2\pi r_1^2}{r_3\sqrt{1-r_2^2}} + i\frac{2\pi r_1}{r_2\sqrt{1-r_2^2}}.$$

So without loss of generality, we assume $r_3 \geqslant r_1 \geqslant r_2$, $k = \frac{r_3}{r_1}$, then

$$\tau = \frac{\omega_2}{\omega_1} = \frac{1}{k^2+1} + i\frac{k}{r_2(k^2+1)}, \text{ with } \{\tau | 0 < \operatorname{Re}\tau \leqslant \frac{1}{2}, |\tau| \geqslant 1\}$$

gives the module space for T_{r_1,r_2,r_3} (see the shaded part of Figure 1).

In particular, for Clifford torus $T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$, $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Now, we are going to consider the second fundamental forms of T_{r_1,r_2,r_3} in $\mathbb{C}P^2$.

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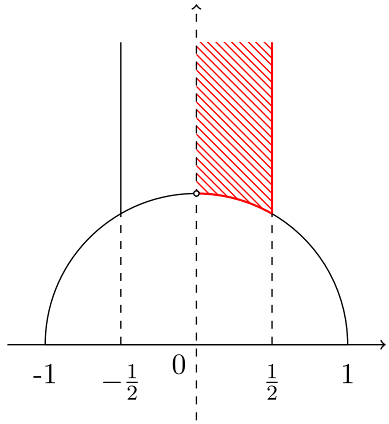

Figure 1. *τ-plane.*

Lemma 4.2. *The second fundamental forms of T_{r_1,r_2,r_3} satisfy*

$$\begin{pmatrix} h^3_{11} & h^3_{12} \\ h^3_{21} & h^3_{22} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-r_2}{\sqrt{1-r_2^2}} \\ \frac{-r_2}{\sqrt{1-r_2^2}} & \frac{r_1^2-r_3^2}{r_1 r_3\sqrt{1-r_2^2}} \end{pmatrix}, \tag{4.3}$$

$$\begin{pmatrix} h^4_{11} & h^4_{12} \\ h^4_{21} & h^4_{22} \end{pmatrix} = \begin{pmatrix} \frac{2r_2^2-1}{r_2\sqrt{1-r_2^2}} & 0 \\ 0 & \frac{r_2}{\sqrt{1-r_2^2}} \end{pmatrix}. \tag{4.4}$$

Proof. Set $\phi = r_2 r_3(\mathrm{d}u + i\mathrm{d}v)$, then the induced metric of T_{r_1,r_2,r_3} can be written as:

$$g = \phi \cdot \bar{\phi}.$$

Let $\{Z_0, Z_1, Z_2\}$ be a unitary frames in $\mathbb{C}^3$. Then $\mathrm{d}Z_0 = \sum_a \psi_{0\bar{a}} Z_a$ with $\psi_{a\bar{b}} = \overline{\psi_{\bar{a}b}}$. From this, we get

$$\begin{aligned} \psi_{0\bar{1}} Z_1 + \psi_{0\bar{2}} Z_2 &= \mathrm{d}Z_0 - \psi_{0\bar{0}} Z_0 \\ &= \mathrm{d}Z_0 - h(dZ_0, Z_0) Z_0 \\ &= \left(\frac{-r_3 - ir_1r_2}{2\sqrt{1-r_2^2}}, \frac{i\sqrt{1-r_2^2}e^{i\varphi}}{2}, \frac{(r_1 - ir_2r_3)e^{i\psi}}{2\sqrt{1-r_2^2}} \right) \phi \\ &\quad + \left(\frac{r_3 - ir_1r_2}{2\sqrt{1-r_2^2}}, \frac{i\sqrt{1-r_2^2}e^{i\varphi}}{2}, \frac{(-r_1 - ir_2r_3)e^{i\psi}}{2\sqrt{1-r_2^2}} \right) \bar{\phi}. \end{aligned} \tag{4.5}$$

And hence we obtain from $|Z_1| = |Z_2| = 1$ that

$$\begin{aligned} Z_1 &= \left(\frac{-r_3 - ir_1r_2}{\sqrt{2(1-r_2^2)}}, \frac{i\sqrt{1-r_2^2}e^{i\varphi}}{\sqrt{2}}, \frac{(r_1 - ir_2r_3)e^{i\psi}}{\sqrt{2(1-r_2^2)}} \right), \\ Z_2 &= \left(\frac{r_3 - ir_1r_2}{\sqrt{2(1-r_2^2)}}, \frac{i\sqrt{1-r_2^2}e^{i\varphi}}{\sqrt{2}}, \frac{(-r_1 - ir_2r_3)e^{i\psi}}{\sqrt{2(1-r_2^2)}} \right). \end{aligned} \tag{4.6}$$

