

HOLOMORPHIC FUNCTIONS WITH IMAGE OF GIVEN LOGARITHMIC OR ELLIPTIC CAPACITY

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Abstract

For holomorphic functions f in the unit disk \mathbb{D} with $f(0) = 0$, we prove a modulus growth bound involving the logarithmic capacity (transfinite diameter) of the image. We show that the pertinent extremal functions map the unit disk conformally onto the interior of an ellipse. We prove a modulus growth bound for elliptically schlicht functions in terms of the elliptic capacity $d_e f(\mathbb{D})$ of the image. We also show that the function $d_e f(r\mathbb{D})/r$ is increasing for $0 < r < 1$.

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1. Introduction

A classical theme in geometric function theory is the study of geometric or analytic properties of a holomorphic function under geometric conditions on the image of the function. The prototype for this theme is the classical lemma of Schwarz: let f be holomorphic in the unit disk \mathbb{D} with $f(0) = 0$. The geometric assumption is that the image $f(\mathbb{D})$ lies in the unit disk. The conclusions are the inequality $|f'(0)| \leq 1$ and the modulus growth bound $|f(z)| \leq |z|$. Several other conditions on $f(\mathbb{D})$ have been studied. We mention some of them here. Landau and Toeplitz considered the diameter condition $\text{Diam} f(\mathbb{D}) = 2$ and proved that $|f'(0)| \leq 1$ and $\text{Diam} f(r\mathbb{D}) \leq 2r$, $0 < r < 1$; here and below $r\mathbb{D} = \{|z| < r\}$. Burckel *et al.* [3] strengthened this result by showing that for a function f holomorphic in \mathbb{D} , the function

$$\Phi_{\text{Diam}}(r) = \frac{\text{Diam} f(r\mathbb{D})}{2r}, \quad 0 < r < 1,$$

is increasing. Moreover, they proved a related modulus growth bound: if f is holomorphic in \mathbb{D} , $\text{Diam}f(\mathbb{D}) = 2$, and $f(0) = 0$, then

$$|f(z)| \leq \frac{2|z|}{1 + \sqrt{1 - |z|^2}}, \quad z \in \mathbb{D}.$$

Another assumption considered in [3] is that the transfinite diameter $d(f(\mathbb{D}))$ of $f(\mathbb{D})$ is given. It is a remarkable theorem of Szegő [14, p. 153] that the transfinite diameter (a Euclidean geometric quantity) of an infinite planar set is equal to its logarithmic capacity (a potential theoretic quantity). We refer to [7, 8, 14] for the basic properties of this quantity.

It follows from a theorem of Pólya (see, for example, [14, p. 141]) that if f is holomorphic in \mathbb{D} , then $|f'(0)| \leq d(f(\mathbb{D}))$. Moreover [3], the function

$$\Phi_d(r) = \frac{d(f(r\mathbb{D}))}{r}, \quad 0 < r < 1, \quad (1.1)$$

is increasing; it follows that

$$d(f(r\mathbb{D})) \leq r d(f(\mathbb{D})), \quad 0 < r < 1.$$

We will prove a related modulus growth bound involving $d(f(\mathbb{D}))$.

THEOREM 1.1. *Let f be a nonconstant bounded holomorphic function in \mathbb{D} with $f(0) = 0$. Then*

$$|f(z)| \leq \frac{4 d(f(\mathbb{D}))}{e^{\mu(|z|)}}, \quad z \in \mathbb{D}. \quad (1.2)$$

Equality holds in (1.2) for some $z \in \mathbb{D} \setminus \{0\}$ if and only if f maps \mathbb{D} conformally onto a domain bounded by an ellipse so that the points 0 and z are mapped onto the foci of the ellipse.

The function μ appearing in (1.2) is a well-studied special function related to the Grötzsch ring capacity. It is defined by

$$\mu(r) = \frac{\pi \mathcal{K}'(r)}{2 \mathcal{K}(r)}, \quad 0 < r < 1,$$

where $\mathcal{K}, \mathcal{K}'$ are the complete elliptic integrals of the first kind:

$$\mathcal{K}(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}, \quad \mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2}), \quad 0 < r < 1.$$

A good source for elliptic integrals and the function μ is [1]. We mention here only that μ is strictly decreasing and maps the interval $(0, 1)$ onto $(0, \infty)$.

Modulus growth bounds for univalent holomorphic functions f with image of given logarithmic capacity and with $f'(0) - 1 = f(0) = 0$ have been studied in [4]. However,

the condition $f'(0) = 1$ makes the problem more complicated and gives a different flavor to that work.

