J. Austral. Math. Soc. 19 (Series A) (1975), 173–179.

ORTHOGONALITY RELATIONS ON ABELIAN GROUPS

G. DAVIS

(Received 18 January 1972; revised 17 July 1972)

Communicated by J. B. Miller

Abstract

An orthogonality relation is an abstract relation on a group having properties similar to the relation on a l-group given by $x \perp y$ if $|x| \wedge |y| = 0$. A group G with an orthogonality relation $\perp$ is isomorphically represented as a subgroup of the group Γ of continuous global sections of a sheaf of groups. If the stalks of the sheaf are torsion-free and G has and element 1 satisfying $1^{\perp} = (0)$ then Γ can be ordered so that it is an l-group and $x \perp y$ if and only if $|x| \wedge |y| = 0$ in Γ. An l-group G is *complemented* if for all $x, y \in G$ there is an $a \in x^{\perp} \oplus x^{\perp\perp}$ with $y \in a^{\perp\perp}$: equivalent conditions are given for G to be complemented.

Introduction

A. I. Veksler (1967) introduced the concept of disjointness relation on a linear space, as an abstraction of the relation $\perp$ on vector lattices defined by $x \perp y$ if $|x| \wedge |y| = 0$, and showed that for certain linear spaces E with a disjointness relation $\perp$ it is possible to define a lattice-order on E such that $x \perp y$ if and if and only if $|x| \wedge |y| = 0$. In this paper the concept of disjointness, now called orthogonality, is extended to abelian groups. It is seen that every group G with an orthogonality relation is (isomorphic with) a subgroup of a group Γ of continous global sections of a sheaf of groups with a Boolean base space. When the group G has an element 1 satisfying $1^{\perp} = (0)$ and the groups comprising the stalks in the sheaf representation are torsion-free then these stalks can be totally-ordered so that Γ is a lattice-ordered group and $x \perp y$ if and only if $|x| \wedge |y| = 0$.

Finally, verious characterizations are given of a class of l-groups that were called "weakly projectable" by Spirason and Strzelecki (to appear).

1. Orthogonality relations

Throughout, let G be an abelian group. A relation $\perp$ on G is a *pre-orthogonality* relation if

(1) $x \perp y$ implies $y \perp x$
(2) $0 \perp x$ for all $x \in G$, where 0 is the identity of G
(3) $x \perp x$ implies $x = 0$
(4) $x \perp a$, $y \perp a$ implies $x + y \perp a$.

For each non-empty subset A of G define $A^{\perp} = \{x \in G: x \perp a)$for all $a \in A\}$, $A^{\perp\perp} = (A^{\perp})^{\perp}$. If $A = \{x\}$ is a singleton set then denote $A^{\perp}$, $A^{\perp\perp}$ by $x^{\perp}$, $x^{\perp\perp}$. An *orthogonality relation* on G is a pre-orthogonality relation $\perp$ for which $x \perp y$ is equivalent to $x^{\perp\perp} \cap y^{\perp\perp} = (0)$. Using Frink's axioms (1941) for a Boolean algebra, it is readily seen that when $\perp$ is an orthogonality relation the class $B(G)$ of subsets of G of the form $A^{\perp}$, ordered by inclusion, is a complete Boolean algebra with $\bigcap_{\alpha} A_{\alpha}^{\perp} = (\cup A\alpha)^{\perp}$ and $A^{\perp\perp}$ as the complement of $A^{\perp}$.

