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On Maps Preserving Products

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Abstract. Maps preserving certain algebraic properties of elements are often studied in Functional Analysis and Linear Algebra. The goal of this paper is to discuss the relationships among these problems from the ring-theoretic point of view.

1 Introduction

The study of linear preserver problems is an active research area in matrix theory and operator theory. Some of these problems concern linear operators on the spaces of matrices or operators which preserve certain relations [23, 24, 28]. Given a relation $\sim$ on an algebra $\mathcal{A}$, one studies the linear preservers of $\sim$, that is, those linear operators ϕ on $\mathcal{A}$ satisfying $\phi(A) \sim \phi(B)$ whenever $A \sim B$. For example, when $\sim$ means commutativity, then the objects will be those linear operators ϕ satisfying $\phi(A)\phi(B) = \phi(B)\phi(A)$ whenever $AB = BA$. Putting $A * B = [A, B] = AB - BA$, the Lie product of A and B, we can also say that this relation $\sim$ means zero Lie product, that is, $A \sim B$ means $[A, B] = 0$, and that ϕ preserves zero Lie products, that is, $[A, B] = 0$ implies $[\phi(A), \phi(B)] = 0$. In fact, one can define $*$ in various ways, for example, $A * B = AB + BA$, the Jordan product, or even $A * B = AB$, the usual product. In the last case, the corresponding linear preservers are usually referred to as zero-product preserving maps.

Much work has been done concerning zero-product preserving maps [11, 13, 15, 20, 29, 33]. In these papers, bijective maps f satisfying the condition $f(x)f(y) = 0$ whenever $xy = 0$ were investigated in the cases of matrix algebras or operator algebras. As a rule, it was proved under certain assumptions that such maps differ from ordinary automorphisms by central elements. However, it is not possible to obtain such a kind of description for algebras which do not have enough zero-divisors. For example, in division rings all additive maps preserve zero products.

Here is another way to look at the zero-product preserving maps. If ϕ preserves zero products, and if $AB = 0 = CD$, then $\phi(A)\phi(B) = 0 = \phi(C)\phi(D)$. In a broader way, we may ask what we can say about ϕ if ϕ preserves "(not necessarily zero) constant products." Or, even more general, we state the following

Problem 1.1 Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, S a nonempty subset of $\mathcal{A}$ and X a relation on S. Suppose that $f\colon S \to \mathcal{B}$ is a map such that

$$f(x)f(y) = f(u)f(v) \text{ whenever } (x, y), (u, v) \in X \text{ with } xy = uv. \tag{1.1}$$

Is it possible to describe f under certain reasonable conditions on $\mathcal{A}$, $\mathcal{B}$, S, X, and f?

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In case $\mathcal{A} = \mathcal{B}$ is a division ring, $S = \mathcal{A}$, $X = \{(x, y) \mid x, y \in \mathcal{A}, xy = k\}$ where k is a nonzero element in $\mathcal{A}$, and $f: \mathcal{A} \to \mathcal{A}$ is a bijective additive map, the preceding problem is specified as

Problem 1.2 Let K be a division algebra and k a nonzero element in K. Suppose that $f: K \to K$ is a bijective additive map such that

$$f(x)f(y) = f(u)f(v) \text{ whenever } xy = uv = k. \tag{1.2}$$

Is it possible to describe f?

Note that if $f = \lambda\varphi$, where λ is a central element in K and $\varphi: K \to K$ is an automorphism, then f certainly satisfies (1.2). If, in addition, k is a central element in K, then $f = \lambda\varphi$ also satisfies (1.2) where $\varphi: K \to K$ is an antiautomorphism. On the other hand, in case the specified element is $k = 1$, the converse is also true. In other words, any bijective additive map $f: K \to K$ such that $f(x)f(y) = f(u)f(v)$ whenever $xy = uv = 1$ is of the form $f = \lambda\varphi$ where λ is a central element in K and $\varphi: K \to K$ is either an automorphism or an antiautomorphism (see Theorem 2.1). This result is equivalent to a theorem of Hua [19] under the additional condition that $f(1) = 1$. In Section 2, we shall remove the condition $f(1) = 1$ in Hua's result and thus solve the problem.

Next, let $\mathcal{A}$ and $\mathcal{B}$ be algebras, S an additive subgroup of $\mathcal{A}$ and $f: S \to \mathcal{B}$ an additive map such that $f(x)f(y) = f(u)f(v)$ for all $x, y, u, v \in S$ with $xy = uv$. It is easy to see that if $\mathcal{A}$ contains an identity 1, then $f(1)$ lies in the centralizer of $f(S)$ in $\mathcal{B}$. If S is multiplicatively closed and contains the identity 1 of $\mathcal{A}$, then $f(1)f(xy) = f(x)f(y)$ for all $x, y \in S$. When $S = \mathcal{A}$, $\mathcal{B}$ has an identity and $\lambda = f(1)$ is invertible in $\mathcal{B}$, we see that $\lambda^{-1}f$ is a homomorphism from $\mathcal{A}$ into $\mathcal{B}$.

In case $\mathcal{A}$ does not have an identity, the situation is a little more difficult. In addition to some restrictions on $f(S)$, we need to make use of a newly developed theory of functional identities. We will state some basic definitions and results on functional identities in Section 3, and then consider in Section 4 the cases when S need not be closed under multiplication (for example, S is a Lie ideal of $\mathcal{A}$, or the set of all skew elements of $\mathcal{A}$ in the presence of involution).

Note that we do not assume additivity of f in considering Problem 1.1. The 2-local automorphisms introduced by Šemrl [30], which are not necessarily additive, are easily seen to satisfy (1.1) in case $X = \{(x, y) \mid x, y \in \mathcal{A}, xy = 1\}$. In Section 5, we prove that all 2-local automorphisms of a finite dimensional division algebra of characteristic zero are automorphisms or antiautomorphisms.