Several monotonicity results for functions similar to (1.1) have recently been proved; the interested reader should look at the papers that cite the influential article [3]. Here we will prove a result that involves elliptic capacity. To introduce this notion, we need some definitions; see [6].

The *antipodal point* of the point $a \in \mathbb{C} \setminus \{0\}$ is the point $a^* = -1/\bar{a}$. The points 0 and ∞ are also antipodal. Two antipodal points in the extended complex plane $\widehat{\mathbb{C}}$ are stereographically projected onto antipodal points of the Riemann sphere. Given a set $E \subset \widehat{\mathbb{C}}$, we define its *antipodal set* (or *elliptic reflection*) $E^* = \{a^* : a \in E\}$. We call a set E *elliptically schlicht* if $E \cap E^* = \emptyset$. At the other extreme E is said to be *diametrically symmetric* if $E = E^*$. A mapping is *diametrically symmetric* if it is defined on a diametrically symmetric set and $f(z^*) = f(z)^*$. A mapping is *elliptically schlicht* if its image is elliptically schlicht. The class of elliptically schlicht conformal mappings was introduced by H. Grunsky. We refer to [6, 10, 12, 16] and references therein for various methods and results related to this class.

Let $E \subset \widehat{\mathbb{C}}$ be a closed elliptically schlicht set which contains infinitely many points. The elliptic transfinite diameter of E is denoted by $d_e(E)$ and is defined in the same way as the usual transfinite diameter by replacing Euclidean distances by elliptic distances. The elliptic distance of $a, b \in \widehat{\mathbb{C}}$ is

$$[a, b]_e = \frac{|a - b|}{|1 + \bar{a}b|}, \quad [a, \infty]_e = \frac{1}{|a|}.$$

The requirement that E is elliptically schlicht ensures that all the elliptic distances between points of E are finite. The elliptic transfinite diameter of an elliptically schlicht set in $\widehat{\mathbb{C}}$ is, by definition, the elliptic transfinite diameter of its closure. It can be proved that the elliptic transfinite diameter is equal to the elliptic capacity which is defined by minimal elliptic energy considerations; see [6]. We will, in fact, use a third equivalent definition which involves extremal length or condenser capacity; the required results will be reviewed in Section 3.

THEOREM 1.2. *Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an elliptically schlicht holomorphic function.*

(a) *The function*

$$\Phi_e(r) = \frac{d_e(f(r\mathbb{D}))}{r}, \quad 0 < r < 1,$$

is increasing. Moreover, it is strictly increasing unless

$$f(z) = \frac{\lambda z + a}{1 - \bar{a}\lambda z} \tag{1.3}$$

for some constants $\lambda \in \overline{\mathbb{D}}$ and $a \in \mathbb{C}$. If f has this form then Φ_e is a constant function.

(b) For $0 < r < 1$,

$$d_e(f(r\mathbb{D})) \leq r d_e(f(\mathbb{D})), \quad 0 < r < 1. \tag{1.4}$$

Equality holds for some r if and only if f is of the form (1.3).

(c) For every $z \in \mathbb{D}$,

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{d_e(f(\mathbb{D}))}{1 - |z|^2}. \tag{1.5}$$

Equality holds for some $z_0 \in \mathbb{D}$ if and only if

$$f(z) = \frac{\lambda\left(\frac{z-z_0}{1-\bar{z}_0 z}\right) + w_o}{1 - \bar{w}_o \lambda\left(\frac{z-z_0}{1-\bar{z}_0 z}\right)}, \tag{1.6}$$

for some $\lambda \in \bar{\mathbb{D}}$ and some $w_o \in \mathbb{C}$.

For univalent functions, inequality (1.5) appeared in [16]. The inequality with 1 in place of $d_e(f(\mathbb{D}))$ was proved in [11].