The Stone space of $B(G)$ (i.e., the set of prime ideals of $B(G)$ equipped with the hull-kernel topology) will be denoted by Q. For each $t \in Q$ a subset G_t of G is defined by $G_t = \{x \in G: x^{\perp\perp} \in t\}$. It is readily seen that each G_t is a subgroup of G and $\bigcap_{t \in Q} G_t = (0)$. Using the method of Dauns and Hoffman (1966) it can be seen that there is a sheaf of groups with base space Q and stalks $G/G_t, t \in Q$, and an isomorphism $\hat{}$ given by $\hat{x}(t) = x + G_t$ mapping G into the group Γ of continuous global sections of this sheaf, such that $x \perp y$ if and only if for each $t \in Q$ either $\hat{x}(t) = 0$ or $\hat{y}(t) = 0$. This sheaf representation is more special than the obvious representation of G as a sub-direct product of the groups G/G_t only in that Q has a topology and the disjoint union of the groups G/G_t is given the finest topology for which each $\hat{x}$ is continuous. Γ then consists of all choice functions $Q \to \cup G/G_t$ that agree locally with some $\hat{x}$. Since Q is compact Hausdorff and has a base for the open sets consisting of the closed open sets $Q_{A_1^{\perp}} = \{t \in Q: A^{\perp} \notin t\}$, $A \subseteq G$, then for any $\sigma \in \Gamma$ there is a finite closed-open partition $Q_{A_1^{\perp}}, \cdots, Q_{A_n^{\perp}}$ of Q, and $x_1, \cdots, x_n \in G$ such that $\sigma = \sum I(Q_{A_i^{\perp}}; x_i)$ where

$$I(Q_{A_i}; x_i)(t) = \begin{cases} \hat{x}(t) & \text{if } t \in Q_{A_i} \\ 0 & \text{if } t \in Q_{A_i}. \end{cases}$$

A necessary and sufficient condition for $\hat{G} = \{\hat{x}: x \in G\}$ and Γ to be isomorphic is $G = A^{\perp} \oplus A^{\perp\perp}$ for all $A^{\perp} \in B(G)$.

2. Lattice-ordering Γ

It will be assumed in this section that $\perp$ is an orthogonality relation on the abelian group G.

LEMMA 2.1. *The quotient groups G/G_t are all torsion-free if and only if $x^{\perp} = (mx)^{\perp}$ for all x in G and all integers $m > 1$.*

PROOF. Assume $x^{\perp} = (mx)^{\perp}$ for all x in G and all $m \geqq 1$. I $mx \in G_t$ then $x^{\perp\perp} = (mx)^{\perp\perp} \in t$ so $x \in G_t$ and G/G_t is torsion free. If each G/G_t is torsion-free and $y \perp mx$ then $y \notin G_t$ implies $mx \in G_t$ so $x \in G_t$ and thus $y \perp x$.

Proposition 2.2. *Let G have an orthogonality relation $\perp$ satisfying $x^{\perp} = (mx)^{\perp}$ for all $x \in G$, $m \geqq 1$. For each $t \in Q$ let P_t be a positive set defining a total order on G/G_t compatible with the group structure. Define a partial order on Γ by $\sigma \geqq 0$ if $\sigma(t) \in Q$. Then Γ is lattice-ordered if for each $x \in G$ the set $Pos(x) = \{t \in Q\colon\ \hat{x}(t) > 0\}$ is open in Q.*

Proof. Each G/G_t is abelian and torsion-free by lemma 2.1 and therefore admits a total order P_t. Suppose that $\mathrm{Pos}(x)$ is open for each $x \in G$. Then the set $\{t \in Q\colon \hat{x}(t) \geqq 0\} = Q \backslash \mathrm{Pos}(-x)$ is closed and contains $\mathrm{Pos}(x)$ yet if $\hat{x}(t_0) = 0$ then there is a basic closed-open neighbourhood $Q_A^{+} = \{t \in Q\colon A^{\perp} \notin t\}$ of t_0 such that $\hat{x}(t) = 0$ for all $t \in Q_A^{+}$ so that $Q_A^{+} \cap \mathrm{Pos}(x)$ is void. Hence $\mathrm{Pos}(x)$ is closed and therefore closed-open. The function $\hat{x} \vee 0$ mapping Q into $\bigcup_{t \in Q} G/G_t)$ according to $t \mapsto \max\{\hat{x}(t), 0\}$ is the supremum of $\hat{x}$ and 0 in the ordered group of all (not necessarily continuous) global sections. However, $\hat{x} \vee 0 = I(\mathrm{Pos}(x); x$ where

