ON THE NUMERICAL RADIUS OF AN ELEMENT OF A NORMED ALGEBRA

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Let A be a unital normed algebra over the complex field $\mathbb{C}$, A' the dual space of A, i.e., the Banach space of all continuous linear functionals on A, and let S be the set of all states on A, i.e.,

$$S = \{s \in A' \mid s(1) = 1 = \|s\|\}.$$

Recall that, for an element $x \in A$, the set

$$V(x) = \{s(x) \mid s \in S\}$$

is called the numerical range of x, and

$$v(x) = \sup_{s \in S} |s(x)|$$

is called the numerical radius of x.

We denote by $G(A)$ the set of invertible elements of A, by $U_r(0)$ the set $\{\lambda \in \mathbb{C} \mid |\lambda| < r\}$, by $\partial U_r(0)$ the set $\{\lambda \in \mathbb{C} \mid |\lambda| = r\}$.

We recall also the following assertions.

LEMMA. *Let $x \in A$. If $v(x) < 1$, then*

$$(1 - \lambda x) \in G(A),$$

$$\|(1 - \lambda x)^{-1}\| \leqq (1 - v(x))^{-1},$$

for all $\lambda \in U_1(0)$.

For the proof of this lemma, see [2] or [3].

The aim of this note is to prove the following theorem and to observe that some results concerning the numerical radius are contained in it.

THEOREM. *If $x \in A$ and if, for all $\lambda \in \overline{U_r(0)}$, $1 - \lambda x \in G(A)$, then*

$$\|x^n\| \leqq \frac{n!}{r^n m(m+1)\dots(m+n-1)} \Big(\sup_{\lambda \in \partial U_r(0)} \|(1 - \lambda x)^{-1}\|\Big)^m \quad (n = 1, 2, \dots; m = 1, 2, \dots).$$

Proof. We may suppose that A is complete.

From the fact that $1 - \lambda x \in G(A)$ for all $\lambda \in \overline{U_r(0)}$, it follows that $\rho(rx) < 1$, where ρ denotes the spectral radius. Hence the series

$$1 + m\lambda x + \frac{m(m+1)}{2} \lambda^2 x^2 + \dots + \frac{m(m+1)\dots(m+n-r)}{n!} \lambda^n x^n + \dots$$

converges uniformly on $\overline{U_r(0)}$ and

$$(1-\lambda x)^{-m} = 1+m\lambda x+\frac{m(m+1)}{2}\lambda^2x^2+\ldots+\frac{m(m+1)\ldots(m+n-1)}{n!}\lambda^n x^n+\ldots.$$

Using the preceding considerations, we deduce that

$$\frac{1}{2\pi i}\int_{\partial U_r(0)}\frac{(1-\lambda x)^{-m}}{\lambda^{n+1}}\,d\lambda = \frac{m(m+1)\ldots(m+n-1)}{n!}x^n$$

and therefore

$$\frac{m(m+1)\ldots(m+n-1)}{n!}\|x^n\| \leqq \frac{1}{r^n}\Big(\sup_{\lambda\in\partial U_r(0)}\|(1-\lambda x)^{-1}\|\Big)^m.$$

This completes the proof.

Corollary 1. (Crabb [4]) *For all* $x\in A$,

$$\|x^n\| \leqq n!(e/n)^n v(x)^n \quad (n=1,2,\ldots).$$

Proof. For $m>n$ and $y=n/(mv(x))x$, $v(y)=n/m<1$ and from the lemma we deduce that $1-\lambda y\in G(A)$ and $\|(1-\lambda y)^{-1}\|\leqq(1-n/m)^{-1}$ for all $\lambda\in\overline{U_1(0)}$. Hence, using the preceding theorem, we have

$$\|x^n\| \leqq \frac{m^n v(x)^n}{n^n}\frac{n!}{m(m+1)\ldots(m+n-1)}\left(1-\frac{n}{m}\right)^{-m}.$$

The assertion follows from this relation on letting $m\to\infty$.

Corollary 2. (Bohnenblust and Karlin [1]) *For all* $x\in A$,

$$e^{-1}\|x\| \leqq v(x) \leqq \|x\|.$$

Corollary 3. (Stampfli [7]) *If* $x\in A$ *is such that* $\rho(x)<2$, $(1+\lambda x)\in G(A)$ *and*

$$\|(1+\lambda x)^{-1}\| \leqq 1$$

for all $\lambda\in\partial U_1(0)$, *then* $x=0$.

Proof. If $|\lambda|\leqq\frac{1}{2}$, then $\rho(\lambda x)<1$ and so $1-\lambda x\in G(A)$. If $\frac{1}{2}<|\lambda|\leqq 1$, then we have $\lambda=t\mu$ with $\frac{1}{2}<t\leqq 1$, $|\mu|=1$ and

$$1-\lambda x = 1-t\mu x = t(1-\mu x)+1-t = t(1-\mu x)(1+\{(1-t)/t\}(1-\mu x)^{-1}).$$

Then $(1-t)/t<1$, and so $\|\{(1-t)/t\}(1-\mu x)^{-1}\|<1$, which implies that $1-\lambda x$ is invertible. It follows from the theorem that $x=0$.

Corollary 4. *If* $x\in A$ *and if, for all* $\lambda\in U_1(0)$, $1-\lambda x\in G(A)$ *and* $\|(1-\lambda x)^{-1}\|\leqq 1$, *then* $x=0$.

Corollary 5. (*See also* [6].) *Let A be a complex Banach algebra with identity. A is commutative if and only if there exists* $K\geqq 1$ *such that*

$$x,y\in A \Rightarrow v(xy)\leqq Kv(yx).$$

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From Corollary 2 it follows that $x \to v(x)$ is a linear norm on A such that

$$x, y \in A \Rightarrow v(xy) \leqq K \| yx \|.$$

From [5, Theorem] it follows that A is commutative.

COROLLARY 6. (Bonsall and Duncan [3]) *Let A be a complex Banach algebra with identity. If there exists $K \geqq 1$ such that*

$$x \in A \Rightarrow v(x) \leqq K\rho(x),$$

then A is commutative.

This assertion follows from the fact that

$$x, y \in A \Rightarrow v(xy) \leqq K\rho(xy) = K\rho(yx) \leqq Kv(yx).$$

COROLLARY 7. *Let A be a complex Banach algebra with identity. If*

$$x \in A, s \in \mathrm{Ex}\, S \Rightarrow s(x^2) = s(x)^2,$$

then A is commutative. Here Ex S *denotes the set of all extreme states.*

From the fact that

$$x \in A \Rightarrow |s(x)| \leqq \rho(x),$$

and the fact that $S = \overline{\mathrm{co}}\, \mathrm{Ex}\, S$, it follows that $v(x) = \rho(x)$.

COROLLARY 8. *Let A be a C^*-algebra with identity. Then A is commutative if and only if, for any pure state p, we have*

$$x \in A \Rightarrow p(x^2) = p(x)^2.$$

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