We will also prove a modulus growth bound for elliptically schlicht functions. The bound will involve the elliptic capacity of the image; it is analogous to Theorem 1.1 with elliptic capacity in place of logarithmic capacity. We need some preparation in order to describe the extremal functions. Let $w_o \in \mathbb{C} \setminus \{0\}$. Consider the doubly connected domain Ω with complementary components $[0, w_o]$ and $[-1/\bar{w}_o, \infty] := \{s(-1/\bar{w}_o) : s \geq 1\} \cup \{\infty\}$. Note that $[-1/\bar{w}_o, \infty] = [0, w_o]^*$. The domain Ω can be mapped conformally onto the annulus $\{d_o < |\zeta| < d_o^{-1}\}$, where $d_o = d_e([0, w_o])$, $0 < d_o < 1$; see [6, Theorem 4]. The mapping function can be written explicitly in terms of elliptic functions; see [1, Ch. 6]. The preimages of the circles $\{|\zeta| = \rho\}$, ($d_o < \rho < d_o^{-1}$), are certain Jordan curves enclosing the segment $[0, w_o]$. We denote these Jordan curves by $\Gamma(\rho, w_o)$. Note that: (a) $\Gamma(1, w_o)$ is a circle and has the property that the sets $[0, w_o]$ and $[-1/\bar{w}_o, \infty]$ are symmetric with respect to this circle; (b) $d_e(\text{interior}(\Gamma(1, w_o))) = 1$; (c) for $d_o < \rho \leq 1$, the interior of $\Gamma(\rho, w_o)$ is an elliptically schlicht Jordan domain.

THEOREM 1.3. *Let f be an elliptically schlicht, nonconstant, holomorphic function in \mathbb{D} with $f(0) = 0$. Then for $z \in \mathbb{D} \setminus \{0\}$,*

$$|f(z)| \leq \left(1 - \mu^{-1} \left(\log \frac{e^{\mu(|z|)}}{d_e(f(\mathbb{D}))}\right)\right)^{-1/2} \mu^{-1} \left(\log \frac{e^{\mu(|z|)}}{d_e(f(\mathbb{D}))}\right). \tag{1.7}$$

Equality holds in (1.7) for some $z_o \in \mathbb{D} \setminus \{0\}$ if and only if f maps \mathbb{D} conformally onto the interior of $\Gamma(\rho, f(z_o))$ for some ρ with $d_e([0, f(z_o)]) < \rho \leq 1$.

Inequality (1.7) is equivalent to

$$e^{\mu(z)} \leq d_e(f(\mathbb{D})) e^{\mu(|f(z)|/\sqrt{1+|f(z)|^2})}. \tag{1.8}$$

Since $f(\mathbb{D})$ is connected, we have $d_e(f(\mathbb{D})) \leq 1$. Therefore, (1.8) and the monotonicity of μ imply that

$$|f(z)| \leq \frac{|z|}{\sqrt{1 - |z|^2}}, \quad z \in \mathbb{D}. \tag{1.9}$$

This modulus growth bound for elliptically schlicht functions has been proved by Shah [17]; see also [9, p. 125], [12]. Equality holds in (1.9) for some $z_0 \in \mathbb{D} \setminus \{0\}$ if and only if f maps \mathbb{D} conformally onto the interior of the circle $\Gamma(1, f(z_0))$; in this case f has the form

$$f(z) = e^{i\phi} \frac{\sqrt{1 - |z_0|^2} z}{1 - \bar{z}_0 z},$$

for some real ϕ .

2. Proof of Theorem 1.1

A basic tool in the proof is the capacity of condensers. A condenser is a pair (A, K) , where A is an open set in the complex plane \mathbb{C} and K is a compact subset of A . We will denote by $\text{cap}(A, K)$ the capacity of the condenser (A, K) . Note that the capacity of the condenser (A, K) is equal to the modulus of the family of curves joining K with ∂A ; see [1, p. 161]. If the domain $A \setminus K$ is regular for the Dirichlet problem, we can consider the harmonic function with boundary values 1 on ∂K and 0 on ∂D . This is the *potential function* of the condenser. If $A \setminus K$ is doubly-connected, then there exists a conformal map of $A \setminus K$ onto an annulus of the form $\{\rho < |w| < \rho^{-1}\}$. It is clear that the pre-images of the circles $\{|w| = r\}$ under this conformal map are the level lines of the potential function of the condenser. We refer to [1, 5, 8] for more information about condensers.

Another tool we will use is Steiner symmetrization. The Steiner symmetrization of an open set $A \subset \mathbb{C}$ with respect to a line ℓ is an open set $S_\ell A$, symmetric with respect to ℓ . We define it by determining the intersection of $S_\ell A$ with every line perpendicular to ℓ . Let γ be such a line. Then $\gamma \cap S_\ell A$ is an open linear segment on γ , symmetric with respect to ℓ and having length equal to $m_1(\gamma \cap A)$; here m_1 denotes the one-dimensional Lebesgue measure. If $\gamma \cap A = \emptyset$, then $\gamma \cap S_\ell A = \emptyset$.

The Steiner symmetrization $S_\ell K$ of a compact set K is defined similarly, with the difference that $\gamma \cap S_\ell K$ is a *closed* segment. Also, if $m_1(\gamma \cap K) = 0$ but $\gamma \cap K \neq \emptyset$, then, by definition, $\gamma \cap S_\ell K$ is the singleton $\ell \cap \gamma$. The set $S_\ell K$ is a compact set, symmetric with respect to the line ℓ .