2 A Generalization of Hua's Theorem

In 1949 Hua [19] proved that every bijective additive map $\alpha: K \to K$ on a division ring K satisfing $\alpha(aba) = \alpha(a)\alpha(b)\alpha(a)$ and $\alpha(1) = 1$ is an automorphism or an antiautomorphism. This result was reformulated by Artin in 1957 as [2, Theorem 1.15]: Any bijective additive map $\alpha: K \to K$ on a division ring K satisfying $\alpha(a^{-1}) = \alpha(a)^{-1}$ and $\alpha(1) = 1$ is an automorphism or an antiautomorphism. A similar result was also established for the $n \times n$ matrix rings over a division ring K [14]. Here we shall extend [2, Theorem 1.15] as

Theorem 2.1 *Let K be a division ring with center Z and $\alpha\colon K \to K$ a bijective additive map such that*

$$\alpha(a^{-1})\alpha(a) = \alpha(b^{-1})\alpha(b) \quad \textit{for all nonzero } a, b \in K.$$

Then $\alpha = \lambda\varphi$, where $\varphi\colon K \to K$ is an automorphism or an antiautomorphism and $\lambda = \alpha(1) \in Z$.

Proof Let $x, y \in K$ be nonzero elements with $xy \neq 1$. Then $x - y^{-1}$ is not zero and neither is $x^{-1} - (x - y^{-1})^{-1}$. Thus we have the following beautiful identity due to Hua:

$$(x^{-1} - (x - y^{-1})^{-1})^{-1} = x - xyx. \tag{2.1}$$

Let $z = \alpha(1^{-1})\alpha(1) \neq 0$; then $z = \alpha(a^{-1})\alpha(a) = \alpha(a)\alpha(a^{-1})$ and so

$$\alpha(a)z = \alpha(a)(\alpha(a^{-1})\alpha(a)) = (\alpha(a)\alpha(a^{-1}))\alpha(a) = z\alpha(a)$$

for all nonzero $a \in K$. Also, it is trivial that $\alpha(0)z = 0 = z\alpha(0)$. Therefore we conclude that $z \in Z$ by the surjectivity of α. Now, applying α to (2.1) and using $\alpha(a^{-1}) = z\alpha(a)^{-1}$, we obtain

$$\begin{aligned}
\alpha(xyx) &= \alpha(x) - \alpha\big((x^{-1} - (x - y^{-1})^{-1})^{-1}\big) \\
&= \alpha(x) - z\alpha\big(x^{-1} - (x - y^{-1})^{-1}\big)^{-1} \\
&= \alpha(x) - z\big(\alpha(x^{-1}) - \alpha\big((x - y^{-1})^{-1}\big)\big)^{-1} \\
&= \alpha(x) - z\big(z\alpha(x)^{-1} - z\big(\alpha(x) - \alpha(y^{-1})\big)^{-1}\big)^{-1} \\
&= \alpha(x) - \big(\alpha(x)^{-1} - \big(\alpha(x) - z\alpha(y)^{-1}\big)^{-1}\big)^{-1} \\
&= \alpha(x) - \big(\alpha(x)^{-1} - \big(\alpha(x) - (z^{-1}\alpha(y))^{-1}\big)^{-1}\big)^{-1} \\
&= \alpha(x) - (\alpha(x) - \alpha(x)z^{-1}\alpha(y)\alpha(x)) \\
&= z^{-1}\alpha(x)\alpha(y)\alpha(x)
\end{aligned}$$

for all nonzero $x, y \in K$ such that $xy \neq 1$. It easy to see that

$$\alpha(xyx) = z^{-1}\alpha(x)\alpha(y)\alpha(x) \tag{2.2}$$

is in fact true for all $x, y \in K$ even if $xy = 0$ or $xy = 1$. Setting $x = 1$ in (2.2) we obtain $\alpha(y) = z^{-1}\alpha(1)\alpha(y)\alpha(1)$, that is, $\alpha(y) = \alpha(1)^{-1}\alpha(y)\alpha(1)$ for all $y \in K$. Hence $\lambda = \alpha(1) \in Z$ since α is surjective. Let $\varphi\colon K \to K$ be defined by $\varphi(a) = \lambda^{-1}\alpha(a)$ for $a \in K$. Clearly φ is a bijective additive map on K, $\varphi(1) = 1$, and $\varphi(a^{-1})\varphi(a) = 1$ for all nonzero $a \in K$. Thus φ is an automorphism or an antiautomorphism in light of Hua's theorem [2, Theorem 1.15] and the proof is then complete. ■

3 Functional Identities and d-Free Sets

The material in this section is taken from the papers [3, 4]. For readers interested in the theory of functional identities, the survey papers by Beidar, Chebotar and Mikhalev [6] and Brešar [8] will be very helpful.

Let S be a nonempty set, Q an algebra with 1, C the center of Q and $\alpha\colon S \to Q$ a map of sets. Let $\mathbb{N}$ be the set of all positive integers and for $n \in \mathbb{N}$ we let S^n denote the n-th Cartesian power of S. The symbol $\mathbf{x}_n$ will be used for $(x_1, \dots, x_n) \in S^n$. For convenience, we use $f\colon S^0 \to Q$ to mean that f is a fixed element in Q.

Let $f\colon S^m \to Q$ be an arbitrary map where $m \in \mathbb{N}$. For $1 \le i < j \le m+2$ define $\hat{f}^i\colon S^{m+1} \to Q$, $\hat{f}^{ij}\colon S^{m+2} \to Q$ and $\hat{f}^{ji}\colon S^{m+2} \to Q$ by

$$\hat{f}^i(\mathbf{x}_{m+1}) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m+1}) \quad \text{and}$$
$$\hat{f}^{ij}(\mathbf{x}_{m+2}) = \hat{f}^{ji}(\mathbf{x}_{m+2}) = f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_{m+2}).$$

