COMPOSITIO MATHEMATICA

Hilbert–Mumford criterion for nodal curves

Jun Li and Xiaowei Wang

Compositio Math. **151** (2015), 2076–2130.

doi:10.1112/S0010437X1500737X

Compositio Math. 151 (2015) 2076–2130
doi:10.1112/S0010437X1500737X

Hilbert–Mumford criterion for nodal curves

Jun Li and Xiaowei Wang

Abstract

We prove by the Hilbert–Mumford criterion that a slope stable polarized weighted pointed nodal curve is Chow asymptotic stable. This generalizes the result of Caporaso on stability of polarized nodal curves and of Hassett on weighted pointed stable curves polarized by the weighted dualizing sheaves. It also solves a question raised by Mumford and Gieseker, to prove the Chow asymptotic stability of stable nodal curves by the Hilbert–Mumford criterion.

Contents

1. Introduction and summary of main result

In the late seventies, Mumford [Mum77] and Gieseker [Gie82] constructed the coarse moduli space $\overline{\mathcal{M}}_g$ of stable curves using Mumford's geometric invariant theory (GIT) (cf. e.g. [HM98]). They proved the GIT stability of smooth curves by verifying the Hilbert–Mumford stability criterion; for nodal curves, they proved the stability indirectly by using semistable replacement and using a numerical criterion to rule out curves with worse than nodal singularities. This construction has been very successful and widely adopted subsequently, for instance, in Caporaso's proof of asymptotic stability of nodal curves [Cap94, BS08].

Received 28 November 2011, accepted in final form 6 October 2014, published online 10 June 2015.
2010 Mathematics Subject Classification 14H10 (primary), 32Q20 (secondary).
Keywords: geometric invariant theory, asymptotic stability, Chow stability, Chow weight, K-stability, Donaldson–Futaki invariant, moduli of curves, one parameter subgroups, Hilbert–Mumford criterion.

In this paper, we will prove the Chow asymptotic stability of weighted pointed nodal curves by verifying the Hilbert–Mumford criterion directly. As an application, we provide a GIT construction of the moduli of weighted pointed stable curves. An interesting consequence of this construction is that the GIT closure of the moduli of weighted pointed smooth curves, using Chow asymptotic stability, is identical to Hassett's coarse moduli of weighted pointed stable curves, while its universal family includes strictly semistable weighted pointed nodal curves.

Another application of our stability study is to show that a polarized nodal curve is K-stable (cf. § 7) if and only if the polarization is numerically proportional to the dualizing sheaf. This generalizes a theorem of Odaka that a stable nodal curve polarized with a dualizing sheaf is K-stable.

The primary goal of this work is towards understanding the GIT compactification of moduli of canonically polarized varieties. The recent works on the relation between various notions of K-stabilities and the existence of *constant scalar curvature Kähler* (cscK) metrics suggest that some deep and interesting geometry is yet to be uncovered in this area. We hope this study will help us understand the stability of high-dimensional singular varieties.

We briefly outline the results proved in this paper. In this paper, we work over a characteristic zero algebraically closed field $\Bbbk$. A *curve* is a proper, reduced pure one-dimensional scheme.

Definition 1.1 [Has03]. A *weighted pointed nodal curve* $(X, \mathbf{x}, \mathbf{a})$ is a connected nodal curve X coupled with n ordered (not necessarily distinct) weighted smooth points

$$\mathbf{x} = (x_1, \dots, x_n) \in X^n \text{ of weights } \mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Q}_+^{\oplus n}$$

such that the total weight at any point is no more than one (i.e. for any $p \in X$, $\sum_{x_i = p} a_i \leqslant 1$). It is *polarized* if it comes with a *very ample* line bundle $\mathcal{O}_X(1)$ of degree d.

In this paper, we will use $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ to denote a *polarized weighted pointed nodal curve.* As $\mathcal{O}_X(1)$ is *very ample*, we form its tautological embedding

$$\imath : X \stackrel{\subset}{\longrightarrow} \mathbb{P}W, \quad W = H^0(\mathcal{O}_X(1))^\vee \tag{1.1}$$

and the Chow point

$$\mathrm{Chow}(X, \mathbf{x}) = (\mathrm{Chow}(X), \mathbf{x}) \in \Xi := \mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2] \times (\mathbb{P}W)^n, \tag{1.2}$$

where $\mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is the space of bi-degree (d, d) hypersurfaces in $(\mathbb{P}W^\vee)^2$; $\mathrm{Chow}(X) \in \mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is the Chow point of $(X, \imath)$ consisting of the set of $(V_1, V_2) \in (\mathbb{P}W^\vee)^2$ such that $V_1 \cap V_2 \cap \imath(X) \neq \emptyset$.

The stability of the Chow point is tested by the positivity of the $\mathbf{a}$-weight of any one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{SL}(W)$. (A one-parameter subgroup, abbreviated to 1-PS, is always non-trivial.) Since $\mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is a projective space, it has a canonical polarization $\mathcal{O}(1)$. We let

$$\mathcal{O}_\Xi(1, \mathbf{a})$$

be the $\mathbb{Q}$-ample line bundle on Ξ that has degree 1 on $\mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ and has degree a_i on the ith copy of the $\mathbb{P}W$ in $(\mathbb{P}W)^n$. The group $\mathrm{SL}(W)$ acts on Ξ, and an integral multiple of $\mathcal{O}_\Xi(1, \mathbf{a})$ is canonically linearized by $\mathrm{SL}(W)$.