Steiner symmetrization reduces the capacity of condensers:

$$\text{cap}(A, K) \geq \text{cap}(S_\ell A, S_\ell K).$$

For more information about Steiner symmetrization, we refer to [5, 8].

We start the proof of Theorem 1.1 with the proof of inequality (1.2). Let $z \in \mathbb{D}$. If $z = 0$, then the inequality is trivially true; so we assume that $z \in \mathbb{D} \setminus \{0\}$. We denote by $[0, z]$ the rectilinear segment with endpoints 0 and z . Since holomorphic functions reduce the capacity of condensers (see [13] and references therein),

$$\text{cap}(\mathbb{D}, [0, z]) \geq \text{cap}(f(\mathbb{D}), f([0, z])). \quad (2.1)$$

Let ℓ be the straight line passing through 0 and $f(z)$. Since Steiner symmetrization reduces the capacity of condensers,

$$\text{cap}(f(\mathbb{D}), f([0, z])) \geq \text{cap}(S_\ell f(\mathbb{D}), S_\ell f([0, z])). \quad (2.2)$$

The set $S_\ell f([0, z])$ is compact, connected, Steiner symmetric in ℓ , and contains the points 0 and $f(z)$. Therefore,

$$[0, f(z)] \subset S_\ell f([0, z]).$$

By a monotonicity property of condenser capacity,

$$\text{cap}(S_\ell f(\mathbb{D}), S_\ell f([0, z])) \geq \text{cap}(S_\ell f(\mathbb{D}), [0, f(z))). \quad (2.3)$$

We use here a Grötzsch-type inequality which can be found in [5]:

$$(\text{cap}(S_\ell f(\mathbb{D}), [0, f(z))))^{-1} \leq \frac{\log d(S_\ell f(\mathbb{D}))}{2\pi} - \frac{\log d([0, f(z)])}{2\pi}. \quad (2.4)$$

It is essentially a basic property of the modulus of curve families (extremal length). Steiner symmetrization also reduces the logarithmic capacity; hence

$$d(S_\ell f(\mathbb{D})) \leq d(f(\mathbb{D})).$$

Therefore

$$\frac{\log d(S_\ell f(\mathbb{D}))}{2\pi} - \frac{\log d([0, f(z)])}{2\pi} \leq \frac{\log d(f(\mathbb{D}))}{2\pi} - \frac{\log d([0, f(z)])}{2\pi}. \quad (2.5)$$

Taking into account (2.1)–(2.5), we infer that

$$\text{cap}(\mathbb{D}, [0, z]) \geq \frac{2\pi}{\log d(f(\mathbb{D})) - \log d([0, f(z)])}. \quad (2.6)$$

The capacity of the condenser $(\mathbb{D}, [0, z])$ can be computed explicitly (see [1, pp. 175, 80]):

$$\text{cap}(\mathbb{D}, [0, z]) = \frac{2\pi}{\mu(|z|)}.$$

The logarithmic capacity of a segment can also computed explicitly (see [14, p. 134]):

$$d([0, f(z)]) = \frac{|f(z)|}{4}.$$

Therefore (2.6) becomes

$$\frac{2\pi}{\mu(|z|)} \geq \frac{2\pi}{\log d(f(\mathbb{D})) - \log(|f(z)|/4)}$$

which is equivalent to (1.2).

We proceed with the proof of the equality statement. Suppose that (1.2) holds with equality for some $z \in \mathbb{D} \setminus \{0\}$. Then inequalities (2.1)–(2.6) become equalities. Equality in (2.1) implies that f is a conformal mapping; see [13] and references therein. Equality in (2.2) implies that $f(\mathbb{D})$ is a simply connected domain, Steiner symmetric in ℓ ; see [5] and references therein. Equality in (2.4) implies that the boundary of $f(\mathbb{D})$ is a level curve of the Green function for the domain $\mathbb{C}_\infty \setminus [0, f(z)]$ with pole at ∞ ; see [5]. By using a Joukowski-type conformal mapping, we see that such a level curve is an ellipse with foci at 0 and $f(z)$.

Conversely, let Ω be a domain bounded by an ellipse with foci at the points 0 and w . Let f be a conformal mapping of \mathbb{D} onto Ω with $f(0) = 0$. Set $z = f^{-1}(w)$. It is now straightforward to show that for this f and this z , we have equality in (1.2).