Now let $I, J \subseteq \{1, 2, \dots, m\}$ where $m \in \mathbb{N}$ and $m \ge 2$, and for each $i \in I$, $j \in J$, let $E_i, F_j\colon S^{m-1} \to Q$ be arbitrary maps. The basic *functional identities* are of the form

$$\sum_{i\in I} \hat{E}_i^i(\mathbf{x}_m)\alpha(x_i) + \sum_{j\in J} \alpha(x_j)\hat{F}_j^j(\mathbf{x}_m) = 0 \quad \text{for all } \mathbf{x}_m \in S^m, \tag{3.1}$$

or a slightly more general one,

$$\sum_{i\in I} \hat{E}_i^i(\mathbf{x}_m)\alpha(x_i) + \sum_{j\in J} \alpha(x_j)\hat{F}_j^j(\mathbf{x}_m) \in C \quad \text{for all } \mathbf{x}_m \in S^m. \tag{3.2}$$

A natural possibility that makes (3.1) (and hence (3.2) too) to be true is when there exist maps $p_{ij}\colon S^{m-2} \to Q$ ($i \in I$, $j \in J$ and $i \ne j$) and $\lambda_k\colon S^{m-1} \to C$ ($k \in I \cup J$) such that

$$\begin{aligned} \hat{E}_i^i(\mathbf{x}_m) &= \sum_{\substack{j\in J\\ j\ne i}} \alpha(x_j)\hat{p}_{ij}^{ij}(\mathbf{x}_m) + \hat{\lambda}_i^i(\mathbf{x}_m), \\ \hat{F}_j^j(\mathbf{x}_m) &= -\sum_{\substack{i\in I\\ i\ne j}} \hat{p}_{ij}^{ij}(\mathbf{x}_m)\alpha(x_i) - \hat{\lambda}_j^j(\mathbf{x}_m), \quad \text{and} \\ \lambda_k &= 0 \quad \text{if} \quad k \notin I \cap J, \end{aligned} \tag{3.3}$$

for all $\mathbf{x}_m \in S^m$, $i \in I$, $j \in J$. Indeed, one can readily check that (3.3) implies (3.1).

Definition 3.1 Notations as above. Let $d \in \mathbb{N}$. The set $\alpha(S)$ is said to be a *d-free* subset of Q if, for all $m \in \mathbb{N}$ and $I, J \subseteq \{1, 2, \dots, m\}$, both of the following conditions are satisfied:

(a) If $\max\{|I|, |J|\} \le d$, (3.1) implies (3.3).
(b) If $\max\{|I|, |J|\} \le d-1$, (3.2) implies (3.3).

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The class of prime rings abounds in d-free subsets as the following two theorems show.

Theorem 3.2 ([3, Theorems 2.4 and 2.20]) *Let A be a prime ring with maximal right quotient ring Q and extended centroid C. Let*

$$\deg(A) = \sup\{\deg(x) \mid x \in A\},$$

where $\deg(x)$ is the degree of x over C if x is algebraic over C, or ∞ if x is not algebraic over C (see [5, Chapter 2]*).*

(1) *If $\deg(A) \geq d$, then A is a d-free subset of Q.*
(2) *If $\deg(A) \geq d+1$, then any noncentral Lie ideal of A is a d-free subset of Q.*
(3) *If A has an involution and $\deg(A) \geq 2(d+1)$, then both the set of skew elements and the set of symmetric elements in A are d-free subsets of Q.*

Theorem 3.3 ([3, Theorem 2.8]) *Let Q be an algebra with 1, C the center of Q, $B \subseteq R$ nonempty subsets of Q and $d \in \mathbb{N}$. If B is d-free, so is R.*

One of the important concepts in the theory of functional identities is that of *quasi-polynomials*. Here we give the definition of the quasi-polynomials in a loose manner and refer the reader to [4] for details.

Let S be an additive group, Q an algebra with 1, C the center of Q and $\alpha\colon S \to Q$ an additive map. We say that a map $E\colon S \to Q$ is an additive quasi-polynomial in α if there exists an element $\lambda \in C$ and an additive map $\mu\colon S \to C$ such that

$$E(x) = \lambda\alpha(x) + \mu(x) \quad \text{for all } x \in S,$$

where λ and μ are called the coefficients of E. In the case when $\mu = 0$, E is said to be without constant coefficient.

Next, a map $E\colon S^2 \to Q$ is said to be a bi-additive quasi-polynomial in α if there exist elements $\lambda_1, \lambda_2 \in C$, additive maps $\mu_1, \mu_2\colon S \to C$ and a bi-additive map $\nu\colon S^2 \to C$ such that

$$E(x,y) = \lambda_1\alpha(x)\alpha(y) + \lambda_2\alpha(y)\alpha(x) + \mu_1(x)\alpha(y) + \mu_2(y)\alpha(x) + \nu(x,y)$$

for all $x, y \in S$. As before, λ_1, λ_2, μ_1, μ_2 and ν are called the coefficients of E, and E is said to be without constant coefficient if $\nu = 0$.

In this way, we can define an m-additive quasi-polynomial in α which involves summands such as

$$\begin{gathered}\lambda\alpha(x_1)\cdots\alpha(x_m),\\ \mu(x_1)\alpha(x_2)\cdots\alpha(x_m),\dots,\mu(x_m)\alpha(x_1)\cdots\alpha(x_{m-1}),\\ \nu(x_1,x_2)\alpha(x_3)\cdots\alpha(x_m),\dots,\nu(x_{m-1},x_m)\alpha(x_1)\cdots\alpha(x_{m-2}),\end{gathered} \tag{3.4}$$

and so on.

Theorem 3.4 ([4, Theorem 1.1]) *Let S be an additive group, Q an algebra with 1, C the center of Q and $\alpha\colon S \to Q$ an additive map. Suppose that $E\colon S^m \to Q$ is an m-additive quasi-polynomial in α such that $E(\mathbf{x}_m) = 0$ for all $\mathbf{x}_m \in S^m$. If $\alpha(S)$ is m-free and E is without constant coefficient, or if $\alpha(S)$ is $m+1$-free, then all the coefficients of E are zero.*

Now, we are ready to continue our investigation.