Definition 1.2. Given $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$, and a 1-PS λ of $\mathrm{SL}(W)$, we let $\zeta = \lim_{t\to 0} \lambda(t) \cdot \mathrm{Chow}(X, \mathbf{x}) \in \Xi$ and define the $\mathbf{a}$-λ-weight $\omega_{\mathbf{a}}(\lambda)$ of $\mathrm{Chow}(X, \mathbf{x}) \in \Xi$ to be the weight of the λ-action on the fiber $\mathcal{O}_\Xi(1, \mathbf{a})|_\zeta$. We define the λ-weight $\omega(\lambda)$ of $\mathrm{Chow}(X) \in \mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ similarly with $\mathrm{Chow}(X, \mathbf{x})$ (respectively $\mathcal{O}_\Xi(1, \mathbf{a})$) replaced by $\mathrm{Chow}(X)$ (respectively $\mathcal{O}(1)$).

Definition 1.3. We say that $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is *stable* (respectively *semistable*) if for any 1-PS λ of $\mathrm{SL}(W)$, the $\mathbf{a}$-λ-weight $\omega_{\mathbf{a}}(\lambda)$ of $\mathrm{Chow}(X, \mathbf{x})$ is positive (respectively non-negative).

To make an analogy with the slope stability of vector bundles, we introduce the notion of slope stable by testing on proper closed subcurves $Y \subsetneq X$. First, with $\mathcal{O}_X(1)$ understood, for subcurve $Y \subseteq X$ we define $\deg Y = \deg \mathcal{O}_X(1)|_Y$. For any proper subcurve $Y \subsetneq X$, we define the number of *linking nodes* of Y to be

$$\ell_Y = |L_Y|, \quad L_Y = Y \cap Y^{\complement}, \quad Y^{\complement} = \overline{X \backslash Y}. \tag{1.3}$$

For simplicity, we abbreviate

$$a_Y = \sum_{x_i \in Y} a_i,$$

and thus $a_X = \sum_{i=1}^n a_i$. We say that $(X, \mathcal{O}_X(1))$ is non-special if $h^1(\mathcal{O}_X(1)) = 0$. We call a subcurve $Y \subset X$ of $(X, \mathcal{O}_X(1), \mathbf{x})$ an *exceptional component* if $Y \cong \mathbb{P}^1$, $Y \cap \mathbf{x} = \emptyset$, $\ell_Y = 2$ and $\deg_Y \mathcal{O}_X(1) = 1$.

Definition 1.4. We say $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is *slope semistable* if $(X, \mathcal{O}_X(1))$ is non-special and for any proper subcurve $Y \subsetneq X$ we have

$$\frac{\deg Y + \ell_Y/2 + a_Y/2}{h^0(\mathcal{O}_Y(1))} \leqslant \frac{\deg X + a_X/2}{h^0(\mathcal{O}_X(1))}. \tag{1.4}$$

We say that it is *stable* if it is semistable and the strict inequality (1.4) holds except when $Y^{\complement}$ is a disjoint union of exceptional components of $(X, \mathcal{O}_X(1), \mathbf{x})$.

In this paper, we will prove by verifying the Hilbert–Mumford criterion the following theorem. For the weight $\mathbf{a}$ and $g(X) = g$, we define

$$\chi_{\mathbf{a},g} := g - 1 + \tfrac{1}{2} a_X \quad \text{and} \quad \chi_{\mathbf{a},g}(X) := \chi_{\mathbf{a},g(X)}. \tag{1.5}$$

Theorem 1.5. *Given g and $\mathbf{a}$ such that $\chi_{\mathbf{a},g} > 0$, there is an $M = M(g, n, \mathbf{a})$ such that a genus g polarized weighed pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ having $\deg X \geqslant M$ is (semi)stable if and only if it is slope (semi)stable.*

By a straightforward extension of [Cap94, Proposition 3.1], Theorem 1.5 can be reformulated (cf. Proposition 5.4) as follows.

Theorem 1.6. *Given g and $\mathbf{a}$ such that $\chi_{\mathbf{a},g} > 0$, there is an $M = M(g, n, \mathbf{a})$ so that a genus g polarized weighed pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ having $\deg X \geqslant M$ is semistable if and only if for any proper subcurve $Y \subsetneq X$, we have*

$$\left| \left(\deg Y + \sum_{x_j \in Y} \frac{a_j}{2} \right) - \frac{\deg_Y \omega_X(\mathbf{a} \cdot \mathbf{x})}{\deg \omega_X(\mathbf{a} \cdot \mathbf{x})} \left(\deg X + \sum_{j=1}^n \frac{a_j}{2} \right) \right| \leqslant \frac{\ell_Y}{2}. \tag{1.6}$$

It is stable if it is semistable (i.e. (1.6) holds) and the strict inequality holds except when Y or $Y^{\complement}$ is a disjoint union of exceptional components of $(X, \mathcal{O}_X(1), \mathbf{x})$.

We remark that the constant M in the theorem can be estimated effectively depending on g, n and $\mathbf{a}$. However, as our approach is unlikely to produce a near optimal bound M, we made no efforts in this paper to trace the dependence of M on g and $\mathbf{a}$. It is certainly an interesting and important question to optimize M, and it is known (cf. [BFMV14]) that the optimal M is $4(2g-2)+\epsilon$ (with $0<\epsilon \ll 1$) when $\mathbf{a}=0$.