Let $\{\omega_i\}$ be an unitary coframe in $\mathbb{C}P^2$ such that restricting to M

$$\omega_1 = s\phi, \ \omega_2 = t\bar{\phi}. \tag{4.7}$$

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Then from (2.8) and (4.5), we have

$$s = t = \frac{\sqrt{2}}{2}. \tag{4.8}$$

Now, using the fact $\psi_{a\bar{b}} = \langle dZ_a, Z_b \rangle$, then from (2.9) we get

$$\omega_{1\bar{1}} = \psi_{0\bar{0}} - \psi_{1\bar{1}} = \frac{(3r_2^2-1)i}{2}d\varphi + \frac{(3r_3^2-1)i}{2}d\psi,$$

$$\omega_{2\bar{2}} = \psi_{0\bar{0}} - \psi_{2\bar{2}} = \frac{(3r_2^2-1)i}{2}d\varphi + \frac{(3r_3^2-1)i}{2}d\psi,$$

$$\omega_{1\bar{2}} = -\psi_{2\bar{1}} = -\frac{(1-r_2^2)i}{2}d\varphi - \frac{2r_1r_2r_2r_3 + i(r_2^2r_3^2 - r_0^2)}{2(1-r_2^2)}d\psi,$$

and so

$$s\bar{t}(\omega_{1\bar{1}} + \omega_{2\bar{2}}) = \left(\frac{2r_3^2 + r_2^2 - 1}{4r_1r_3\sqrt{(1-r_2^2)}} + \frac{i(3r_2^2-1)}{4r_2\sqrt{1-r_2^2}}\right)\phi + \left(\frac{-(2r_3^2 + r_2^2 - 1)}{4r_1r_3\sqrt{(1-r_2^2)}} + \frac{i(3r_2^2-1)}{4r_2\sqrt{1-r_2^2}}\right)\overline{\phi},$$

$$\omega_{1\bar{2}} = \left(\frac{-(2r_3^2 + r_2^2 - 1)}{4r_1r_3\sqrt{(1-r_2^2)}} + \frac{i(3r_2^2-1)}{4r_2\sqrt{1-r_2^2}}\right)\phi + \left(\frac{2r_3^2 + r_2^2 - 1}{4r_1r_3\sqrt{(1-r_2^2)}} - \frac{i(r_2^2+1)}{4r_2\sqrt{1-r_2^2}}\right)\overline{\phi}.$$

Thus, we obtain from (2.17) and (2.18) that the coefficients of the complex-valued second fundamental forms (see (2.19)) of T_{r_1,r_2,r_3} are

$$\begin{cases} a = \dfrac{r_3^2 - r_1^2}{4r_1r_3\sqrt{1-r_2^2}} + \dfrac{i(3r_2^2-1)}{4r_2\sqrt{1-r_2^2}}, \\ b = \dfrac{r_1^2 - r_3^2}{4r_1r_3\sqrt{1-r_2^2}} + \dfrac{i(3r_2^2-1)}{4r_2\sqrt{1-r_2^2}}, \\ c = \dfrac{r_3^2 - r_1^2}{4r_1r_3\sqrt{1-r_2^2}} - \dfrac{i(r_2^2+1)}{4r_2\sqrt{1-r_2^2}}. \end{cases} \tag{4.9}$$

Using Lemma 2.1, (4.3) and (4.4) follow directly. □

Remark 4.3. *The above proof shows that* $s = t = \frac{\sqrt{2}}{2}$, *that is,* $\cos\alpha = \frac{\sqrt{2}}{2}$ *and the Kähler function* $C = \cos\alpha = 0$, *which also implies that the homogeneous torus in* $\mathbb{C}P^2$ *is Lagrangian.*

Now we discuss the classification the homogeneous Helfrich tori in $\mathbb{C}P^2$.

Theorem 4.4. T_{r_1,r_2,r_3} *is a Helfrich surface if and only if*

1. *When* $0 \leqslant \lambda_1 \leqslant \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$.
2. *When* $\lambda_1 > \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$ *or* $T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$.

Proof. By Lemma 4.2, we get the coefficients of the mean curvature of T_{r_1,r_2,r_3} as follows:

$$H^3 = \frac{r_1^2 - r_3^2}{2r_1 r_3\sqrt{1-r_2^2}}, \quad H^4 = \frac{(3r_2^2 - 1)}{2r_2\sqrt{1-r_2^2}}. \tag{4.10}$$

And hence the norm square of the mean curvature is

$$|H|^2 = \frac{(1-r_1^2)(1-r_2^2)(1-r_3^2)}{4r_1^2 r_2^2 r_3^2} - 2. \tag{4.11}$$