4 Equal-Product Preserving

Proposition 4.1 *Let A and Q be associative rings with $A^2 = A$, and C the center of Q which is a field. Suppose that $\alpha\colon A \to Q$ is an additive map such that $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. If $\alpha(A)$ is a 3-free subset of Q, then $\alpha = \lambda\varphi$ where $\lambda \in C$ and $\varphi\colon A \to Q$ is a homomorphism.*

Proof It suffices to consider the case when α is a nonzero map. We first note that

$$\alpha(xy)\alpha(z) = \alpha(x)\alpha(yz) \quad \text{for all } x, y, z \in A, \tag{4.1}$$

and so $\alpha(xy)$ is a bi-additive quasi-polynomial in α by [4, Theorem 1.2]. That is,

$$\alpha(xy) = \lambda_1\alpha(x)\alpha(y) + \lambda_2\alpha(y)\alpha(x) + \mu_1(x)\alpha(y) + \mu_2(y)\alpha(x) + \nu(x, y), \tag{4.2}$$

for some elements $\lambda_1, \lambda_2 \in C$, additive maps $\mu_1, \mu_2\colon A \to C$ and biadditive map $\nu\colon A^2 \to C$. Substituting (4.2) into (4.1) we obtain a tri-additive quasi-polynomial in α without constant coefficient

$$\begin{aligned}&\lambda_2\alpha(x)\alpha(z)\alpha(y) - \lambda_2\alpha(y)\alpha(x)\alpha(z) + \mu_2(z)\alpha(x)\alpha(y)\\ &\quad + [\mu_1(y) - \mu_2(y)]\alpha(x)\alpha(z) - \mu_1(x)\alpha(y)\alpha(z) + \nu(y, z)\alpha(x) - \nu(x, y)\alpha(z)\end{aligned}$$

which vanishes for all $x, y, z \in A$. Since $\alpha(A)$ is a 3-free subset of Q, it follows from Theorem 3.4 that $\lambda_2 = \mu_1 = \mu_2 = \nu = 0$. Thus $\alpha(xy) = \lambda_1\alpha(x)\alpha(y)$ for all $x, y \in A$ and so $\varphi = \lambda_1\alpha$ is a homomorphism. Since $A^2 = A$ and α is nonzero, we have $\lambda_1 \neq 0$. Therefore $\alpha = \lambda\varphi$ where $\lambda = \lambda_1^{-1}$ and so the proof is complete. ■

Lemma 4.2 *Let R and Q be associative rings, A a Lie ideal of R, and C the center of Q which is a field of characteristic not 2. Suppose that $\alpha\colon A \to Q$ is an additive map satisfying the properties that (1) $\alpha([A, A])$ is not contained in C, and (2) $\sum_i \alpha(x_i)\alpha(y_i) = 0$ for all $x_i, y_i \in A$ with $\sum_i x_i y_i = 0$. If $\alpha(A)$ is a 5-free subset of Q, then $\alpha = \lambda\varphi$, where $\lambda \in C$ and $\varphi\colon A \to Q$ is a Lie homomorphism.*

Proof Given $x, y, z \in A$, we have

$$[x, y]z - z[x, y] + [y, z]x - x[y, z] - [x, z]y + y[x, z] = 0,$$

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and so

$$\alpha([x,y])\alpha(z) - \alpha(z)\alpha([x,y]) + \alpha([y,z])\alpha(x) \\ - \alpha(x)\alpha([y,z]) - \alpha([x,z])\alpha(y) + \alpha(y)\alpha([x,z]) = 0.$$

In light of [4, Theorem 1.2], we conclude that $\alpha([x,y])$ is a bi-additive quasi-polynomial in α, that is,

$$\alpha([x,y]) = \lambda_1\alpha(x)\alpha(y) + \lambda_2\alpha(y)\alpha(x) + \mu_1(x)\alpha(y) + \mu_2(y)\alpha(x) + \nu(x,y), \tag{4.3}$$

for some elements $\lambda_1, \lambda_2 \in C$, additive maps $\mu_1, \mu_2 \colon A \to C$ and bi-additive map $\nu \colon A^2 \to C$. Our goal is to show that $\lambda_2 = -\lambda_1 \neq 0$ and $\mu_1 = \mu_2 = \nu = 0$.

Since $\alpha([x,y]) + \alpha([y,x]) = 0$, we obtain from (4.3) a bi-additive quasi-polynomial in α

$$(\lambda_1 + \lambda_2)\alpha(x)\alpha(y) + (\lambda_1 + \lambda_2)\alpha(y)\alpha(x) + (\mu_1(x) + \mu_2(x))\alpha(y) \\ + (\mu_1(y) + \mu_2(y))\alpha(x) + (\nu(x,y) + \nu(y,x))$$

which vanishes for all $x, y \in A$. Since $\alpha(A)$ is 5-free and *a fortiori* 3-free, it follows from Theorem 3.4 that $\lambda_1 + \lambda_2 = 0$, $\mu_1(x) + \mu_2(x) = 0$ and $\nu(x,y) + \nu(y,x) = 0$ for all $x, y \in A$. Thus (4.3) can be rewritten as

$$\alpha([x,y]) = \lambda_1[\alpha(x), \alpha(y)] + \mu_1(x)\alpha(y) - \mu_1(y)\alpha(x) + \nu(x,y). \tag{4.4}$$

For $x, y, u, v \in A$, we have

$$x[y,uv] + [x,uv]y + u[v,xy] + [u,xy]v = [xy,uv] + [uv,xy] = 0,$$

and so

$$\alpha(x)\alpha([y,uv]) + \alpha([x,uv])\alpha(y) + \alpha(u)\alpha([v,xy]) + \alpha([u,xy])\alpha(v) = 0. \tag{4.5}$$

Now, we can conclude from [4, Theorem 2.6] that $\alpha([u,xy])$ is a tri-additive quasi-polynomial in α. After substituting the quasi-polynomial expression of $\alpha([u,xy])$ into (4.5) as we did above for $\alpha([x,y])$, and making use of Theorem 3.4, we will obtain (with some tedious computations) that

$$\alpha([u,xy]) = c[\alpha(u), \alpha(x)\alpha(y)] \tag{4.6}$$

for some $c \in C$. From this we have

$$\alpha([u,[x,y]]) = \alpha([u,xy]) - \alpha([u,yx]) \\ = c[\alpha(u), \alpha(x)\alpha(y)] - c[\alpha(u), \alpha(y)\alpha(x)] = c[\alpha(u), [\alpha(x), \alpha(y)]].$$