$$I(\mathrm{Pos}(x);\, x)\,(t) \;=\; \begin{cases} \hat{x}(t) & \text{if } t \in \mathrm{Pos}(x) \\ 0 & \text{if } t \in \mathrm{Pos}(x), \end{cases}$$

so $\hat{x} \vee 0 \in \Gamma$. For $\sigma = \sum_i I(Q_{A_i}^{+}\colon x_i) \in \Gamma$, where $\{Q_{A_i}^{+}\}$ is a finite closed-open partition of Q,

$$\begin{aligned} \sigma \vee 0(t) &= \max\{\sigma(t), 0\} \\ &= \sum_i I(Q_{A_i}^{+} \cap \mathrm{Pos}(x_i);\, x_i) \end{aligned}$$

is in Γ, so that Γ is lattice-ordered.

Theorem 2.3. *If G has an orthogonality relation $\perp$ satisfying $x^{\perp} = (mx)^{\perp}$ for all x in G, $m \geqq 1$ and an element 1 satisfying $1^{\perp} = (0)$ then the group Γ can be lattice-ordered so that $\sigma \perp \tau$ if and only if $|\sigma| \wedge |\tau| = 0$ for σ, τ in Γ.*

Proof. Since $1^{\perp} = (0)$ then $1 \notin G_t$ for all $t \in Q$. For each $t \in Q$ there is a positive set in G/G_t such that for each $x \in G$ the set $\mathrm{Pos}(x) = \{t \in Q\colon \hat{x}(t) > 0$ in $G/G_t\}$ is open. Consider the positive sets $\{m1 + G_t\colon m = 0, 1, 2, \cdots\}$: for $x \in G$, $\mathrm{Pos}(x) = \bigcup_{m=0}^{\infty} \{t \in Q\colon \widehat{x - n}1(t) = 0\}$ which, as a union of open sets, is open. The class K of strings (P_t) such that each P_t is a positive set in G/G_t containing $1 + G_t$ and such that $\mathrm{Pos}(x) = \{t \in Q\colon 0 \neq x + G_t \in P_t\}$ is open for each $x \in G$ is inductive when ordered by $(P_t)_{t \in Q} \geqq (P_t')_{t \in Q}$ if $P_t \supseteq P_t'$ for all $t \in Q$, so that maximal elements of K follow from the axiom of choice. Let $(P_t)_{t \in Q}$ be maximal in K. For $x \in G$ the closure $\overline{\mathrm{Pos}(x)}$ of $\mathrm{Pos}(x)$ is closed-open in Q. If $t \in Q$ define $P_t' = \{x + G_t\colon t \in \overline{\mathrm{Pos}(x)}\}$. If $x + G_t \in P_t' \cap (-P_t')$ then $t \in \overline{\mathrm{Pos}(x)} \cap \overline{\mathrm{Pos}(-x)}$ which, however, is void since $\mathrm{Pos}(x)$, $\mathrm{Pos}(-x)$ are disjoint open sets with open closures. For each $t \in Q$ the set $P_t' \cup \{0\}$ is therefore a positive set in G/G_t containing P_t. Furthermore, for $x \in G$,

$$\begin{aligned}\operatorname{Pos}'(x) &= \{t\in Q\colon x+G_t\in P'_t\}\\ &= \{t\in Q\colon t\in\overline{\operatorname{Pos}(x)}\} = \overline{\operatorname{Pos}(x)} \text{ is closed-open.}\end{aligned}$$

The maximality of $(P_t)_{t\in Q}$ in K therefore gives that $\operatorname{Pos}(x)$ is closed-open for each $x\in G$. Now suppose that for some $t_0\in Q$, P_{t_0} does not define an isolated order on G/G_{t_0}, so there is an $a+G_{t_0}\in P_{t_0}$ and an integer $m_0\geqq 2$ such that $m_0a+G_{t_0}\in P_{t_0}$.

