LETTERS TO THE EDITOR
FURTHER COMMENTS ON A PAPER BY H. J. WEINER

(Sequential random packing in the plane. *J. Appl. Prob.* **15** (1978), 803–814.)

Dear Editor,

An alternative argument for Lemma 3 *and Theorem* 1 *of* [9]

The Rényi packing scheme is as in [9]. Let

(1.1) $M(a,b)$ = mean number of unit squares which can be sequentially randomly parked on an $a \times b$ rectangle, $a, b > 1$, in accord with a Rényi model.

(1.2) $m(a)$ = mean number of unit length segments which may be sequentially randomly parked on a segment of length $a > 1$.

Then we have the following result.

Lemma.

$$M(a+1,b) \geqq M(a,b) \tag{1.3}$$

$$M(a+1,b) \leqq M(a,b) + m(b) \tag{1.4}$$

$$M(a+2,b) \geqq M(a,b) + M(b). \tag{1.5}$$

Proof. To show (1.3), consider any final configuration of unit squares in an $a \times b$ rectangle. This configuration is mapped into an $(a+1) \times b$ rectangle as follows. If both rectangles have lower left corner at $(0,0)$, then (x,y) is mapped to $((a+1)x/a, y)$ $0 < x < a$; $0 < y < b$. The new cars of size $((a+1)/a, 1)$ in the $(a+1,b)$ rectangle are then shrunk to unit square size, where their centers are left at the same coordinates. Then, any spaces which may accommodate a unit square in the $(a+1,b)$ rectangle are filled in as in the Rényi model for an $(a+1,b)$ rectangle. Similarly, any final configuration of unit squares in an $(a+1,b)$ rectangle corresponds to a final configuration of unit squares in an $a \times b$ rectangle by the mapping (x,y) to $(ax/(a+1), y)$. The new cars of size $(a/(a+1), 1)$ in the (a,b) rectangle are then increased (in the x-dimension) to

Received 25 October 1979.

unit squares with the same centers. Any of these unit squares that may now overlap are now removed such that the minimal numbers are removed, and replaced, if possible, by at most the same number of unit squares, by the Rényi parking scheme for unit squares on an $(a \times b)$ rectangle. The mapping is thus not unique, but each configuration in the larger rectangle corresponds to a configuration of at most that number of cars in the smaller rectangle, and vice versa, establishing (1.3). Similar arguments, along with Lemma 2 of [9] establish (1.4), (1.5) for the Rényi model, completing the lemma. If unit square cars are replaced by $\alpha \times \beta$ size cars, or if the Solomon model is considered, similar results hold.

Theorem. For $a, b \gg 1$,

$$M(a,b) \leqq m(a) + m(b) - 1 + \frac{4}{(a-1)(b-1)} \int_0^{a-1} \int_0^{b-1} M(\xi,\eta)\, d\xi\, d\eta. \tag{1.6}$$

$$\begin{aligned} M(a+2,b+2) \geqq{}& m(a) + m(b) - 1 \\ &+ \frac{4}{(a-1)(b-1)} \int_0^{a-1} \int_0^{b-1} M(\xi,\eta)\, d\xi\, d\eta. \end{aligned} \tag{1.7}$$

Proof. To prove (1.6), the first unit square car is parked in an $a \times b$ rectangle with lower left coordinates at (ξ, η), $0 < \xi < a-1$, $0 < \eta < b-1$. The two rectangular strips with coordinates $(\xi, 0)$, $(\xi+1, 0)$, (ξ, b), $(\xi+1, b)$ and $(0, \eta)$, $(0, \eta+1)$, (a, η), $(a+1, \eta)$, respectively, are placed on the $a \times b$ rectangle. Each are parked with unit square cars vertically or horizontally, respectively, in accord with a one-dimensional Rényi scheme for unit length cars on a b-length or a-length segment respectively, omitting one of the two cars which overlap in the space (ξ, η), $(\xi+1, \eta)$, $(\xi, \eta+1)$, $(\xi+1, \eta+1)$. Hence $m(a) + m(b) - 1$ cars are so parked. In the remaining (at most four) rectangular parking rectangles which remain, each is parked with unit squares in accord with a Rényi scheme (e.g. a remaining (γ, δ) rectangle is parked in accord with a Rényi scheme for a (γ, δ) rectangle with unit square cars). An induction on (1.4) yields (1.6). Similarly, an induction on (1.5) on an $(a+2, b+2)$ rectangle yields (1.7). This completes the theorem.

Let, for $a, b > 1$,

$$R(a,b) \equiv m(a)m(b). \tag{1.8}$$

From [1], Equation (1.2), $R(a,b)$ satisfies

$$R(a,b) = m(a) + m(b) - 1 + \frac{4}{(a-1)(b-1)} \int_0^{a-1} \int_0^{b-1} R(\xi,\eta)\, d\xi\, d\eta. \tag{1.9}$$

By Lemma 1 of [9], and the previous theorem,

$$R(a,b) \leqq M(a+2,b+2) \leqq R(a+2,b+2), \tag{1.10}$$

from which it follows that (see [9])

$$\lim_{a,b\to\infty}(ab)^{-1}M(a,b)=\left(\lim_{a\to\infty}a^{-1}m(a)\right)^2\equiv\eta^2, \tag{1.11}$$

which is the Palásti conjecture for the Rényi model. Similar results hold for the Solomon model.

University of California at Davis

Yours sincerely,
HOWARD WEINER

Dear Editor,

A critique of Weiner's work on Palásti's conjecture

Weiner [9] presented an argument purporting to prove Palásti's conjecture for Rényi and Solomon random parking schemes. This argument was criticized in letters to the editor by Tory and Pickard [8], Tanemura [7], and Hori [4], and Weiner responded [10]. Weiner [11] has now offered an alternative argument. In this letter, we point out various instances of the fundamental error which Weiner makes. This error invalidates not only his specific results but also his entire approach. Indeed, it is now apparent that very little of Weiner's work on Palásti's conjecture can withstand close scrutiny.

The notion of exchangeability is germane. A collection of random variables, indexed by I, is exchangeable if the likelihood is invariant under permutations of I. By exchangeability for sequential random packing schemes we shall mean that all realizations leading to the same final configuration have the same likelihood; i.e. the invariance is under permutations of the order in which the particles are packed. Tory and Pickard have pointed out the rather obvious fact that, in this sense, the Rényi model is not exchangeable (see Figure 2 and the related discussion in Tory and Pickard [8]). Clearly, neither is the Solomon model.

Weiner's fundamental error. To compare averages for different parking procedures, Weiner repeatedly (in both [9] and [11]) argues solely on the basis of related individual final configurations. That is, he consistently ignores the role in determining such averages played by the relative frequencies with which these configurations occur. These frequencies are different for the different models, so arguments relating averages via realizations must also involve the frequencies. Weiner's arguments do not. Furthermore, since the parking procedures are non-exchangeable, the relative frequencies are far too complicated to permit such comparisons. Consequently, Professor Weiner's work cannot be 'corrected'

Received 4 January 1980.