3. Proof of Theorem 1.2

We denote by $\lambda(A, B)$ the extremal distance between two compact disjoint sets in $\widehat{\mathbb{C}}$. We first review some known facts on the connection of elliptic capacity with extremal length and condenser capacity. Suppose that E is a compact, connected, elliptically schlicht set in \mathbb{C} . Let E^* be its elliptic reflection which is clearly connected too. Let Ω be the (unique) component of $\widehat{\mathbb{C}} \setminus (E \cup E^*)$ which borders both E and E^* . We denote by $\lambda(E, E^*)$ the extremal distance of the sets E and E^* ; that is, $\lambda(E, E^*)$ is the extremal length of the family of curves in Ω that join E with E^* . We refer to [7, Ch. 4] for the basic properties of extremal distance. It follows from the results in [2, 6] that

$$d_e(E) = \exp\{-\pi\lambda(E, E^*)\}. \quad (3.1)$$

We will need the following special symmetrization lemma.

LEMMA 3.1. *Let K be a compact, connected, elliptically schlicht set in \mathbb{C} and let A be an elliptically schlicht domain containing K . Let K^\sharp be the closed disk centered at the origin and having radius equal to $d_e(K)$. Let A^\sharp be the open disk centered at the origin and having radius equal to $d_e(A)$. Then*

$$\text{cap}(A, K) \geq \text{cap}(A^\sharp, K^\sharp) \quad (3.2)$$

with equality if and only if the boundary of A is a level curve for the potential function of the condenser $(\widehat{\mathbb{C}} \setminus K^*, K)$.

PROOF. It suffices to prove that

$$\lambda(\partial A, K) \leq \lambda(\partial A^\sharp, K^\sharp). \quad (3.3)$$

By a basic property of extremal distance [7, p. 135],

$$\lambda(K, K^*) \geq \lambda(K, \partial A) + \lambda(\partial A, \partial A^*) + \lambda(\partial A^*, K^*). \quad (3.4)$$

But $\lambda(K, \partial A) = \lambda(K^*, \partial A^*)$. Hence

$$2\lambda(\partial A, K) \leq \lambda(K, K^*) - \lambda(\partial A, \partial A^*). \quad (3.5)$$

Similarly,

$$2\lambda(\partial A^\sharp, K^\sharp) = \lambda(K^\sharp, (K^\sharp)^*) - \lambda(\partial A^\sharp, (\partial A^\sharp)^*). \tag{3.6}$$

By the definition of K^\sharp and A^\sharp ,

$$\lambda(K, K^*) = -\frac{1}{\pi} \log d_e(K) = -\frac{1}{\pi} \log d_e(K^\sharp) = \lambda(K^\sharp, (K^\sharp)^*), \tag{3.7}$$

and similarly

$$\lambda(\partial A, \partial A^*) = \lambda(\partial A^\sharp, (\partial A^\sharp)^*). \tag{3.8}$$

Now (3.3) follows from (3.5)–(3.8).

If we have equality in (3.2), then we have equality in (3.4). Therefore ([5, p. 9], [9, p. 22]), the boundary of A is a level curve for the potential function of the condenser $(\widehat{\mathbb{C}} \setminus K^*, K)$. The converse is proved similarly. \square

The proof of Theorem 1.2 will be given in five parts.

Part 1. In part 1 we will prove that the function Φ_e is increasing under the additional assumption that $f(0) = 0$.

For $0 < \rho < 1$, we denote by G_ρ the complement of the unbounded complementary component of $f(\rho\mathbb{D})$; namely, G_ρ is the simply connected domain that we obtain if we fill up the holes of $f(\rho\mathbb{D})$. We also set $L_\rho = \overline{G_\rho}$ (closure in $\widehat{\mathbb{C}}$) and

$$d_\rho = d_e(\overline{f(\rho\mathbb{D})}) = d_e(L_\rho).$$

Consider the doubly connected domain Ω_ρ with complementary components L_ρ and L_ρ^* . Clearly, Ω_ρ is diametrically symmetric.