On the other hand we obtain from (4.4) that

$$\begin{aligned}\alpha([u,[x,y]]) = \lambda_1\big[\alpha(u), \lambda_1[\alpha(x),\alpha(y)] + \mu_1(x)\alpha(y) - \mu_1(y)\alpha(x) + \nu(x,y)\big]\\ + \mu_1(u)\big(\lambda_1[\alpha(x),\alpha(y)] + \mu_1(x)\alpha(y) - \mu_1(y)\alpha(x) + \nu(x,y)\big)\\ - \mu_1([x,y])\alpha(u) + \nu(u,[x,y]).\end{aligned}$$

Comparing both expressions for $\alpha([u,[x,y]])$, we obtain a tri-additive quasi-polynomial in α,

$$\begin{aligned}(\lambda_1^2 - c)[\alpha(u),[\alpha(x),\alpha(y)]] + \lambda_1\mu_1(u)[\alpha(x),\alpha(y)]\\ + \lambda_1\mu_1(x)[\alpha(u),\alpha(y)] - \lambda_1\mu_1(y)[\alpha(u),\alpha(x)] - \mu_1([x,y])\alpha(u)\\ - \mu_1(u)\mu_1(y)\alpha(x) + \mu_1(u)\mu_1(x)\alpha(y) + \nu(u,[x,y])\end{aligned}$$

which vanishes for all $u, x, y \in A$. Since $\alpha(A)$ is 5-free and *a fortiori* 4-free, it follows from Theorem 3.4 that $\lambda_1^2 = c$ and $\mu_1 = 0$. Thus (4.4) and (4.6) can be rewritten respectively as

$$\alpha([x,y]) = \lambda_1[\alpha(x),\alpha(y)] + \nu(x,y) \tag{4.7}$$

and

$$\alpha([u,xy]) = \lambda_1^2[\alpha(u),\alpha(x)\alpha(y)]. \tag{4.8}$$

If $\lambda_1 = 0$, then (4.7) would imply that $\alpha([A,A])$ is contained in C, contradicting our hypothesis. Therefore, $\lambda_1 \neq 0$.

It remains to show that $\nu = 0$. Since $[x[y,u],v] + [[x,u]y,v] = [[xy,u],v]$, we have

$$\alpha([x[y,u],v]) + \alpha([[x,u]y,v]) = \alpha([[xy,u],v]) \quad \text{for all } x,y,u,v \in A.$$

By (4.7) and (4.8), the last equation becomes

$$\begin{aligned}\lambda_1^2[\alpha(x)(\lambda_1[\alpha(y),\alpha(u)] + \nu(y,u)),\alpha(v)]\\ + \lambda_1^2[(\lambda_1[\alpha(x),\alpha(u)] + \nu(x,u))\alpha(y),\alpha(v)]\\ = \lambda_1^3[[\alpha(x)\alpha(y),\alpha(u)],\alpha(v)] + \nu([xy,u],v).\end{aligned}$$

Thus we obtain a 4-additive quasi-polynomial in α

$$\lambda_1^2\nu(y,u)[\alpha(x),\alpha(v)] + \lambda_1^2\nu(x,u)[\alpha(y),\alpha(v)] - \nu([xy,u],v)$$

which vanishes for all $u, v, x, y \in A$. By Theorem 3.4 again, we have $\nu = 0$ since $\alpha(A)$ is 5-free and $\lambda_1 \neq 0$. Thus $\alpha([x,y]) = \lambda_1[\alpha(x),\alpha(y)]$ for all $x, y \in A$. Then $\varphi = \lambda_1\alpha$ is a Lie homomorphism and $\alpha = \lambda\varphi$, where $\lambda = \lambda_1^{-1}$. This completes the proof. ∎

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In Lemma 4.2 we impose the condition on a Lie ideal A of R that

$$(4.9)\qquad \sum_i \alpha(x_i)\alpha(y_i) = 0 \quad \text{for all } x_i, y_i \in A \text{ with } \sum_i x_i y_i = 0.$$

This implies in particular that α is equal-product preserving, that is, $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. The converse implication is true when $R = M_n(F)$, the algebra of all $n \times n$ matrices over a field F, and $A = [R, R]$. Since $[R, R]$ is spanned by $E_{i,j} = e_{i,j}$, $i \neq j$ and $E_{i,i} = e_{i,i} - e_{1,1}$, $i \neq 1$, in order to verify (4.9) it suffices to show that

$$(4.10)\qquad \sum_{i,j,k,l} \alpha(a_{i,j}E_{i,j})\alpha(b_{k,l}E_{k,l}) = 0 \quad \text{whenever } \sum_{i,j,k,l} a_{i,j}E_{i,j}b_{k,l}E_{k,l} = 0,$$

which can be done by routine, but somewhat tedious, matrix computations. Then we have the following

Corollary 4.3 *Let Q be an associative ring with center C a field of characteristic not 2, R a matrix algebra over a field F and $A = [R, R]$. Suppose that $\alpha\colon A \to Q$ is an additive map satisfying the properties that (1) $\alpha([A, A])$ is not contained in the center C, and (2) $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. If $\alpha(A)$ is a 5-free subset of Q, then $\alpha = \lambda\varphi$ for some nonzero $\lambda \in C$ and some Lie homomorphism $\varphi\colon A \to Q$.*

In light of the above corollary, it is natural to ask

Problem 4.4 Let Q be an associative ring with center C a field of characteristic not 2, R a prime ring and A a Lie ideal of R. Suppose that $\alpha\colon A \to Q$ is an additive map such that $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. If $\alpha(A)$ is a d-free subset of Q with sufficiently large d, is it possible to describe α in terms of a Lie homomorphism?