Then, $t_0\in\operatorname{Pos}(m_0a)$ so there is an open neighbourhood $Q_A{}^+$ of t_0 contained in $\operatorname{Pos}(m_0a)$. Also, $t_0\notin\operatorname{Pos}(a)$ so there is an open neighbourhood $Q_B{}^+$ of t_0 contained in $Q\backslash\operatorname{Pos}(a)$. Then for $t\in Q_A{}^+\cap Q_B{}^+$, P_t does not define an isolated order on G/G_t. Consider the string $(P'_t)_{t\in Q}$ with

$$P'_t = \begin{cases} P_t \text{ if } t\in Q_A{}^+\cap Q_B{}^+,\\ \{ma+y+G_t\colon m=0,1,2,\cdots, y+G_t\in P_t\}\\ \text{if } t\in Q_A{}^+\cap Q_B{}^+\end{cases}$$

Each P'_t is a semigroup properly containing P_t. If P'_t is not a positive set for some $t\in Q_A{}^+\cap Q_B{}^+$ then $n_1a+y_1+G_t=-n_2a-y_2+G_t$ for some integers n_1, $n_2\geqq 1$, and $y_1+G_t, y_2+G_t\in P_t$. Then $(n_1+n_2)a+G_t=-(y_1+y_2)+G_t\in -P_t$ and $m_0(n_1+n_2)a+G_t\in P_t$ (since $m_0a(t)>0$ for $t\in Q_A{}^+\cap Q_B{}^+$) so that $m_0(n_1+n_2)a+G_t=0$ which means $a\in G_t$ since G/G_t is torsion-free. This contradicts the choice of $a\in G$ so that each P'_t is a positive set in G/G_t. For $x\in G$ put $\operatorname{Pos}'(x)=\{t\in Q\colon 0\neq x+G_t\in P'_t\}$. Suppose $t_1\in\operatorname{Pos}'(x)$. If $t_1\in Q_A{}^+\cap Q_B{}^+$ then $t_1\in\operatorname{Pos}(x)$ so there is an open neighbourhood of t_1 contained in $\operatorname{Pos}(x)$ and hence in $\operatorname{Pos}'(x)$. If $t_1\in Q_A{}^+\cap Q_B{}^+$ then $\hat{x}(t_1)=\widehat{ma+y}(t_1)\neq 0$, so there is an open neighbourhood $Q_C{}^+$ of t_1 such that $\hat{x}(t)=\widehat{ma+y}(t)\neq 0$ for $t\in Q_C^+$. Also, there is an open neighbourhood $Q_D{}^+$ of t_1 such that $\hat{y}(t)>0$ for $t\in Q_D{}^+$. Thus $Q_C{}^+\cap Q_D{}^+\supseteq\operatorname{Pos}'(x)$. The set $\operatorname{Pos}'(x)$ is therefore open for each $x\in G$. This contradicts the maximality of $(P_t)_{t\in Q}$ in K so that each P_t defines an isolated order on G/G_t. Now suppose that for some $t\in Q$, P_t does not define a total order on G/G_t, so that for some $a\in G$, $a+G_t\in P_t\cup(-P_t)$. That is $t\in\operatorname{Pos}(x)\cap\operatorname{Pos}(-x)$ which is an open set, so for t_1 in some open neighbourhood $Q_A{}^+$ of t, P_t does not define a total order on G/G_{t_1}. As before, consider the string $(P'_t)_{t\in Q}$ with

$$P'_t = \begin{cases} P_t \text{ if } t\in Q_A{}^+,\\ \{ma+y+G_t\colon m=0,1,2,\cdots, y+G_t\in P_t\}\\ \qquad\qquad\qquad \text{if } t\in Q_A{}^+\end{cases}$$