The case $\mathbf{x}=\emptyset$ is a theorem of Caporaso [Cap94] on the stability of polarized nodal curves. The case of the asymptotic Hilbert stability of smooth weighted pointed curves is a theorem of Swinarski [Swi12] (see also [Mor09]).

We now sketch the main ingredients of our proof. Our starting point is a theorem of Mumford that expresses the $\mathbf{a}$-λ-weight of $\operatorname{Chow}(X, \mathbf{x})$ in terms of the leading coefficient of the Hilbert–Samuel polynomial of an ideal $\mathcal{I} \subset \mathcal{O}_{X\times\mathbb{A}^1}(1)$ (cf. Proposition 2.1). Our observation is that this leading coefficient can be evaluated by the leading coefficient of the Hilber–Samuel polynomial of the pullback $\tilde{\mathcal{I}}$ of $\mathcal{I}$ to the normalization $\tilde{X}$ of X. This transforms the evaluation of the $\mathbf{a}$-λ-weight to the calculation of the areas of a class of Newton polygons associated to the pullback sheaf $\tilde{\mathcal{I}}$. We then obtain an effective bound of the areas of these Newton polygons and thus a bound of the $\mathbf{a}$-λ-weight of $\operatorname{Chow}(X, \mathbf{x})$. Since this bound is linear in the weights of λ, we can apply linear programing to complete a proof of Theorem 1.5.

Our GIT construction of the moduli of weighted pointed stable curves goes as follows. We form the Hilbert scheme $\mathcal{H}$ of pointed one-dimensional subschemes of $\mathbb{P}^m$ of fixed degree. Let $\psi: \mathcal{H} \to \mathcal{C}$ be the equivariant Hilbert–Chow morphism (map) to the Chow variety of pointed one-dimensional cycles in $\mathbb{P}^m$ of the same degree. Applying our main theorem, we conclude that in the case where the degree is sufficiently large, the preimage under ψ of the set $\mathcal{C}^{\mathrm{ss}} \subset \mathcal{C}$ of GIT-semistable points is the set of semistable polarized weighted pointed nodal curves. Let $\mathcal{K} \subset \mathcal{H}$ be the subset of canonically polarized weighted pointed smooth curves. We prove that the GIT-quotient of the closure $\overline{\mathcal{K}}$ is isomorphic to Hassett's moduli of weighted pointed stable curves. An interesting observation is that the complement $\overline{\mathcal{K}} - \mathcal{K}$ contains polarized semistable but not canonically polarized weighted pointed nodal curves. Thus though GIT gives the same compactification as that of Hassett of the moduli of canonically polarized weighted pointed smooth curves, the geometric objects added to obtain the compactification in the mentioned two constructions are different. It is worth pursuing to see how this extends in the high-dimensional case.

In the end, using a result of Stoppa and the fact that the Donaldson–Futaki invariants can be expressed as the limit of normalized Chow weights under a 1-PS, we apply our main theorem to prove that a polarized nodal curve $(X, \mathcal{O}_X(1))$ is K-stable if and only if $\mathcal{O}_X(1)$ is numerically proportional to ω_X (cf. Theorem 7.1). This implies that GIT compactification is same as the compactification of smooth curves using K-stability.

The paper is organized as follows. In § 2, we show that the weights can be evaluated via the leading coefficients of the Hilbert–Samuel polynomial of a sheaf on the normalization $\tilde{X}$. In § 3, we reduce our study to a particular class of 1-PS: the staircase 1-PS. We will derive a sharp bound for each irreducible component in § 4. We complete the proof of our main theorems in § 5. The last two sections include the applications of our stability study to constructing moduli of weighted pointed nodal curves and to study the K-stability of polarized curves.