Theorem 4.5 *Let Q be an associative ring with center C a field of characteristic not 2, R a ring with involution and A the Lie ring of all skew elements of R. Suppose that $\alpha\colon A \to Q$ is an additive map satisfying (1) $\alpha\big([aba, a]\big) \neq 0$ for some $a, b \in A$, and (2) $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. If $\alpha(A)$ is a 4-free subset of Q, then $\alpha = \gamma\varphi$ for some nonzero $\gamma \in C$ and Lie homomorphism $\varphi\colon A \to Q$.*

Proof Since for any $x, y \in A$, xyx and yxy are elements of K, we see that

$$(4.11)\qquad \alpha(xyx)\alpha(y) = \alpha(x)\alpha(yxy) \quad \text{for all } x, y \in A.$$

Linearizing (4.11) on x and y we get

$$(4.12)\qquad [\alpha(xyu) + \alpha(uyx)]\alpha(v) + [\alpha(xvu) + \alpha(uvx)]\alpha(y) = \alpha(x)[\alpha(yuv) + \alpha(vuy)] + \alpha(u)[\alpha(yxv) + \alpha(vxy)]$$

for all $x, y, u, v \in A$. By [4, Theorem 2.6] we conclude that $\alpha(xyu) + \alpha(uyx)$ is a tri-additive quasi-polynomial in α . Therefore, we can write

$$\begin{aligned}\alpha(xyx) = \lambda_1\alpha(x)\alpha(y)\alpha(x) + \lambda_2\alpha(x)^2\alpha(y) + \lambda_3\alpha(y)\alpha(x)^2 \\ + \mu_1(x)\alpha(y)\alpha(x) + \mu_2(x)\alpha(x)\alpha(y) + \mu_3(y)\alpha(x)^2 \\ + \nu_1(x, y)\alpha(x) + \nu_2(x, x)\alpha(y) + \omega(x, y, x)\end{aligned} \tag{4.13}$$

for some elements $\lambda_1, \lambda_2, \lambda_3 \in C$, additive maps $\mu_1, \mu_2, \mu_3 \colon A \to C$, bi-additive maps $\nu_1, \nu_2 \colon A^2 \to C$ and tri-additive map $\omega \colon A^3 \to C$. Using (4.13) to rewrite (4.11) and replacing x and y by $x + u$ and $y + v$ respectively, we get a 4-additive quasi-polynomial in α without constant coefficient which vanishes for all $x, y, u, v \in A$. And applying Theorem 3.4 to the resulting identity, we see that $\lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = \nu_2 = \omega = 0$ and $\nu_1(x, y) = \nu_1(y, x)$ for all $x, y \in A$. Thus (4.13) becomes

$$\alpha(xyx) = \lambda\alpha(x)\alpha(y)\alpha(x) + \nu(x, y)\alpha(x), \tag{4.14}$$

and linearization on x yields

$$\begin{aligned}\alpha(xyz + zyx) = \lambda\alpha(x)\alpha(y)\alpha(z) + \lambda\alpha(z)\alpha(y)\alpha(x) \\ + \nu(x, y)\alpha(z) + \nu(z, y)\alpha(x),\end{aligned} \tag{4.15}$$

where $\lambda = \lambda_1$ and $\nu = \nu_1$. Since

$$[x, y]yx + xy[x, y] + [y, x]xy + yx[y, x] = [x, y](yx - xy) + (xy - yx)[x, y] = 0,$$

we see that

$$\begin{aligned}\lambda\alpha([x, y])\alpha(y)\alpha(x) + \lambda\alpha(x)\alpha(y)\alpha([x, y]) + \lambda\alpha([y, x])\alpha(x)\alpha(y) \\ + \lambda\alpha(y)\alpha(x)\alpha([y, x]) + \nu([x, y], y)\alpha(x) + \nu([y, x], x)\alpha(y) = 0,\end{aligned} \tag{4.16}$$

or equivalently,

$$\lambda\big[[\alpha(x), \alpha(y)], \alpha([x, y])\big] + \nu([x, y], y)\alpha(x) + \nu([y, x], x)\alpha(y) = 0, \tag{4.17}$$

because $\nu(x, y)\alpha([x, y]) + \nu(y, x)\alpha([y, x]) = 0$.

Suppose $\lambda = 0$; then (4.17) reduces to $\nu([x, y], y)\alpha(x) + \nu([y, x], x)\alpha(y) = 0$. Linearizing on x, y we have

$$\begin{aligned}\nu([x, y], v)\alpha(u) + \nu([u, y], v)\alpha(x) + \nu([x, v], y)\alpha(u) \\ + \nu([u, v], y)\alpha(x)\nu([y, x], u)\alpha(v) + \nu([y, u], x)\alpha(v) \\ + \nu([v, x], u)\alpha(y) + \nu([v, u], x)\alpha(y) = 0.\end{aligned} \tag{4.18}$$

Then, by Theorem 3.4, we have $\nu([y, x], u) + \nu([y, u], x) = 0$ for all $x, y, u \in A$ and so $\nu([y, x], x) = 0$ for all $x, y \in A$. We infer from (4.14) that $\alpha([xyx, x]) =$

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$\alpha(x[y,x]x) = \nu(x,[y,x])\alpha(x) = \nu([y,x],x)\alpha(x) = 0$ for all $x, y \in A$, contradicting our hypothesis (1).

Suppose $\lambda \neq 0$. Then, with [4, Theorem 1.2], we conclude from the linearized form of (4.17) that $\alpha([x,y])$ is a bi-additive quasi-polynomial in α, that is,

$$\alpha([x,y]) = \gamma_1\alpha(x)\alpha(y) + \gamma_2\alpha(y)\alpha(x) + \eta_1(x)\alpha(y) + \eta_2(y)\alpha(x) + \tau(x,y)$$

for some elements $\gamma_1, \gamma_2 \in C$, additive maps $\eta_1, \eta_2 \colon A \to C$ and bi-additive map $\tau \colon A^2 \to C$. Using Theorem 3.4 and the fact $\alpha([x,y]) + \alpha([y,x]) = 0$, we conclude that $\gamma_2 = -\gamma_1$, $\eta_2 = -\eta_1$ and $\tau(x,y) = -\tau(y,x)$ for all $x, y \in A$. Therefore,

$$\alpha([x,y]) = \gamma_1[\alpha(x),\alpha(y)] + \eta_1(x)\alpha(y) - \eta_1(y)\alpha(x) + \tau(x,y). \tag{4.19}$$