Let $0 < r < s < 1$. Since holomorphic functions reduce the capacity of condensers (see [13] and references therein),

$$\text{cap}(s\mathbb{D}, \overline{r\mathbb{D}}) \geq \text{cap}(f(s\mathbb{D}), \overline{f(r\mathbb{D})}). \tag{3.9}$$

Since f is continuous, $\overline{f(r\mathbb{D})} \supset \overline{f(r\mathbb{D})}$. Also $f(s\mathbb{D}) \subset G_s$. Therefore, by the domain monotonicity of condenser capacity,

$$\text{cap}(f(s\mathbb{D}), \overline{f(r\mathbb{D})}) \geq \text{cap}(G_s, \overline{f(r\mathbb{D})}). \tag{3.10}$$

By the definition of condenser capacity (which involves the Dirichlet integral),

$$\text{cap}(G_s, \overline{f(r\mathbb{D})}) \geq \text{cap}(G_s, L_r). \tag{3.11}$$

Note that $0 < d_r < 1$ because L_r is connected; see [6, p. 320]. The domain Ω_r is diametrically symmetric. Hence [6, Theorem 4] there exists a diametrically symmetric conformal mapping g of Ω_r onto the annulus $\{d_r < |w| < d_r^{-1}\}$ so that ∂L_r corresponds to the circle $\{|w| = d_r\}$ and ∂L_r^* corresponds to the circle $\{|w| = d_r^{-1}\}$. By the conformal invariance of condenser capacity,

$$\text{cap}(G_s, L_r) = \text{cap}(g(G_s \setminus L_r) \cup \overline{d_r\mathbb{D}}, \overline{d_r\mathbb{D}}). \tag{3.12}$$

Let

$$d_s^\# := d_e(g(G_s \setminus L_r) \cup \overline{d_r\mathbb{D}}).$$

By Lemma 3.1,

$$\text{cap}(g(G_s \setminus L_r) \cup \overline{d_r\mathbb{D}}, \overline{d_r\mathbb{D}}) \geq \text{cap}(d_s^\#\mathbb{D}, \overline{d_r\mathbb{D}}). \tag{3.13}$$

Since g is diametrically symmetric on the doubly connected domain Ω_r and $\Omega_s \subset \Omega_r$, we have (see [6, p. 323])

$$d_s = d_e(L_s) = d_e(g(G_s \setminus L_r) \cup \overline{d_r\mathbb{D}}) = d_s^\#. \tag{3.14}$$

Inequalities (3.9)–(3.14) imply that

$$\text{cap}(s\mathbb{D}, \overline{r\mathbb{D}}) \geq \text{cap}(d_s\mathbb{D}, \overline{d_r\mathbb{D}}). \tag{3.15}$$

For annular condensers,

$$\text{cap}(\rho_1\mathbb{D}, \overline{\rho_2\mathbb{D}}) = \left(2\pi \log \frac{\rho_1}{\rho_2}\right)^{-1}.$$

Therefore (3.15) gives

$$\frac{s}{r} \leq \frac{d_s}{d_r}$$

which is equivalent to $\Phi_e(r) \leq \Phi_e(s)$. Thus the function Φ_e is increasing.

Part 2. In this part we continue to assume that $f(0) = 0$ and prove that the function Φ_e is strictly increasing unless $f(z) = \lambda z$ for some constant $\lambda \in \overline{\mathbb{D}}$, in which case Φ_e is a constant function.

Suppose that $\Phi_e(r) = \Phi_e(s)$ for some $0 < r < s < 1$. Then we have equality in (3.9) and therefore (see [13]) and references therein) f is univalent in $s\mathbb{D}$. We also have equality in (3.13) which comes from Lemma 3.1. By the equality statement of Lemma 3.1,

$$g(\Omega_s) \cup \overline{d_r\mathbb{D}} = d_s^\#\mathbb{D} = d_s\mathbb{D}.$$

Hence the function $g \circ f$ maps the annulus $\{r < |z| < s\}$ onto the annulus $\{d_r < |w| < d_s\}$ and each circle $\{|z| = \rho\}$, $r < \rho < s$, onto a circle centered at the origin. By the annulus theorem [15, Ch. 9],

$$g \circ f(z) = \frac{d_s}{s}z, \quad r < |z| < s,$$

or equivalently,

$$g^{-1}(w) = f\left(\frac{s}{d_s}w\right), \quad d_r < |w| < d_s. \tag{3.16}$$

Recall that, by its definition, g^{-1} is a univalent function in the annulus $\{d_r < |w| < d_r^{-1}\}$. Since f is defined and holomorphic in \mathbb{D} , equality (3.16) extends g^{-1} to a holomorphic function in the disk $\{|w| < d_r^{-1}\}$. By the argument principle the extended g^{-1} remains univalent. We further extend g^{-1} on $\{|w| \geq d_r^{-1}\}$ by setting

$$g^{-1}(w) = (g^{-1}(w^*))^*, \quad |w| \geq d_r^{-1}.$$

We thus obtain a univalent function $g^{-1} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with $g^{-1}(0) = 0, g^{-1}(\infty) = \infty$. It follows that $g^{-1}(w) = bw$ for some complex constant b . By (3.16),

$$f(z) = g^{-1}\left(\frac{d_s}{s}z\right) = b\frac{d_s}{s}z, \quad z \in \mathbb{D}. \tag{3.17}$$