List of notation

$\mathcal{I}(\lambda)$; $\tilde{\mathcal{I}}(\lambda)$	$(t^{\rho_0}s_0,\dots)\subset\mathcal{O}_{X\times\mathbb{A}^1}(1)$; similarly defined on $\tilde{X}$	(2.3)
$e(\mathcal{I}(\lambda))$; $e(\tilde{\mathcal{I}})$	n.l.c. $\chi(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda)^k)$; similarly defined on $\tilde{X}$	(2.4); after (2.6)
$e(\tilde{\mathcal{I}})_q$; $e(\tilde{\mathcal{I}}_\alpha)$	contribution of $e(\tilde{\mathcal{I}})$ at $q\in X$; along $\tilde{X}_\alpha$	(2.12)
$\omega(\lambda)$	λ-Chow weight	Proposition 2.1
$v(\tilde{s}_i,q)$	the vanishing order of $\tilde{s}_i$ at q	(2.8)
$\hbar(q)$	$\max\{i\mid v(\tilde{s}_i,q)\neq\infty\}$	(2.9)
$\hbar_\alpha$	$\min_i\{i\mid \tilde{s}_j\vert_{\tilde{X}_\alpha}=0,\ \text{for } j\geqslant i+1\}$	(2.16)
Δ_q	Newton polygon supported at $q\in\tilde{X}$	Definition 2.7
$\mathcal{E}_i=\mathcal{E}(\lambda)_i$	$(s_i,s_{i+1},\dots,s_m)\subset\mathcal{O}_X(1)$	(3.1)
$\Lambda_\alpha(\lambda)$; $\Lambda(\lambda)$	$\{q\in X_\alpha\mid s_{\hbar_\alpha}(q)=0\}$; $\Lambda(\lambda)=\bigcup_{\alpha=1}^r\Lambda_\alpha(\lambda)$	Definition 3.1
$\delta(\tilde{s}_i,p)$	$\text{length}(\tilde{\mathcal{E}}_i/\tilde{\mathcal{E}}_{i+1})_p$ or $=0$	Definition 3.2
$\text{inc}_\alpha(\tilde{s}_i)$	$\sum_{p\in\tilde{X}_\alpha}\delta(\tilde{s}_i,p)p$ and $\text{inc}(\tilde{s}_i)=\sum_\alpha\text{inc}_\alpha(\tilde{s}_i)$	Definition 3.1
$\delta_\alpha(\tilde{s}_i)$; $\delta(\tilde{s}_i)$	$\sum_{p\in\tilde{X}_\alpha}\delta(\tilde{s}_i,p)$; $\delta(\tilde{s}_i)=\sum_\alpha\delta_\alpha(\tilde{s}_i)$	Definition 3.1
$w(\tilde{\mathcal{E}}_i,p)$; $w_\alpha(\tilde{\mathcal{E}}_i)$	$\text{length}(\mathcal{O}_{\tilde{X}}(1)/\tilde{\mathcal{E}}_i)_p$; $w_\alpha(\tilde{\mathcal{E}}_i)=\sum_{p\in\tilde{X}_\alpha}w(\tilde{\mathcal{E}}_i,p)$	Definition 3.2
$\mathbb{I}_\alpha=\mathbb{I}_\alpha(\lambda)$	$\{i\in\mathbb{I}\mid\text{inc}(\tilde{s}_i)\cap\tilde{X}_\alpha\neq\emptyset\text{ or } i=\hbar_\alpha\}$	(3.3)
L_Y; L_α; $\tilde{L}_Y$; $\tilde{L}_\alpha$	$Y\cap Y^{\complement}$; L_{X_α}; $\pi^{-1}(L_Y)\cap\tilde{Y}$; $\tilde{L}_{X_\alpha}$	(1.3) and (3.9)
$\tilde{N}_Y$; N_α; $\tilde{N}_\alpha$	$\pi^{-1}(N_Y)\cap\tilde{Y}$; N_{X_α}; $\tilde{N}_{X_\alpha}$	(3.8)
ℓ_α; $\ell_{\alpha,\beta}$; $\ell_{\alpha,\alpha}$	$\lvert L_\alpha\rvert$; $\lvert X_\alpha\cap X_\beta\rvert$; $-\lvert L_\alpha\rvert$	(3.9); (6.10)
$\mathbb{I}_\alpha^{\text{pri}}$	$\{i\in\mathbb{I}_\alpha\mid w_\alpha(\tilde{\mathcal{E}}_{i+1})\leqslant\deg X_\alpha-2g(X_\alpha)-\ell_\alpha-1\}$	Definition 3.10
$E_\alpha^\epsilon(\rho)$	upper bound of $e(\mathcal{I})_\alpha$	(4.3)
$W_i=W_i(\lambda)$	$\{v\in W\mid s_i(v)=\cdots=s_m(v)=0\}\subset W$	(5.8)
$\omega_{\mathbf{a}}(\lambda)$	$\omega(\lambda)+\mu_{\mathbf{a}}(\lambda)$	(5.11)
$\Phi:\mathcal{H}\to\mathcal{C}$	Hilbert–Chow map	before Proposition 6.2
$\mathcal{K},\bar{\mathcal{K}}\subset\mathcal{H}$	slice polarized by $\omega_{\mathcal{X}/\mathcal{H}}(\mathbf{a}\cdot\mathbf{x})$	before (6.5)
$\vec{\delta}(\mathcal{L})$	degree class for the line bundle $\mathcal{L}$	after (6.9)

2. Chow stability, Chow weight and Newton polygon

In this section, we first recall some basic facts from [Mum77] on stability of a polarized curve; we then localize the calculation of the weight of $\text{Chow}(X)$ to a divisor on the normalization of X and interpret the contribution from each point of the divisor as the area of a generalized Newton polytope.

Let $(X,\mathcal{O}_X(1))$ be a polarized connected nodal curve, together with its associated embedding $\imath:X\to\mathbb{P}W$ (cf. (1.1)) and its Chow point $\text{Chow}(X)$. We will reserve the symbol λ for a 1-PS of $\text{SL}(W)$; for such λ, we diagonalize its action by choosing

$$\mathbf{s}=\{s_0,\dots,s_m\}\text{ a basis of } W^\vee \tag{2.1}$$

so that under its dual bases the action λ is given by

$$\lambda(t):=\text{diag}[t^{\rho_0},\dots,t^{\rho_m}]\cdot t^{-\rho_{\text{ave}}},\quad \rho_0\geqslant\rho_1\geqslant\cdots\geqslant\rho_m=0, \tag{2.2}$$

and $\rho_{\text{ave}}=(1/(m+1))\sum\rho_i$. We will call $\mathbf{s}$ a diagonalizing basis of λ.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