Substituting this expression into the linearized form of (4.17), we have, by Theorem 3.4 again, $\eta_1 = 0$. Note that

$$\begin{aligned}\alpha([xyz+zyx,u]) = {}&\alpha([x,u]yz + zy[x,u])\\ &+ \alpha(x[y,u]z + z[y,u]x) + \alpha(xy[z,u] + [z,u]yx).\end{aligned}$$

Using (4.15) and (4.19), we can derive from this identity that $\tau = 0$, and so $\alpha([x,y]) = \gamma_1[\alpha(x),\alpha(y)]$ for all $x, y \in A$ where $\gamma_1 \neq 0$. By setting $\varphi = \gamma_1\alpha$, which is certainly a Lie homomorphism, we have $\alpha = \gamma\varphi$ where $\gamma = \gamma_1^{-1}$, completing the proof. ∎

Corollary 4.6 *Let Q be an associative ring with center C a field of characteristic not 2, R a prime ring with involution that does not satisfy the standard identity $s_8(x_1,\dots,x_8)$ of degree 8, and A the Lie ring of all skew elements of R. Suppose that $\alpha \colon A \to Q$ is an injective additive map with $\alpha(x)\alpha(y) = \alpha(u)\alpha(v)$ for all $x, y, u, v \in A$ with $xy = uv$. If $\alpha(A)$ is a 4-free subset of Q, then $\alpha = \zeta\varphi$ for some nonzero $\zeta \in C$ and Lie homomorphism $\varphi \colon A \to Q$.*

Proof In view of Theorem 4.5, it suffices to show that $[aba,a] \neq 0$ for some $a, b \in A$ since α is injective. Assume on the contrary that $[xyx,x] = xyx^2 - x^2yx = 0$ for all $x, y \in A$. Then the set A of all skew elements in R satisfies a polynomial identity, namely $p(x,y) = xyx^2 - x^2yx$, of degree 4 and so R satisfies the standard identity $s_8(x_1,\dots,x_8)$ of degree 8 by [17, Theorem 6.5.2], contradicting our hypothesis. This completes the proof. ∎

5 Local Automorphisms on Division Algebras

Let A be an algebra over a field F. Following Larson and Sourour [22] we say that a linear map $f \colon A \to A$ is a *local automorphism* if for every $a \in A$, there is an F-automorphism $\theta_a \colon A \to A$, depending on a, such that $f(a) = \theta_a(a)$. Local automorphisms were studied by many authors [9, 10, 12, 18, 21, 22, 25, 26, 31, 32] for different classes of operator algebras.

It would be interesting to consider the following problem.

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Problem 5.1 Let K be a central division algebra and $f\colon K \to K$ a local automorphism. Is it true that f is either an automorphism or an antiautomorphism?

Note that if we could prove that $f(k^{-1}) = f(k)^{-1}$ for all nonzero $k \in K$, then the result would follow from Theorem 2.1. Another approach could be to prove that $[f(k^2), f(k)] = 0$ for all $k \in K$, and then the result would follow from a theorem due to Brešar [7].

On the other hand, it is true that in the case of quaternions, all local automorphisms are automorphisms or antiautomorphisms.

Proposition 5.2 *Let $\mathbb{H}$ be the quaternion algebra over the field $\mathbb{R}$ of real numbers. Then any local automorphism of $\mathbb{H}$ is either an automorphism or an antiautomorphism.*

Proof Let $f\colon \mathbb{H} \to \mathbb{H}$ be a local automorphism. It is easy to see that f is injective and hence is bijective by its linearity since $\mathbb{H}$ is finite-dimensional over $\mathbb{R}$. In view of a classical theorem due to Ancochea [1], we need to show that $f(x^2) = f(x)^2$ for all $x \in \mathbb{H}$. First $f(1) = \theta_1(1) = 1$ for some automorphism θ_1 on $\mathbb{H}$. Hence $f(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$ by the linearity of f. Let $x \in \mathbb{H}$. Then $x^2 = a + bx$ for some a and b in $\mathbb{R}$, and so $f(x^2) = a + bf(x)$. On the other hand, we have $f(x) = \theta_x(x)$ for some automorphism θ_x on $\mathbb{H}$. By the Noether-Skolem theorem [16, Theorem 4.3.1] there exists a nonzero element $y \in \mathbb{H}$ such that $\theta_x(x) = yxy^{-1}$. Thus $f(x)^2 = \theta_x(x)^2 = (yxy^{-1})^2 = yx^2y^{-1} = y(a+bx)y^{-1} = a+byxy^{-1} = a+bf(x) = f(x^2)$ as required. ■

In [30] Šemrl introduced the concept of 2-local automorphisms. These are the (not necessarily additive) mappings $f\colon A \to A$ such that for every $a, b \in A$ there is an F-automorphism $\theta_{a,b}\colon A \to A$, depending on a and b, such that $f(a) = \theta_{a,b}(a)$ and $f(b) = \theta_{a,b}(b)$. These 2-local automorphisms were studied for different operator algebras [25, 26, 27]. Using ideas from Molnar's paper [26], we shall describe 2-local automorphisms of finite-dimensional division algebras.

