SERIES SOLUTION OF A FUNCTIONAL EQUATION[1]

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1. Preliminaries

In this paper we will study analytic solutions to the linear functional equation

$$g(f(z))-x \cdot g(z) = h(z)$$

where f and h are given functions, x is a given complex number and the function g is to be found. This is a generalization of Schröder's functional equation. The results obtained are global in nature and the solutions holomorphic. The equation will be viewed from the standpoint of linear operator theory. When studied in this manner one arrives at a general operator inversion formula.

If U be a plane domain whose boundary contains at least two members, $\mathfrak{H}(U)$ will denote the linear topological space of functions holomorphic on U with the topology of continuous convergence. $\mathfrak{G}(U)$ will be the group of conformal homeomorphisms of U onto U and $\mathfrak{I}(U)$ will be the semi-group (under composition) of members of $\mathfrak{H}$ which have values in U. We reserve the letters f, g, h, q, and s for functions. Juxtaposition of these symbols indicates composition of functions; $g \cdot f$ means multiplication. By fz we mean the value of the function f at the complex number z. Following K. Menger [1] we give a name to the identity function, i.e. by the function j we mean that function defined by $jz = z$ for each complex number z. We reserve the symbol L to stand for an operator and L^0 to stand for the identity operator. We will be indiscriminate and use the remaining letters to stand for constant functions or the corresponding complex numbers. If x is the constant function of value x by $x \cdot L$ we mean the operator which yields the function $x \cdot Lf$ for each function f.

With each function $f \in \mathfrak{I}$ we associate the translation operator L_f which takes a function $g \in \mathfrak{H}$ into $Lg = gf \in \mathfrak{H}$. Because of the properties of right substitution each operator L is linear. In operator notation we will solve

$$(L-x \cdot L^0)g = h$$

for g by interpolating on the set $S(L)$, the discrete spectrum of L.

[1] This paper forms part of a thesis written under the direction of Prof. M. McKiernan.

2. The operator inversion

THEOREM 1. *Let $f \in \mathfrak{F}-\mathfrak{G}$ and suppose there is an $a \in U$ such that $fa = a$. Then $S(L_f) = \{1, c, c^2, \cdots\}$ where $c = Dfa$, the derivative of f at a. If $x \notin \bar{S}$ (the closure of S) the linear operator $(L-x \cdot L^0)$ can be uniquely inverted over $\mathfrak{H}$ in the form*

$$-\frac{L^0}{(x-1)}-\frac{(L-L^0)}{(x-1)(x-c)}-\frac{(L-L^0)(L-cL^0)}{(x-1)(x-c)(x-c^2)}\cdots. \tag{1}$$

The combined results of M. Heins [2] for multiply-connected and A. Denjoy and J. Wolff for simply-connected domains [2] implies that $|c| < 1$ and the sequence of iterates of f converges in the topology of $\mathfrak{H}$ to the constant function of value a. We first prove

LEMMA 1. *Let $f \in \mathfrak{F}-\mathfrak{G}$ with $fa = a \in U$ and $h \in \mathfrak{H}$ with $x \neq 0$. If g is regular at $z = a$ and in a neighbourhood of $z = a$ satisfies $(L-x \cdot L^0)g = h$ then g can be continued analytically throughout U and $(L-x \cdot L^0)g = h$ holds on U.*

PROOF. If $d \in U$ there is a domain $N_{ad} \subset U$ whose closure is compact in U and which contains both a and d. Suppose that $h \in \mathfrak{H}$ and that g is regular on a neighborhood $N_a \subset U$ of $z = a$ with $(L-x \cdot L^0)g = h$. We define the natural iterates of f recursively $f_0 = j$, $f_1 = f$, $f_{n+1} = ff_n$, $n = 1, 2, \cdots$. There is a positive integer k such that $f_k(N_{ad}) \subset N_a$ because of the uniform convergence of the sequence $\{f_r\}$ on the closure of N_{ad}. Put $U_1 = f(N_{ad})$, $U_2 = f(U_1)$, etc. These are all domains containing the point $z = a$. On U_{k-1} put $g_{k-1} = (gf-h)/x$; since $fz \in U_k$ for $z \in U_{k-1}$ this is well defined. On U_{k-2} put $g_{k-2} = (g_{k-1}f-h)/x$, etc. until the original regular element is extended to all of N_{ad}. As $d \in U$ was arbitrary and each extension is a bona fide analytic continuation the result is analytic on U and satisfies thereon $(L-x \cdot L^0)g = h$.