In [Mum77], Mumford introduced a subsheaf

$$\mathcal{I}(\lambda) = (t^{\rho_0}s_0, \dots, t^{\rho_m}s_m) \subset \mathcal{O}_{X\times\mathbb{A}^1}(1) := p_X^*\mathcal{O}_X(1) \tag{2.3}$$

generated by the sections in the parentheses, where $p_X : X \times \mathbb{A}^1 \to X$ is the projection. Let $e(\mathcal{I}(\lambda))$ be the normalized leading coefficient (abbreviated to n.l.c.) of the Hilbert–Samuel polynomial:

$$\chi(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda)^k) = e(\mathcal{I}(\lambda))\frac{k^2}{2} + \text{lower order terms.} \tag{2.4}$$

Proposition 2.1 (Mumford). *The λ-weight of* $\mathrm{Chow}(X)$ *is*

$$\omega(\lambda) = \frac{2\deg X}{m+1}\sum_{i=0}^{m}\rho_i - e(\mathcal{I}(\lambda)).$$

In the following, when the 1-PS λ and its diagonalizing basis $\mathbf{s}$ are understood, we will drop λ from $\mathcal{I}(\lambda)$ and abbreviate $\mathcal{I}(\lambda)$ to $\mathcal{I}$. Our first step is to lift the calculation of $e(\mathcal{I})$ $(= e(\mathcal{I}(\lambda)))$ to the normalization

$$\pi : \tilde{X} \longrightarrow X.$$

We let

$$\tilde{s}_i = \pi^* s_i \in \mathcal{O}_{\tilde{X}}(1) := \mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}} \tag{2.5}$$

and let $\tilde{\mathcal{I}}$ be the pullback of $\mathcal{I}$:

$$\tilde{\mathcal{I}} = (t^{\rho_0}\tilde{s}_0, \dots, t^{\rho_m}\tilde{s}_m) \subset \mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(1) = \mathcal{O}_{\tilde{X}}(1) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X}\times\mathbb{A}^1}. \tag{2.6}$$

We define $e(\tilde{\mathcal{I}}) = \text{n.l.c.}\,\chi(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(k)/\tilde{\mathcal{I}}^k)$. We have the following special case of [Mum77, Lemma 5.6] which enables us to lift the evaluation of $e(\mathcal{I})$ to $\tilde{X}$. As [Mum77, Lemma 5.6] was not proved in [Mum77], we give a proof here shortly.

Proposition 2.2. *We have* $e(\mathcal{I}) = e(\tilde{\mathcal{I}})$.

Our next step is to localize the evaluation of $e(\tilde{\mathcal{I}})$ to individual $q \in \tilde{X}$. Let z be a uniformizing parameter of $\tilde{X}$ at q; let t be the standard coordinates of $\mathbb{A}^1$. We denote by $\hat{\mathcal{O}}_{\tilde{X},q}$ the formal completion of the local ring $\mathcal{O}_{\tilde{X},q}$ at its maximal ideal. We fix an isomorphism of $\hat{\mathcal{O}}_{\tilde{X},q}$-modules (the first isomorphism below):

$$\varphi_q : \mathcal{O}_{\tilde{X}}(1) \otimes_{\mathcal{O}_{\tilde{X}}} \hat{\mathcal{O}}_{\tilde{X},q} \cong \hat{\mathcal{O}}_{\tilde{X},q} \cong \Bbbk[\![z]\!], \tag{2.7}$$

where the second isomorphism is induced by the choice of z.

Definition 2.3. Let $\tilde{s}_i \in H^0(\mathcal{O}_{\tilde{X}}(1))$ be as in (2.5). We define

$$v(\tilde{s}_i, q) = \text{the vanishing order of } \tilde{s}_i \text{ at } q; \tag{2.8}$$

in the case $\tilde{s}_i \equiv 0$ near q, we define $v(\tilde{s}_i, q) = \infty$. We set

$$\hbar(q) = \max\{i \mid v(\tilde{s}_i, q) \neq \infty\} \quad \text{and} \quad w(\tilde{\mathcal{I}}, q) = v(\tilde{s}_{\hbar(q)}, q). \tag{2.9}$$

The quantity $w(\tilde{\mathcal{I}},q)$ is the width of the polygon Δ_q associated to $\tilde{\mathcal{I}}$ (at q) to be defined later.

We now look at the image of $\tilde{\mathcal{I}}$ under $\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(1)\to\hat{\mathcal{O}}_{\tilde{X}\times\mathbb{A}^1,(q,0)}$. We let

$$I_q=(z^{v(\tilde{s}_0,q)}t^{\rho_0},\ldots,z^{v(\tilde{s}_{m-1},q)}t^{\rho_{m-1}},z^{v(\tilde{s}_m,q)}t^{\rho_m})\subset R=\Bbbk[\![z,t]\!], \tag{2.10}$$

agreeing $z^\infty=0$. By construction, φ_q induces an isomorphism

$$(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(k)/\tilde{\mathcal{I}}^k)\otimes_{\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}}\hat{\mathcal{O}}_{\tilde{X}\times\mathbb{A}^1,(q,0)}\cong R/I_q^k. \tag{2.11}$$

Notice that the right-hand side is not a finite module when $\hbar(q)<m$. Since for all i we have $t^{\rho_i}\cdot\varphi_q(\tilde{s}_i)\in(t^{\rho_{\hbar(q)}})\subset R$, the induced homomorphism $(t^{k\cdot\rho_{\hbar(q)}})/I_q^k\to R/I_q^k$ is injective, and $(t^{k\cdot\rho_{\hbar(q)}})/I_q^k$ is a finite module. We define

$$e(\tilde{\mathcal{I}})_q=\text{n.l.c.}\dim((t^{k\cdot\rho_{\hbar(q)}})/I_q^k)+2\rho_{\hbar(q)}\cdot w(\tilde{\mathcal{I}},q). \tag{2.12}$$

We have the following formula, independently obtained by Swinarski; a special case can be found in [Sch91, p. 300].