Therefore,

$$d_r = d_e(f(r\mathbb{D})) = d_e\left(\frac{|b|d_s r}{s}\mathbb{D}\right) = \frac{|b|d_s r}{s} = \frac{|b|d_r r}{r} = |b|d_r.$$

Hence $|b| = 1$. Note also that $d_s \leq s$ because (by the monotonicity of Φ_e proved in part 1),

$$\frac{d_s}{s} \leq \lim_{\rho \rightarrow 1} \frac{d_e(f(\rho\mathbb{D}))}{\rho} \leq \lim_{\rho \rightarrow 1} \frac{1}{\rho} = 1.$$

We set $\lambda = bd_s/s$ and infer from (3.17) that $f(z) = \lambda z$ with $|\lambda| \leq 1$.

Part 3. In this part we prove part (a) of the theorem in its general form. Set $a := f(0) \in \mathbb{C}$ and consider the linear fractional transformation

$$T(z) = \frac{z - a}{1 + \bar{a}z}.$$

This is a diametrically symmetric function and an elliptic isometry [16]. The function $T \circ f$ is holomorphic in \mathbb{D} and $T \circ f(0) = 0$. It is also elliptically schlicht as the composition of a diametrically symmetric and an elliptically schlicht function. So we can apply part 1 of this proof and conclude that the function

$$\Phi_e^T(r) = \frac{d_e(T \circ f(r\mathbb{D}))}{r}, \quad 0 < r < 1,$$

is increasing. But T , as elliptic isometry, preserves the elliptic transfinite diameter. Hence $\Phi_e^T = \Phi_e$ and so Φ_e is increasing.

Suppose that $\Phi_e(r) = \Phi_e(s)$ for some $0 < r < s < 1$. Then, by part 2 of this proof, $T \circ f(z) = \lambda z$ for some constant $\lambda \in \overline{\mathbb{D}}$. Hence

$$f(z) = T^{-1}(\lambda z) = \frac{\lambda z + a}{1 - \bar{a}\lambda z}.$$

Conversely, if f has this form, then it follows easily that $\Phi_e \equiv |\lambda|$.

Part 4. We prove part (b) of the theorem. Fix $0 < r < 1$. By (a) which we have already proved,

$$\frac{d_e(f(r\mathbb{D}))}{r} \leq \lim_{\rho \rightarrow 1^-} \frac{d_e(f(\rho\mathbb{D}))}{\rho} \leq d_e(f(\mathbb{D}))$$

and this implies (1.4). The equality statement in (b) follows from the strict monotonicity statement in (a).

Part 5. We finally prove (c). Assume first that $f(0) = 0$. For $0 < r < 1$, let

$$m(r) := \min\{|f(z)| : |z| = r\}$$

and let z_r be a point with $|z_r| = r$ for which the minimum is attained. The disk $\Delta_r := \{|w| < m(r)\}$ lies in $f(r\mathbb{D})$. This fact and (b) yield

$$d_e(f(\mathbb{D})) \geq \frac{d_e(f(r\mathbb{D}))}{r} \geq \frac{d_e(\Delta_r)}{r} = \frac{m(r)}{r} = \frac{|f(z_r)|}{r}. \tag{3.18}$$

But

$$\lim_{r \rightarrow 0} \frac{|f(z_r)|}{r} = |f'(0)|$$

and therefore

$$|f'(0)| \leq d_e(f(\mathbb{D})).$$

If this holds with equality, then (3.18) implies that $\lim_{r \rightarrow 0} d_e(f(r\mathbb{D}))/r = d_e(f(\mathbb{D}))$ and therefore the function Φ_e is constant. Because of (a), we infer that $f(z) = \lambda z$ for some $\lambda \in \mathbb{D}$.

We now remove the assumption $f(0) = 0$ and prove (c) in general. Fix $z_o \in \mathbb{D}$ and set $w_o = f(z_o)$. Consider the linear fractional transformations

$$T_1(z) = \frac{z + z_o}{1 + \overline{z_o}z}, \quad T_2(w) = \frac{w - w_o}{1 + \overline{w_o}w}.$$

Set $g = T_2 \circ f \circ T_1$ and note that g is an elliptically schlicht function with $g(0) = 0$. By what we have proved so far,

$$|g'(0)| \leq d_e(g(\mathbb{D})) = d_e(f(\mathbb{D}))$$

and this is equivalent to (1.5). Equality holds in (1.5) if and only if $g(z) = \lambda z$, or, equivalently $f = T_2^{-1} \circ (\lambda z) \circ T_1^{-1}$; that is, when f has the form (1.6).