To proceed with the proof of theorem 1 we first suppose that $c \neq 0$. The results on the solutions to Schröder's functional equation (c.f. Fatou [3], Ch. 2) imply that $S = \{1, c, c^2, \cdots\}$, i.e. there is no function $g \in \mathfrak{H}$ (in fact no function regular at $z = a$) for which $(L-x \cdot L^0)g = 0$ save for $g = 0$ unless x has one of the values $1, c, c^2, \cdots$. Furthermore, there exists a unique function $s \in \mathfrak{H}$ with the properties that $(L-c \cdot L^0)s = 0$, $sa = 0$ and $Dsa = 1$. If $x = c^r$ then every solution to $(L-x \cdot L^0)g = 0$ in $\mathfrak{H}$ is a constant function times s^r, i.e. the corresponding space of characteristic functions is spanned by the r-th power of the function s.

Put $p_0 = 1$, $p_n = (j-c^{n-1}) \cdot p_{n-1}$, $n = 1, 2, \cdots$. Then the operators p_nL satisfy $L(p_nL)-c^n \cdot p_nL = p_{n+1}L$; using this formula it is a straightforward matter to show that the series (1) applied to h formally satisfies the equation $(L-x\cdot L^0)g = h$, and this depends only on the fact that the series converges for hz and h/z. We show that in fact (1) applied to any $h \in \mathfrak{H}$ gives rise to a series which converges absolutely and uniformly on a neighborhood of $z = a$ in U. Since $h \in \mathfrak{H}$ means h is regular at $z = a$ and $Dsa = 1$ means s is locally invertible at $z = a$, hs^{-1} is regular at $z = 0 = sa$, (where s^{-1} represents the local inverse to s at zero). We can then say that $h = \sum_{r=0}^{\infty} b_r \cdot s^r$ where the b_r are uniquely determined by h. Near $z = a$ we have

$$p_n Lh(z) = \sum_{k=n}^{\infty} p_n c^k \cdot b_k \cdot (sz)^k \tag{2}$$

since $p_m c^n = 0$ for $n < m$. Also for $k \geqq n$

$$p_n c^{k+1}/p_n c^k = c^n \cdot (1-c^{-k-1})/(1-c^{n-k-1})$$

means

$$\begin{aligned} |p_n c^{k+1}/p_n c^k| &= |(1-c^{k+1})/(1-c^{k+1-n})| \\ &\leqq (1+|c|)/(1-|c|) \end{aligned}$$

since $0 < |c| < 1$. Using this and (2) we have

$$|p_n Lh(z)| \leqq |p_n c^n| \cdot |sz|^n \cdot \sum_{k=n}^{\infty} |b_k| \cdot \{(1+|c|) \cdot |sz|/(1-|c|)\}^{k-n}.$$

But since $sa = 0$ the series

$$\sum_{k=0}^{\infty} |b_k| \cdot \{|sz| \cdot (1+|c|)/(1-|c|)\}^k$$

is bounded uniformly in z near $z = a$ by M say. Thus

$$|p_n Lh(z)| \leqq |p_n c^n| \cdot \{(1-|c|)/(1+|c|)\}^n \cdot M$$

and

$$|p_n Lh(z)/p_{n+1}x| \leqq |p_n c^n/p_{n+1}x| \cdot ((1-|c|)/(1+|c|))^n \cdot M.$$

Now if $x \notin \bar{S}$ we can show $|p_{n+1}x| \geqq |x|^{n+1} \cdot M' > 0$, M' independent of n. For

$$|p_{n+1}x| = |x|^{n+1} \cdot \prod_{r=0}^{n} |1-c^r \cdot x^{-1}|$$

and since the product $\prod_{r=0}^{\infty} |1-c^r \cdot x^{-1}|$ is absolutely convergent hence non-null and bounded away from zero the assertion follows. Further,

$$|p_n c^n| = |c|^{n(n-1)/2} \cdot \prod_{r=1}^{n} |1-c^r| < |c|^{n(n-1)/2} \cdot M''$$