Lemma 2.4. *We have the summation formula* $e(\tilde{\mathcal{I}})=\sum_{q\in\tilde{X}}e(\tilde{\mathcal{I}})_q$.

Proof of Proposition 2.2. Let $p_1,\ldots,p_l$ be the nodes of X; let $\xi=\pi\times 1_{\mathbb{A}^1}:\tilde{X}\times\mathbb{A}^1\to X\times\mathbb{A}^1$ be the projection. Tensoring the exact sequence

$$0\longrightarrow\mathcal{O}_{X\times\mathbb{A}^1}\longrightarrow\xi_*\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}\longrightarrow\bigoplus_{j=1}^{l}\mathcal{O}_{p_j\times\mathbb{A}^1}\longrightarrow 0$$

with $\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k$, we obtain an exact sequence

$$\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k\xrightarrow{f_k}(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\xi_*\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}\longrightarrow\bigoplus_{\alpha=1}^{r}(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)|_{p_j\times\mathbb{A}^1}\longrightarrow 0.$$

By projection formula, we have

$$\xi_*(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(k)/\tilde{\mathcal{I}}^k)=\xi_*(\xi^*(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k))=(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\xi_*\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}.$$

Thus

$$e(\tilde{\mathcal{I}})=\text{n.l.c.}\,\chi(\xi_*(\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}(k)/\tilde{\mathcal{I}}^k))=\text{n.l.c.}\,\chi((\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\xi_*\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}),$$

which equals

$$\text{n.l.c.}\,(\chi(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)-\dim\ker f_k+\sum_{i=1}^{l}\chi((\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)|_{p_j\times\mathbb{A}^1})).$$

We claim that both

$$\chi((\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\mathcal{O}_{p_j\times\mathbb{A}^1})\quad\text{and}\quad\dim\ker f_k \tag{2.13}$$

are linear in k. This will prove the Proposition.

We begin with the first claim. We let q be one of the nodes of X; let q^+ and q^- be the preimages $\pi^{-1}(q)\subset\tilde{X}$, and let x and y be uniformizing parameters of $\tilde{X}$ at q^+ and q^-, respectively. Then, after fixing an isomorphism $\mathcal{O}_X(1)\otimes_{\mathcal{O}_X}\mathcal{O}_{X,q}\cong\mathcal{O}_{X,q}$ near q and defining $R=\Bbbk[\![x,y]\!]/(xy)$, we have the isomorphism

$$(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}^k)\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\mathcal{O}_{q\times\mathbb{A}^1}\cong(R[t]/I^k)\otimes_{R[t]}R[t]/(x,y),\tag{2.14}$$

where $I\subset R[t]$ is the ideal generated by $t^{\rho_i}\hat{s}_i$, $i=0,\ldots,m$, and $\hat{s}_i$ are formal germs of s_i at q as elements in R. Since for some i the value $s_i(q)\neq 0$, $i_q=\max\{i\mid s_i(q)\neq 0\}$ is finite. Thus the right-hand side of (2.14) is isomorphic to $R[t]/(I^k,x,y)=\Bbbk[t]/(t^{k\cdot i_q})$ whose dimension is linear in k. This proves the first claim.

For the second claim, since the kernel of f_k consists of torsion elements supported on the union of $p_1\times\mathbb{A}^1,\ldots,p_l\times\mathbb{A}^1$. Hence, to prove the claim, we only need to study the kernel of an analogous homomorphism

$$\bar{f}_k: R[t]/I^k\longrightarrow(R[t]/I^k)\otimes_{R[t]}(\Bbbk[\![x]\!][t]\oplus\Bbbk[\![y]\!][t]),$$

where I is as in the previous paragraph and $R[t]\to\Bbbk[\![x]\!][t]\oplus\Bbbk[\![y]\!][t]$ is the normalization homomorphism $g(x,y,t)\mapsto(g(x,0,t),g(0,y,t))$. Since the domain and the target of $\bar{f}_k$ are t-graded rings and $\bar{f}_k$ is a homomorphism of graded rings, as vector spaces,

$$\ker\bar{f}_k=\bigoplus_{j\geqslant 1}\ker\{(\bar{f}_k)_j: t^jR/(I^k\cap t^jR)\to(t^jR/(I^k\cap t^jR))\otimes_R(\Bbbk[\![x]\!]\oplus\Bbbk[\![y]\!])\}.$$

Because $R=\Bbbk[\![x,y]\!]/(xy)$, as R-modules, each $t^jR/(I^k\cap t^jR)$ is isomorphic to R/J for J being one of the ideals in the list:

$$R,(0),(x^e),(y^e),(x^e,y^{e'}),(x^e+y^{e'})\quad\text{where } e,e'\in\mathbb{N}.$$