4. Proof of Theorem 1.3

Let $z \in \mathbb{D} \setminus \{0\}$. Since holomorphic functions decrease the capacity of condensers,

$$\text{cap}(\mathbb{D}, [0, z]) \geq \text{cap}(f(\mathbb{D}), f([0, z])). \tag{4.1}$$

Let α be the ray emanating from the origin and passing through the point $f(z)$. Let S^α denote circular symmetrization with respect to this ray; see [5, 8] for the necessary definitions and basic results about circular symmetrization. Since circular symmetrization reduces the capacity of condensers,

$$\begin{aligned} \text{cap}(f(\mathbb{D}), f([0, z])) &\geq \text{cap}(S^\alpha f(\mathbb{D}), S^\alpha f([0, z])) \\ &\geq \text{cap}(S^\alpha f(\mathbb{D}), [0, f(z)]). \end{aligned} \tag{4.2}$$

Set $d = d_e(f(\mathbb{D}))$ and $t = d_e([0, f(z)])$. Circular symmetrization with respect to a ray emanating from the origin reduces elliptic capacity; see [2, Theorem 5]. Hence

$$d_e(S^\alpha f(\mathbb{D})) \leq d_e(f(\mathbb{D})) = d.$$

Therefore the domain monotonicity of condenser capacity and Lemma 3.1 yield

$$\text{cap}(S^\alpha f(\mathbb{D}), [0, f(z)]) \geq \text{cap}(d\mathbb{D}, \overline{t\mathbb{D}}). \tag{4.3}$$

By (4.1)–(4.3),

$$\text{cap}(\mathbb{D}, [0, z]) \geq \text{cap}(d\mathbb{D}, \overline{t\mathbb{D}}). \tag{4.4}$$

The elliptic capacity of a rectilinear segment can be computed explicitly via formula (3.1) and formulae for Teichmüller’s ring domain [1, pp. 87, 167]:

$$\begin{aligned} t &= d_e([0, f(z)]) = \exp\{-\pi\lambda([0, |f(z)|], (-\infty, -|f(z)|])\} \\ &= \exp\left\{-\mu\left(\frac{|f(z)|}{\sqrt{1 + |f(z)|}}\right)\right\}. \end{aligned}$$

Thus (4.4) gives

$$\frac{2\pi}{\mu(|z|)} \geq \frac{2\pi}{\log(d/t)} = \frac{2\pi}{\log\{d_e(f(\mathbb{D})) \exp(\mu(|f(z)|/\sqrt{1 + |f(z)|^2}))\}}$$

which implies that

$$\log\left(\frac{e^{\mu(|z|)}}{d_e(f(\mathbb{D}))}\right) \leq \mu\left(\frac{|f(z)|}{\sqrt{1 + |f(z)|^2}}\right). \tag{4.5}$$

Elementary calculations show that (4.5) is equivalent to (1.7).

Suppose now that (1.7) holds with equality for some $z_o \in \mathbb{D} \setminus \{0\}$. Then for this z_o , inequalities (4.1)–(4.3) become equalities. Equality in (4.1) implies that f is univalent (see [13]). Equality in (4.2) implies that $f(\mathbb{D})$ is circularly symmetric with respect to the ray α ; see [5] and references therein. Equality in (4.3) implies (see the equality statement in Lemma 3.1) that the boundary of $f(\mathbb{D})$ is a level curve of the potential function of the condenser

$$(\widehat{\mathbb{C}} \setminus [-1/\overline{f(z_o)}, \infty], [0, f(z_o)]).$$

Therefore the boundary of $f(\mathbb{D})$ is one of the curves $\Gamma(\rho, f(z_o))$ with

$$d_e([0, f(z_o)]) < \rho \leq 1;$$

note that for $\rho > 1$, the interior of $\Gamma(\rho, f(z_o))$ is not elliptically schlicht.

Conversely, let $w_o \in \mathbb{C} \setminus \{0\}$ and consider the Jordan curve $\Gamma(\rho, w_o)$ for some ρ with $d_e([0, w_o]) < \rho \leq 1$. Let f be a conformal mapping of \mathbb{D} onto the interior of $\Gamma(\rho, w_o)$ with $f(0) = 0$. Let $z_o = f^{-1}(w_o)$. Then, by looking at the proof of (1.7) above, it is straightforward to show that (1.7) holds with equality.

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