M'' independent of n since the product $\prod_{r=1}^{\infty} |1-c^r|$ is absolutely convergent. Collecting these parts

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$$\left|\frac{p_n L h(z)}{p_{n+1} x}\right| \leqq \frac{|c|^{n(n-1)/2} \cdot M}{|x|^{n+1} \cdot M'} \cdot M'' \cdot \left\{\frac{1-|c|}{1+|c|}\right\}^n \tag{3}$$
$$\leqq \left\{\frac{1-|c|}{1+|c|} \cdot \frac{|c|^{n-1/2}}{|x|}\right\}^n \cdot M_1,$$

M_1 independent of n and z for z sufficiently near to $z = a$. As $|c| < 1$, from and after some value of n the quantity $\left((1-|c|) \cdot |c|^{(n-1)/2}\right)/\left((1+|c|) \cdot |x|\right)$ is less than one. Summing (3) on n, the right hand sum is dominated by a convergent geometric series, hence (1) applied to h converges absolutely and uniformly on a sufficiently small neighborhood of $z = a$.

We have thus generated a local regular (at a) solution to the equation $(L - x \cdot L^0)g = h$ for $x \notin \bar{S}$ and $h \in \mathfrak{H}$. Lemma 1 implies that g is analytic on U and if U be simply connected we know $g \in \mathfrak{H}$. But even if U not have connectivity one, g is still holomorphic on U. For let g_a be the regular element of the analytic g generated by (1). Suppose that for some $d \in U$ the process in lemma 1 applied to g_a leads to more than one regular element of g above d. Let g_d and g_d^* represent these two elements of g. By our assumption both must satisfy $(L - x \cdot L^0)g = h$ near $z = d$, i.e. $gf - x \cdot g = h$. Then the difference $g^* = g_d - g_d^*$ is regular at $z = d$ and can be continued throughout U. Also, since h is holomorphic $g^*f - x \cdot g^* = 0$ near $z = d$ and this must hold throughout U for any continuation of g^* and g^*f since f is holomorphic on U. But then the continuation of g^* to $z = a$ implies a regular solution to $(L - x \cdot L^0)g_a^* = 0$ at $z = a$. As $x \notin \bar{S}$ this can only be the zero function. Thus g^* is the sero function on all of U and $g \in \mathfrak{H}$. This reasoning is, of course, false if $x \notin \bar{S}$; and even if there is a local solution to $(L - x \cdot L^0)g = h$ for $x \in \bar{S}$ it will generally not be holomorphic on U as is evidenced say by $x = 0$. This completes the proof of theorem 1 for the case $c \neq 0$.

If $c = Dfa = 0$ the function f has the form near a

$$a + \sum_{n=k}^{\infty} a_n \cdot (j - a)^n$$

with $a_k \neq 0$, where k is an integer exceeding one (we assume f is not a constant function). We easily verify that the only solutions g to $(L - x \cdot L^0)g = 0$ with $g \neq 0$ and $g \in \mathfrak{H}$ occur when $x = 1$. The functions g corresponding to $x = 1$ are all constant. In this situation we use the solutions to Böttcher's functional equation (see e.g. Fatou [3], Ch. 2). There is a function $q \in \mathfrak{H}$ with the properties; $qa = 0$, $Dqa \neq 0$, and $qf = q^k$. Furthermore, the only other functions satisfying all of these conditions have the form $\omega \cdot q$ where ω is a $(k-1)$st root of unity. Noticing that q is locally invertible at $z = a$ we see that for $h \in \mathfrak{H}$ we have, near $z = a$

$$h = \sum_{r=0}^{\infty} b_r \cdot q^r.$$

The interpolation polynomials p_{n+1} become $(j-1)\cdot j^n$ and $p_{n+1}L$ is $L^{n+1}-L^n$, as $c=0$. We find

$$\begin{aligned}(L^{n+1}-L^n)h &= \sum_{r=0}^{\infty} b_r\cdot(L^{n+1}q)^r-\sum_{r=0}^{\infty} b_r\cdot(L^nq)^r\\ &= (b_0+b_1\cdot q^{k(n+1)}+\cdots)-(b_0+b_1\cdot q^{nk}+\cdots)\\ &= q^{nk}\cdot(b_1\cdot q^k+\cdots)-q^{nk}\cdot(b_1+\cdots).\end{aligned}$$