Each P'_t is a semigroup containing P_t in G/G_t. If P'_t is not a positive set for some $t\in Q_A{}^+$ then, as before, $ma+G_t\in -P_t$ for some $m\geqq 1$. Since P_t defines an isolated order on G/G_t then $a + G_t \in - P_t$, contrary to assumption. The maximality of $(P_t)_{t \in Q}$ therefore gives that each P_t defines a total order on G/G_t. The group Γ is therefore lattice-ordered by $\sigma \geqq 0$ if $\sigma(t) \in P_t$ for all $t \in Q$. Since $|\sigma|(t) = \max\{\sigma(t), -\sigma(t)\}$ in G/G_t and $\sigma \wedge \tau(t) = \min\{\sigma(t), \tau(t)\}$ in G/G_t, then $|\sigma| \wedge |\tau| = 0$ if only if for each $t \in Q$ either $\sigma(t) = 0$ or $\tau(t) = 0$.

3. Lattice-groups

In this section G will denote an abelian lattice-group. The relation defined on G by $x \perp y$ if $|x| \wedge |y| = 0$ is an orthogonality relation with

$$x^{\perp\perp} \cap y^{\perp\perp} = (|x| \wedge |y|)^{\perp\perp}$$

$$x^{\perp\perp} \vee y^{\perp\perp} = (|x| \vee |y|)^{\perp\perp},$$

and the subgroups G_t are prime lattice-ideals. The results 3.1, 3.3 are due to Spirason and Strzelecki (to appear).

PROPOSITION 3.1. *A subgroup $I \subseteq G$ is of the form G_t for some $t \in Q$ if and only if*

(1) *$x \perp y$ implies $x \in I$ or $y \in I$*
(2) *$x \in I$ implies $x^{\perp\perp} \subseteq I$*
(3) *$x \in I$* implies *$x^{\perp} \neq (0)$*

An immediate corollary to this is that every minimal prime lattice-ideal of G is of the form G_t, for some $t \in Q$. Recall that a prime lattice-ideal M is *minimal prime* if M is minimal in the class of prime lattice-ideals of G: a necessary and sufficient condition for a prime lattice-ideal M to be minimal prime is that for each $x \in M$ there is a $y \notin M$ with $x \perp y$.

DEFINITION 3.2. The lattice-group G is said to be *complemented* if for all $x, y \in G$ there is an $a \in x^{\perp\perp} \oplus x^{\perp}$ such that $y \in a^{\perp\perp}$.

THEOREM 3.3. *G is complemented if and only if every subgroup $G_t \neq G$ is a minimal prime lattice-ideal.*

The class of minimal prime lattice ideals of G will be denoted by $M(G)$. For $x \in G$ the class $M(x) \subseteq M(G)$ is defined by $M(x) = \{M \in M(G): x \notin M\}$. The set $\mu_G = \{M(x): x \in G\}$ forms a closed-open base for the open sets for a Hausdorff topology on $M(G)$. In fact, for $x, y \in G$,

$$M(x) \cap M(y) = M(|x| \wedge |y|)$$

and

$$M(x) \cup M(y) = M(|x| \vee |y|)$$

https://doi.org/10.1017/S1446788700029475 Published online by Cambridge University Press

The class of subgroups $G_t \neq G$ is denoted by $V(G)$. For $x \in G$, the class $V(x)$ is defined by $V(x) = \{G_t \neq G: x \notin G_t\}$. For $x, y \in G$,

$$V(x) \cap V(y) = V(|x| \wedge |y|)$$

and

$$V(x) \cup V(y) = V(|x| \vee |y|).$$

The set $v_G = \{V(x): x \in G\}$ then forms a compact-open base for the open sets for a topology on $V(G)$, which is compact if and only if G has an element 1 satisfying $1^{\perp} = (0)$. Further it is readily seen that for $x, y \in G$, $V(x) = V(y)$ if and only if $x^{\perp} = y^{\perp}$.