One can check that for J of the first five kinds, $\ker(\bar{f}_k)_j=0$; for J of the last kind, $\ker(\bar{f}_k)_j\cong\Bbbk$. Thus we always have $\dim\ker(\bar{f}_k)_j\leqslant 1$. On the other hand, since $s_{i_q}(q)\neq 0$, $t^{\rho_{i_q}}\in I$ and $t^{k\rho_{i_q}}\in I^k$. Thus $\ker(\bar{f}_k)_j=0$ for $j\geqslant ki_q$. This proves that $\dim\ker f_k$ is at most linear in k. This proves the proposition. □

Because of this proposition, we will work over the normalization $\tilde{X}$ of X subsequently. To avoid possible confusion, we will reserve '$\tilde{\ }$' to denote the associated objects lifted to $\tilde{X}$. For instance, we will denote by $X_1,\ldots,X_r$ the irreducible components of X and denote by $\tilde{X}_1$, $\ldots,\tilde{X}_r$ their respective normalizations. For the sections $t^{\rho_i}s_i$ in $\mathcal{I}$, $t^{\rho_i}\tilde{s}_i$ are their lifts in $\tilde{\mathcal{I}}=\mathcal{I}\otimes_{\mathcal{O}_{X\times\mathbb{A}^1}}\mathcal{O}_{\tilde{X}\times\mathbb{A}^1}$. For consistency, we reserve subindices i for the sections s_i and reserve the Greek α for the indices of the irreducible components $\{X_\alpha\}_{1\leqslant\alpha\leqslant r}$.

Proof of Lemma 2.4. Letting $\tilde{\mathcal{I}}_\alpha=\tilde{\mathcal{I}}|_{\tilde{X}_\alpha\times\mathbb{A}^1}\subset\mathcal{O}_{\tilde{X}_\alpha\times\mathbb{A}^1}(1)$, then

$$e(\tilde{\mathcal{I}})=\sum_{\alpha=1}^r \text{n.l.c.}\,\chi(\mathcal{O}_{\tilde{X}_\alpha\times\mathbb{A}^1}(k)/\tilde{\mathcal{I}}_\alpha^k)=\sum_{\alpha=1}^r e(\tilde{\mathcal{I}}_\alpha).\tag{2.15}$$

For $q\in\tilde{X}_\alpha$, we define $e(\tilde{\mathcal{I}}_\alpha)_q=e(\tilde{\mathcal{I}})_q$. Thus to prove the lemma we only need to show that

$$e(\tilde{\mathcal{I}}_\alpha)=\sum_{q\in\tilde{X}_\alpha}e(\tilde{\mathcal{I}}_\alpha)_q.$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

To proceed, we first note that $\hbar(q)$ (cf. (2.9)) is locally constant on $\tilde{X}$ and hence constant on each individual component $\tilde{X}_\alpha \subset X$. We let $\hbar_\alpha = \hbar(q)$ for some $q \in \tilde{X}_\alpha$. Then we have

$$\hbar_\alpha = \max_i \{ i \mid \tilde{s}_j|_{\tilde{X}_\alpha} = 0, \text{for } j \geqslant i+1 \}. \tag{2.16}$$

We claim $t^{\rho_{\hbar_\alpha}}$ divides $t^{\rho_i}\tilde{s}_i$ for all i. Indeed, the case $i > \hbar_\alpha$ follows from $\tilde{s}_i|_{\tilde{X}_\alpha} \equiv 0$; the case $i \leqslant \hbar_\alpha$ follows from $\rho_i \geqslant \rho_{\hbar_\alpha}$. We let $\bar{\rho}_i = \rho_i - \rho_{\hbar_\alpha}$, and introduce the ideal

$$\tilde{\mathcal{R}}_\alpha = (t^{\bar{\rho}_0}\tilde{s}_0, t^{\bar{\rho}_1}\tilde{s}_1, \ldots, t^{\bar{\rho}_{\hbar_\alpha}}\tilde{s}_{\hbar_\alpha}) \subset \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(1).$$

This way, $\tilde{\mathcal{I}}_\alpha = t^{\rho_{\hbar_\alpha}}\tilde{\mathcal{R}}_\alpha \subset t^{\rho_{\hbar_\alpha}}\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(1)$.

We let $(t^{k\rho_{\hbar_\alpha}}) = t^{k\rho_{\hbar_\alpha}}\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)$; it belongs to the exact sequence

$$0 \longrightarrow (t^{k\cdot\rho_{\hbar_\alpha}})/\tilde{\mathcal{I}}_\alpha^k \longrightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{I}}_\alpha^k \longrightarrow \mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\cdot\rho_{\hbar_\alpha}}) \longrightarrow 0.$$

Since $(t^{k\cdot\rho_{\hbar_\alpha}})/\tilde{\mathcal{I}}_\alpha^k = t^{k\cdot\rho_{\hbar_\alpha}}(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k)$ and $\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k$ is finite, we have

$$\chi(\mathcal{O}_{\tilde{X}\times \mathbb{A}^1}(k)/\tilde{\mathcal{I}}_\alpha^k) = \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k) + \chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\cdot\rho_{\hbar_\alpha}})).$$

Taking the n.l.c. of individual terms, and using

$$\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/(t^{k\cdot\rho_{\hbar_\alpha}})) = k\,\rho_{\hbar_\alpha} \cdot \chi(\mathcal{O}_{\tilde{X}_\alpha}(k)) = k^2\,\rho_{\hbar_\alpha} \cdot \deg X_\alpha + O(k),$$

we obtain

$$e(\tilde{\mathcal{I}}_\alpha) = \text{n.l.c.}\,\chi(\mathcal{O}_{\tilde{X}\times \mathbb{A}^1}(k)/\tilde{\mathcal{I}}_\alpha^k) = \text{n.l.c.}\,\chi(\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k) + 2\rho_{\hbar_\alpha}\cdot \deg \tilde{X}_\alpha. \tag{2.17}$$