Thus $|L^{n+1}hz-L^nhz|\leqq|(qz)^{nk}|\cdot M$ where M is independent of z and n if z be sufficiently near a. Now we get

$$\left|\frac{(L^{n+1}-L^n)hz}{(x-1)\cdot x^{n+1}}\right|\leqq\left|\frac{(qz)^{nk}\cdot M}{(x-1)\cdot x^{n+1}}\right|=\left|\frac{(qz)^k}{x}\right|^n\cdot M'.$$

As $qa=0$ and $k\geqq 2$, for z near enough to a we have $|(qz)^k/x|<u<1$ where we suppose that x is not zero or one. Thus

$$\left|\frac{(L^{n+1}-L^n)hz}{(x-1)\cdot x^{n+1}}\right|\leqq u^n\cdot M'. \tag{4}$$

Summing (4) on n we obtain a dominant geometric series on the right, proving that for $x\neq 0$ or 1 and $h\in\mathfrak{H}$ the series (1) converges absolutely and uniformly near $z=a$. We obtain then a local solution to $(L-x\cdot L^0)g=h$. Again, lemma 1 insures that g is analytic on U and satisfies $(L-x\cdot L^0)g=h$ and precisely the same reasoning as in the case $c\neq 0$ allows us to conclude that $g\in\mathfrak{H}$ whatever the connectivity of U provided only that $x\neq 0$ or 1. This completes the proof of theorem 1.

The class of functions $\mathfrak{H}$ in theorem 1 can be enlarged to the class $\mathfrak{M}$ of functions meromorphic on U provided we restrict $c=Dfa$ to be non-zero. If $f\in\mathfrak{F}-\mathfrak{G}$ and $fa=a\in U$ with $c=Dfa\neq 0$ then on $\mathfrak{M}$, $S(L_f)=\{c^r\}_{-\infty}^{\infty}$ and as before each characteristic space is spanned by s^r where s is the principal solution to Schröder's equation. Let $\mathfrak{M}_r\subset\mathfrak{M}$ be the class of functions meromorphic on U which have a pole of order no greater than r at $z=a$. If $h\in\mathfrak{M}_r$ and we wish to solve $gf-x\cdot g=h$ we use the standard procedure and multiply through by $c^r\cdot s^r=(sf)^r=(Ls)^r$. Then

$$c^r\cdot s^r\cdot gf-x\cdot c^r\cdot g\cdot s^r=h\cdot s^r\cdot c^r. \tag{5}$$

Putting $g^*=g\cdot s^r$, $h^*=h\cdot s^r\cdot c^r$ we get from (5)

$$g^*f-x\cdot c^r\cdot g^*=h^*=(L-x\cdot c^r\cdot L^0)g^*.$$

But now h^* is regular at $z=a$. Our results in theorem 1 give for $x\cdot c^r\neq 0, 1, c,\cdots$

$$g^*=-\frac{h^*}{(x\cdot c^r-1)}-\frac{h^*f-h^*}{(x\cdot c^r-1)\cdot(x\cdot c^r-c)}-\cdots. \tag{6}$$

But in terms of g and h (6) becomes

$$g = -\left\{\frac{L^0}{(x-c^{-r})} + \frac{(L-c^{-r}\cdot L^0)}{(x-c^{-r})\cdot(x-c^{-r+1})} + \cdots\right\} h. \tag{7}$$

To extend our local polar element to a member of $\mathfrak{M}$ we need a lemma.