Thus, substituting (4.10) and (4.11) into the Euler–Lagrange equation (3.18) and using the fact $C = \cos\alpha \equiv 0$, we get that

$$\begin{cases} \left\{\left(12 - 4\lambda_1 + (4\lambda_1 - 8)r_2^2\right) r_1^2 r_2^2 r_3^2 + 2r_2^2(r_1^4 + r_3^4) - (1-r_1^2)(1-r_2^2)^2(1-r_3^2)\right\}(r_1^2 - r_3^2) = 0, \\ \left\{\left(10 - 4\lambda_1 + (4\lambda_1 - 8)r_2^2\right) r_1^2 r_2^2 r_3^2 + 2r_1^2 r_3^2 - (1-r_1^2)(1-r_2^2)^2(1-r_3^2)\right\}(3r_2^2 - 1) + 2r_2^4(r_1^2 - r_3^2) = 0, \\ r_1^2 + r_2^2 + r_3^2 = 1. \end{cases}$$

By solving the equation above, we obtain

1. When $0 \leqslant \lambda_1 \leqslant \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$.
2. When $\lambda_1 > \frac{5}{2}$, $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$ or $T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$. □

4.2. The Helfrich energy of homogeneous tori

Proposition 4.5. *The Helfrich energy of T_{r_1,r_2,r_3} is*

$$\begin{aligned} \mathcal{H}_{\lambda_1,\lambda_2}(f) &= \int_M (|H|^2 + \lambda_1 + \lambda_2 C^2)\mathrm{d}M \\ &= \frac{((1-r_1^2)(1-r_2^2)(1-r_3^2) + (4\lambda_1 - 8)r_1^2 r_2^2 r_3^2)\pi^2}{r_1 r_2 r_3}. \end{aligned} \tag{4.12}$$

Proof. From Lemma 4.1, we have $\mathrm{d}T_{r_1,r_2,r_3} = r_1 r_2 r_3 \mathrm{d}\varphi\mathrm{d}\psi$. By using of (4.11), (4.12) follows directly. □

Now we consider the lower bound of $\mathcal{H}_{\lambda_1,\lambda_2}(f)$ for T_{r_1,r_2,r_3}.

Theorem 4.6. *The Helfrich energy of T_{r_1,r_2,r_3} satisfies*

1. *When $0 \leqslant \lambda_1 \leqslant 3$,*
$$\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{4\lambda_1\pi^2}{3\sqrt{3}},$$
and the equality holds if and only if $r_1 = r_2 = r_3 = \frac{\sqrt{3}}{3}$.
2. *When $\lambda_1 > 3$,*
$$\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{(4\lambda_1 - 8)\pi^2}{\sqrt{4\lambda_1 - 9}},$$
and the equality holds if and only if $r_1 = \sqrt{\frac{1}{4\lambda_1 - 9}}$, $r_2 = r_3 = \sqrt{\frac{2\lambda_1 - 5}{4\lambda_1 - 9}}$.

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Proof. Computing the extreme value of (4.12) under the constraint $r_1^2 + r_2^2 + r_3^2 = 1$ yields

$$\begin{cases} (4\lambda_1 - 8)(r_1^2 - r_2^2)r_1^2 r_2^2 r_3^2 + (1 + r_1^2)(1 - r_2^2)(1 - r_3^2)r_2^2 - (1 - r_1^2)(1 + r_2^2)(1 - r_3^2)r_1^2 = 0, \\ (4\lambda_1 - 8)(r_1^2 - r_3^2)r_1^2 r_2^2 r_3^2 + (1 + r_1^2)(1 - r_2^2)(1 - r_3^2)r_3^2 - (1 - r_1^2)(1 - r_2^2)(1 + r_3^2)r_1^2 = 0, \\ r_1^2 + r_2^2 + r_3^2 = 1. \end{cases}$$

A straightforward calculation shows that

1. When $0 \leqslant \lambda_1 \leqslant \frac{5}{2}$,
$$T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}},\ \mathcal{H}_{\lambda_1,\lambda_2}(f) = \frac{4\lambda_1\pi^2}{3\sqrt{3}}.$$
2. When $\lambda_1 > \frac{5}{2}$,
$$T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}},\ \mathcal{H}_{\lambda_1,\lambda_2}(f) = \frac{4\lambda_1\pi^2}{3\sqrt{3}},$$
or
$$T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}},\ \mathcal{H}_{\lambda_1,\lambda_2}(f) = \frac{(4\lambda_1-8)\pi^2}{\sqrt{4\lambda_1-9}}.$$