THEOREM 3.4. *The following are equivalent*:

(1) *G is complemented*
(2) *$V(G) = M(G)$*
(3) *v_G is relatively complemented*
(4) *$V(G)$ is a Hausdorff space*
(5) *Each $V(x) \in v_G$ is closed in $V(G)$.*

PROOF. If every $G_t \neq G$ is minimal prime and $G_{t_1} \neq G_{t_2} \notin V(G)$ then there is an $x \in G_{t_1}$ such that $x \notin G_{t_2}$, and a $y \in G_{t_1}$ such that $x \perp y$. Then $G_{t_1} \in V(y)$, $G_{t_2} \in V(x)$ and $V(y) \cap V(x)$ is void. Conversely, if $V(G) \neq M(G)$ then there is a $G_t \neq G$ that is not minimal prime. Then G_t contains a minimal prime G_{t_1}, and G_t, G_{t_1} cannot be Hausdorff-separated. Thus (2) is equivalent to (4). Suppose that G is complemented. Take $V(x) \in v_G$ and $V(y) \subseteq V(x)$ so that $y^{\perp\perp} \subseteq x^{\perp\perp}$. Then there exist $a \in y^{\perp}$, $b \in y^{\perp\perp}$ such that $x^{\perp\perp} \subseteq (a+b)^{\perp\perp} = a^{\perp\perp} \vee b^{\perp\perp}$. Then

$$(|x| \wedge |a|)^{\perp\perp} \subseteq x^{\perp\perp} \text{ and } (|x| \wedge |a|)^{\perp\perp} \vee y^{\perp\perp} = (x^{\perp\perp} \wedge a^{\perp\perp}) \vee y^{\perp\perp}$$

$$= (x^{\perp\perp} \vee y^{\perp\perp}) \wedge (a^{\perp\perp} \vee y^{\perp\perp}) = x^{\perp\perp}.$$

Also, $|x| \wedge |a| \perp y$ so that $(|x| \wedge |a|) \cap V(y)$ is void, and $V(|x| \wedge |a|) \cup V(y) = V(x)$. That is v_G is relatively complemented. On the other hand if v_G is relatively complemented and $x, y \in G$ then $V(x) \subseteq V(|x| \wedge |y|)$ so there is an $x' \in G$ such that $x' \perp x$ and $(x')^{\perp\perp} \vee x^{\perp\perp} = x^{\perp\perp} \wedge y^{\perp\perp}$. Then $y \in (x')^{\perp\perp} \vee x^{\perp\perp} = (x'+x)^{\perp\perp}$ so that G is complemented. That is (1) is equivalent to (3).

Suppose that each $V(x)$ in v_G is closed and $G_t \neq G$. If $x \in G_t$ then $t \in V(x)$ so there is a basic open set $V(y)$ with $t \in V(y) \subseteq V(G) \backslash V(x)$. That is $y \perp x$ and $y \notin G_t$, so that G_t is minimal prime. If each $G_t \neq G$ is minimal prime then $V(G) = M(G)$ so that each $V(x) = M(x)$ is closed.

References

J. Dauns and K. H. Hofmann (1966), 'The representation of biregular rings by sheaves', *Math. Z.* **91**, 103–123.

G. T. Spirason and E. Strezelcki, 'A note on *Pi-ideals*' (to appear).

A. I. Veksler (1967), 'Linear spaces with disjoint elements and their conversion into vector lattices', *LLeningrad Gos. Ped. Inst. Ucen. Zap.* **328**, 19–43.

O. Frink (1941), 'Representations of Boolean algebras', *Bull. Amer. Math. Soc.* **47**, 755–756.

La Trobe University
Bundoora 3083 Australia