LEMMA 2. *Suppose $f \in \mathfrak{F}-\mathfrak{G}$ with $fa = a \in U$ and that $Z \subset U$ is isolated in U. Put $Z_0 = Z$ and $Z_n = f^{-1}[Z_{n-1}] \cap U$, $n = 1, 2, \cdots$ and $Z^* = \bigcup_0^\infty Z_n$. Then $U-Z^* = U^*$ is a domain with the property that $f[U^*] \subset U^*$.*

PROOF. U^* has the property that $f[U^*] \subset U^*$, for if $z \in U^*$ with $fz \notin U^*$ then $fz \in Z^*$ but this in turn means $z \in Z^*$ which we assumed was not the case. Each Z_n has no limit points in U; for suppose the assertion holds for some $n \geqq 0$. We show it holds for $n+1$. If $t \in Z_{n+1}$ is a limit point of Z_{n+1} in U there is a sequence $\{t_k\} \subset Z_{n+1}$ such that $t_k \to t \in U$. But then $ft_k \to ft$ and since $ft_k \in Z_n$, $ft \in U$. This is impossible unless $ft_k = ft$ for all k from and after some value. But unless the points t_k are themselves identical from and after some point this is impossible because there is a neighborhood of t in which f never again assumes the value ft. Therefore, each Z_n has limit points belonging exclusively to the frontier of U. This implies that if $t \in U$ is a limit point of Z^* then any neighborhood of t must intersect infinitely many of the sets Z_n. Now suppose $t \in U$ is a limit point of Z^*. Then there is a sequence of disks $X_1 \supset X_2 \supset X_3 \supset \cdots$ in U, centers t with diameters contracting to zero and there is a point in X_1, $t_{n_1} \in Z_{n_1}$, $t_{n_1} \neq t$ for some n_1. By the previous, in X_2 there is a point $t_{n_2} \in Z_{n_2}$ with $n_1 < n_2$. We continue in this fashion determining a sequence of distinct points $\{t_{n_k}\}$, $n_1 < n_2 < n_3 < \cdots$ converging to $t \in U$. As $t_{n_k} \in Z_{n_k}$, $f_{n_k} t_{n_k} \in Z_0$, $k = 1, 2, \cdots$. As $\{f_r\}$ converges continuously to $z = a$ on U, $f_{n_k} t_{n_k} \to a$. This is a contradiction for then Z_0 itself would have a limit point in U. This proves U^* is open. It is connected since Z^* is isolated in U.

We identify Z in lemma 2 with the poles of h^* (this does not include $z = a$ as h^* is regular there). On U^*, h^* is holomorphic. We can apply lemma 1 as $f[U^*] \subset U^*$ and $fa = a \in U^*$, to guarantee that g^* is holomorphic on U^*. Thus g is meromorphic on U having possible poles on the complete f-inverse image of the set of poles of h.

THEOREM 2. *Let $f \in \mathfrak{F}-\mathfrak{G}$ with $fa = a \in U$ and $c = Dfa \neq 0$. Then $S(L_f) = \{c^r\}_{-\infty}^\infty$ on the field $\mathfrak{M}$. If $\mathfrak{M}_r$ is the class of functions having a pole of at most order r at $z = a$ and meromorphic on U, then $x \notin \{0, c^{-r}, c^{-r+1}, \cdots\}$ means $(L-x\cdot L^0)$ is uniquely invertible on $\mathfrak{M}_r$ as*

$$-\frac{L^0}{(x-c^{-r})} - \frac{(L-c^{-r}\cdot L^0)}{(x-c^{-r})\cdot(x-c^{-r+1})} - \cdots. \tag{8}$$

3. Discussion and applications

The assumptions concerning the function f in theorem 1 are natural as the work of M. Heins [2] shows; for $fz = z$ has at most one solution if $f \in \mathfrak{J}$, $f \neq j$. If f has no fixed point in U the derived set of $\{f_r\}$ constitutes a class of constant functions whose values belong to the boundary of U and form a continuum there. In this case the discussion of the operator $(L - x \cdot L^0)$ is difficult, though one can show that in many instances (1) provides a solution to $(L - x \cdot L^0)g = h$. For example if U is the right half-plane and f is $j+1$ then $Lg = g(j+1)$ and we are dealing with the finite difference situation. Formula (1) is well known in this instance.