For the case of $0 \leqslant \lambda_1 \leqslant \frac{5}{2}$, it is obvious that $\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{4\lambda_1\pi^2}{3\sqrt{3}}$ and equality holds if and only if $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$. In the second case, if $\frac{5}{2} < \lambda_1 \leqslant 3$, since $\frac{4\lambda_1\pi^2}{3\sqrt{3}} \leqslant \frac{(4\lambda_1-8)\pi^2}{\sqrt{4\lambda_1-9}}$, we have that $\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{4\lambda_1\pi^2}{3\sqrt{3}}$ and equality holds if and only if $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$. If $\lambda_1 > 3$, $\frac{4\lambda_1\pi^2}{3\sqrt{3}} > \frac{(4\lambda_1-8)\pi^2}{\sqrt{4\lambda_1-9}}$, and hence $\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{(4\lambda_1-8)\pi^2}{\sqrt{4\lambda_1-9}}$, the equality holds if and only if $T_{r_1,r_2,r_3} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$. □

Remark 4.7. *For the case of $\lambda_1 = 3$ in the above proof, $T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}} = T_{\sqrt{\frac{1}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}},\sqrt{\frac{2\lambda_1-5}{4\lambda_1-9}}}$, This implies that $\mathcal{H}_{\lambda_1,\lambda_2}(f) \geqslant \frac{4\lambda_1\pi^2}{3\sqrt{3}}$, and equality holds if and only if $T_{r_1,r_2,r_3} = T_{\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}}$.*

Remark 4.8. *In* [14], *Montiel and Urbano introduced the conformal invariants $W^-(F)$ and $W^+(F)$ for compact surfaces in $\mathbb{C}P^2$. In view of the Helfrich functional $\mathcal{H}_{\lambda_1,\lambda_2}(f)$, we have*

1. *When $\lambda_1 = 2, \lambda_2 = 0$, $\mathcal{H}_{2,0}(F) = W^-(F) = \int_M (|H|^2 + 2)\mathrm{d}M$;*
2. *When $\lambda_1 = 0, \lambda_2 = 6$, $\mathcal{H}_{0,6}(F) = W^+(F) = \int_M (|H|^2 + 6C^2)\mathrm{d}M$;*
3. *When $\lambda_1 = 1$, $\lambda_2 = 3$, $\mathcal{H}_{1,3}(F) = W(F) = \int_M (|H|^2 + 1 + 3C^2)\mathrm{d}M$, that is, the Willmore functional.*

Thus, by using the Euler–Lagrange equation of $\mathcal{H}_{\lambda_1,\lambda_2}(F)$, we have the following corollary.

Corollary 4.9. ([8]) *The Euler–Lagrange equation of $W(F)$ is*

$$\begin{cases} \triangle^{\perp}H^3 + (3 + 3C^2 - 2|H|^2)H^3 + \sum_{\beta,i,j} h_{ij}^3 h_{ij}^{\beta} H^{\beta} - 6\sqrt{1-C^2}C_{,1} = 0, \\ \triangle^{\perp}H^4 + (3 + 3C^2 - 2|H|^2)H^4 + \sum_{\beta,i,j} h_{ij}^4 h_{ij}^{\beta} H^{\beta} - 6\sqrt{1-C^2}C_{,2} = 0. \end{cases}$$

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Corollary 4.10. ([8]) *The Euler–Lagrange equation of* $W^-(F)$ *is*

$$\triangle^\perp H^\alpha + (1 - 3C^2 - 2|H|^2)H^\alpha + \sum_{\beta,i,j} h^\alpha_{ij} h^\beta_{ij} H^\beta = 0.$$

Corollary 4.11. *The Euler–Lagrange equation of* $W^+(F)$ *is*

$$\begin{cases} \triangle^\perp H^3 + (5 + 9C^2 - 2|H|^2)H^3 + \sum_{\beta,i,j} h^3_{ij} h^\beta_{ij} H^\beta - 12\sqrt{1 - C^2}C_{,1} = 0, \\ \triangle^\perp H^4 + (3 + 3C^2 - 2|H|^2)H^4 + \sum_{\beta,i,j} h^4_{ij} h^\beta_{ij} H^\beta - 12\sqrt{1 - C^2}C_{,2} = 0. \end{cases}$$

Let us consider the homogeneous tori T_{r_1,r_2,r_3} in $\mathbb{C}P^2$, then it follows from Theorems 4.4 and 4.6 that

Corollary 4.12. T_{r_1,r_2,r_3} *is a critical surface for* $W(F)$*,* $W^-(F)$*, or* $W^+(F)$ *if and only if it is the Cliford torus* $T_{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}}$.

Corollary 4.13. *Considering the homogeneous tori* T_{r_1,r_2,r_3} *in* $\mathbb{C}P^2$*. Then* $W^-(F) \geqslant \frac{8\pi^2}{3\sqrt{3}}$ *(see also* [14]*) and* $W^+(F) \geqslant 0$*, and the equalities hold if and only if* T_{r_1,r_2,r_3} *is the Cliford torus* $T_{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}}$.

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