Theorem 5.3 *Let K be a finite-dimensional division algebra over its center Z. Suppose that the characteristic of K is zero. Then every 2-local automorphism is an automorphism or an antiautomorphism.*

Proof Let $f\colon K \to K$ be a 2-local automorphism and $\mathrm{tr}\colon K \to Z$ a reduced trace of K. We first assert that

$$\mathrm{tr}(f(x)f(y)) = \mathrm{tr}(xy) \quad \text{for all } x, y \in K. \tag{5.1}$$

For any $x, y \in K$ there exists a Z-automorphism $\theta_{x,y}$ on K such that $f(x) = \theta_{x,y}(x)$ and $f(y) = \theta_{x,y}(y)$. By [16, Theorem 4.3.1], there exists a nonzero $c \in K$ such that $\theta_{x,y}(x) = cxc^{-1}$ and $\theta_{x,y}(y) = cyc^{-1}$. Thus

$$\mathrm{tr}(f(x)f(y)) = \mathrm{tr}(\theta_{x,y}(x)\theta_{x,y}(y)) = \mathrm{tr}(cxc^{-1}cyc^{-1}) = \mathrm{tr}(cxyc^{-1}) = \mathrm{tr}(xy).$$

Let $\{k_1, k_2, \dots, k_{n^2}\}$ be a basis of K over Z. We claim that $f(k_1)$, $f(k_2), \dots, f(k_{n^2})$ are linearly independent over Z. Assume otherwise that there exist $\lambda_1, \lambda_2, \dots \lambda_{n^2} \in Z$, not all zero, such that $\sum_{i=1}^{n^2} \lambda_i f(k_i) = 0$. Let $\sum_{j=1}^{n^2} \mu_j k_j$ be the inverse of

$$\sum_{i=1}^{n^2} \lambda_i k_i \neq 0.$$

It follows from (5.1) that

$$\begin{aligned}
0 &= \operatorname{tr}\Big(\Big[\sum_{i=1}^{n^2} \lambda_i f(k_i)\Big]\Big[\sum_{j=1}^{n^2} \mu_j f(k_j)\Big]\Big) = \sum_{i=1}^{n^2}\sum_{j=1}^{n^2} \lambda_i \mu_j \operatorname{tr}\big(f(k_i) f(k_j)\big) \\
&= \sum_{i=1}^{n^2}\sum_{j=1}^{n^2} \lambda_i \mu_j \operatorname{tr}(k_i \cdot k_j) = \operatorname{tr}\Big(\Big[\sum_{i=1}^{n^2} \lambda_i k_i\Big]\Big[\sum_{j=1}^{n^2} \mu_j k_j\Big]\Big) \\
&= \operatorname{tr}(1),
\end{aligned}$$

a contradiction. Therefore $f(k_1), f(k_2), \dots, f(k_{n^2})$ are linearly independent over Z and hence span K over Z.

Now we can prove the linearity of f over Z. For any $u, v \in K$ and for each $i \in \{1, 2, \dots, n^2\}$, we have from (5.1) that

$$\begin{aligned}
\operatorname{tr}\big(f(u+v) f(k_i)\big) &= \operatorname{tr}\big((u+v)k_i\big) = \operatorname{tr}(uk_i) + \operatorname{tr}(vk_i) = \\
&\operatorname{tr}\big(f(u) f(k_i)\big) + \operatorname{tr}(f(v) f(k_i)) = \operatorname{tr}\big(f(u) f(k_i) + f(v) f(k_i)\big).
\end{aligned}$$

Since the $f(k_i)$'s span K over Z, by the linearity of the reduced trace, we have

$$\operatorname{tr}\big(f(u+v)x\big) = \operatorname{tr}\big(f(u)x + f(v)x\big) \quad \text{for all } x, u, v \in K.$$

Equivalently, we have

$$\operatorname{tr}\big(\big(f(u+v) - f(u) - f(v)\big)x\big) = 0 \quad \text{for all } x, u, v \in K. \tag{5.2}$$

Assume that $f(u + v) - f(u) - f(v) = y \neq 0$ for some $u, v \in K$. Setting $x = y^{-1}$, we obtain from (5.2) that $0 = \operatorname{tr}(yx) = \operatorname{tr}(1)$, a contradiction. Therefore, $f(u + v) = f(u) + f(v)$ for all $u, v \in K$. In a similar way, we can show that $\operatorname{tr}\big(\big(f(\alpha u) - \alpha f(u)\big)x\big) = 0$ for all $\alpha \in Z$ and $u, x \in K$ and accordingly $f(\alpha u) = \alpha f(u)$ for all $\alpha \in Z$ and $u \in K$. Thus f is a linear map on K over Z. Being a 2-local automorphism, f is injective and hence is surjective since K is finite-dimensional over Z.

Finally, for each $u \in K$ there exists an automorphism θ_{u,u^2} such that $f(u) = \theta_{u,u^2}(u)$ and $f(u^2) = \theta_{u,u^2}(u^2)$. Then $f(u^2) = \theta_{u,u^2}(u^2) = \theta_{u,u^2}(u)^2 = f(u)^2$ for all $u \in K$, and so f is indeed an automorphism or an antiautomorphism by Ancochea's theorem [1]. ■

For all cases considered in [25, 26, 27, 30], 2-local automorphisms are just automorphisms. This is not true in general since for finite-dimensional division algebras some antiautomorphisms can also be 2-local automorphisms.

Example 5.4 Let $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\boldsymbol{i} \oplus \mathbb{R}\boldsymbol{j} \oplus \mathbb{R}\boldsymbol{k}$ be the quaternion algebra over the field $\mathbb{R}$ of real numbers. Then the conjugation $\alpha\colon a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$ is a 2-local automorphism of $\mathbb{H}$.

Proof Let $x = a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k}$ and $y = a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k}$ be two arbitrary elements of the algebra $\mathbb{H}$. We shall find a nonzero element $c = u\boldsymbol{i}+v\boldsymbol{j}+w\boldsymbol{k} \in \mathbb{H}$ such that $cxc^{-1} = \alpha(x)$ and $cyc^{-1} = \alpha(y)$, or equivalently, $cx = \alpha(x)c$ and $cy = \alpha(y)c$. From the two equations

$$(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k})(a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{j} + d_1\boldsymbol{k}) = (a_1 - b_1\boldsymbol{i} - c_1\boldsymbol{j} - d_1\boldsymbol{k})(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}),$$
$$(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k})(a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{j} + d_2\boldsymbol{k}) = (a_2 - b_2\boldsymbol{i} - c_2\boldsymbol{j} - d_2\boldsymbol{k})(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}),$$

comparing the coefficients, we obtain

$$\begin{cases} b_1u + c_1v + d_1w = 0, \\ b_2u + c_2v + d_2w = 0. \end{cases}$$

This system always has a nonzero solution (u, v, w), and so α is indeed a 2-local automorphism. ■

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