It is interesting that the analogue of theorem 2 for $c = 0$ does not arise; a fact one readily deduces by taking U as the unit disk $|z| < 1$, $f = j^2$, $h = 1/j$ and the equation $(L - x \cdot L^0)g = 1/j = gj^2 - x \cdot g$. In fact there is no regular or polar element at $a = 0$ satisfying this equation. Moreover, any further extension of theorems 1 and 2 to a still larger class of functions must of necessity be of a different character from the results herein stated. For Pincherle (see e.g. Walsh [4]) has shown that if $0 < c < 1$ there is, for any complex number $1/x$ a function g^* meromorphic on the plane punctured at $z = 0$ and satisfying

$$g^*(j/c) - 1/x \cdot g^* = 0.$$

One has some degree of liberty in specifying the zeros and poles of g^* which makes it useful. If the function s is the basic Schröder function for f with $c = Dfa$, $fa = a$; we have, upon setting $g = g^* s$

$$gf = g^* sf = g^*(c \cdot s) = x \cdot g^* s = x \cdot g$$

or

$$gf - x \cdot g = 0 = (L - x \cdot L^0)g.$$

As g^* is meromorphic on the plane punctured at zero g is meromorphic on a disk punctured at $z = a$. This shows that when the value of the derivative at the fixed point is real and positive the spectrum of the translation operator L_f associated with the function f will include every complex number if we allow the class of functions acted on by L_f to have arbitrary behavior at $z = a$, i.e. if we allow arbitrary isolated singular points.

It is worth noting that even for $x = 0$, (1) provides a local solution to $gf = h$ (for $c \neq 0$), i.e. essentially inverts f locally, concerning this see the recent paper of M. McKiernan [7]. Also, we may mention some facts concerning the case $x \in S(L)$ relative to theorem 1.

THEOREM 3. *Let* $f \in \mathfrak{J} - \mathfrak{G}$ *with* $fa = a \in U$.

a) If $c = Dfa \neq 0$ *with* $x = c^m$ *for* m *a non-negative integer and* $h \in \mathfrak{H}$, *there is no solution* g *to* $(L - x \cdot L^0)g = h$ *regular at* $z = a$ *unless* $b_m = 0$

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in the expansion $h = \sum b_r \cdot s^r$ *where* s *is the unique function satisfying* $(L - c \cdot L^0)s = 0$, $sa = 0$, $Dsa = 1$. *If* $b_m = 0$ *there is a function* g *satisfying* $(L - x \cdot L^0)g = h$ *and analytic on* U. *Any two regular elements satisfying this equation at* $z = a$ *differ by a function of the form* $b \cdot s^m$, b *a constant function.*

b) If $c = 0$ *and* $x = 1$, *there is no solution to* $(L - L^0)g = h$ *regular at* $z = a$ *unless* $b_r = 0$ *for* $r \equiv 0 \pmod k$, $r \neq 0$ *in the expansion* $h = \sum b_r \cdot q^r$ *where* q *is one of the functions satisfying* $qf = q^k$, $qa = 0$, $Dqa \neq 0$ *and* $f = a + \sum_{n=k}^{\infty} a_n \cdot (j - a)^n$, $a_k \neq 0$, $k \geqq 2$. *If these conditions are met there is a solution* g *to* $(L - L^0)g = h$ *analytic on* U. *Any two regular elements of* g *satisfying the equation at* $z = a$ *differ by a constant function.*

If U is the unit disk about the origin and $f = j^2$ with $h = j$ then theorem 1 implies there exists a unique function g regular on $|z| < 1$ with $gj^2 - x \cdot g = j$ (we take $x > 1$). This is Weierstrasses example of a function not continuable beyond the rim $|z| = 1$ and a very rapid way of verifying this is by using the functional equation itself. The function g is continuous on $\overline{U}$. If we put $u(\theta) = \mathrm{Re}\,(g(e^{i \cdot \theta}))$ the function u is Weierstrasses example of a continuous, real-valued, nowhere differentiable function. This function satisfies the system

$$u(2 \cdot j) - x \cdot u = \cos$$
$$u(j + 2 \cdot \pi) - u = 0,$$

both examples of the general equation herein studied. G. Julia [5] has given a general theorem concerning the existence of non-continuable solutions to $gf - x \cdot g = h$ for rational functions f and the role of the same equation in constructing non-differentiable continuous functions is emphasized in the work of G. de Rham [6].

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