Next, let $\{q_1, \ldots, q_l\}$ be the support of $(\tilde{s}_{\hbar_\alpha} = 0) \cap \tilde{X}_\alpha$. Following the convention in (2.11), we have an isomorphism

$$\mathcal{O}_{\tilde{X}_\alpha \times \mathbb{A}^1}(k)/\tilde{\mathcal{R}}_\alpha^k \xrightarrow{\cong} \bigoplus_{a=1}^{l} t^{k\cdot\rho_{\hbar_\alpha}} R/(I_{q_a} \cap t^{\rho_{\hbar_\alpha}}R)^k,$$

induced by restricting to germs at q_a after multiplying by $t^{k\cdot\rho_{\hbar_\alpha}}$. Adding that $\deg \tilde{X}_\alpha = \dim \mathcal{O}_{\tilde{X}_\alpha}(1)/(\tilde{s}_{\hbar_\alpha}) = \sum_{a=1}^{l} w(\tilde{\mathcal{I}}, q_a)$, (2.17) gives us

$$e(\tilde{\mathcal{I}}_\alpha) = \sum_{a=1}^{l} (\text{n.l.c.}\ \dim(t^{k\cdot\rho_{\hbar_\alpha}} R/(I_{q_a} \cap t^{\rho_{\hbar_\alpha}}R)^k) + 2\rho_{\hbar_\alpha}\cdot w(\tilde{\mathcal{I}}, q_a)) = \sum_{q \in \tilde{X}_\alpha} e(\tilde{\mathcal{I}})_q.$$

This proves the lemma. □

Example 2.5. Let λ be a 1-PS with diagonalizing basis $\{s_i\}$ and weights $\rho_0 = 1 > \rho_1 = \cdots = \rho_m = 0$. Suppose $(s_1 = \cdots = s_m = 0)$ is a reduced point $q \in X$. Then $e(\mathcal{I}(\lambda)) = 1$ and $\omega(\lambda) = (2\deg X)/(m+1) - 1$ (respectively $e(\mathcal{I}(\lambda)) = 2$ and $\omega(\lambda) = (2\deg X)/(m+1) - 2$) when q is a smooth point (respectively nodal point) of X.

We give a useful geometric interpretation of the quantity $e(\tilde{\mathcal{I}})_q$. Let $I \subset \Bbbk[z_1, z_2]$ be a monomial ideal and let Γ be the set of exponents of monomials in I; namely, I is the linear span of the monomials $\{x^\gamma \mid \gamma \in \Gamma\}$, and thus

$$\Gamma \subset (\mathbb{N} \cup \{0\})^2 \subset \mathbb{R}^2_{\geqslant 0} := (\mathbb{R}_{\geqslant 0})^2 \subset \mathbb{R}^2.$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

We then form the *closed convex hull* $\mathrm{Conv}(\mathbb{R}^2_{\geqslant 0}+\Gamma)\subset\mathbb{R}^2$ of $\mathbb{R}^2_{\geqslant 0}+\Gamma$ and let $\bar\Gamma=\mathrm{Conv}(\mathbb{R}^2_{\geqslant 0}+\Gamma)\cap\mathbb{N}^2$; then the integral closure $\bar I$ of I is the ideal generated by $\{x^\gamma \mid \gamma\in\bar\Gamma\}$ (cf. [Eis95, Exercise 4.23, p. 141]).

We let $\Delta(I)$ be the Newton polygon of I:

$$\Delta(I)=\mathbb{R}^2_{\geqslant 0}-\mathrm{Conv}(\mathbb{R}^2_{\geqslant 0}+\Gamma)\subset\mathbb{R}^2_{\geqslant 0}.$$

LEMMA 2.6. *Let $|\Delta(I)|$ be the area of the $\Delta(I)$. Then*

$$\dim \Bbbk[z_1,z_2]/I^k=|\Delta(I)|\cdot k^2+O(k).$$

Proof. Since $\bar I$ is the integral closure of I, by the Briançon–Skoda theorem [Laz04, Theorem 9.6.26], $I^k\subset\bar I^k\subset I^{k-1}$ for k sufficiently large. Since $\dim I^{k-1}/I^k$ is bounded from above by a linear function in k, $\dim\Bbbk[z_1,z_2]/I^k=\dim\Bbbk[z_1,z_2]/\bar I^k+O(k)$.

Further, $\dim\Bbbk[z_1,z_2]/\bar I^k$ is precisely the number of lattice points in $k\Delta(\bar I)=k\Delta(I)$. From the work of Kantor and Khovanskii [KK93, Don02], the number of lattice points inside the polygon is given by $|\Delta(I)|\cdot k^2+O(1)$. This proves the lemma. □

We now come back to the 1-PS λ and its diagonalizing basis $\mathbf{s}=\{s_i\}$.

DEFINITION 2.7. For any $q\in\tilde X$, we define

$$\Gamma_q=\{(v(\tilde s_i,q),\rho_i)\mid i=0,\ldots,m;v(\tilde s_i,\rho_i)<\infty\}\subset(\mathbb{N}\cup\{0\})^2;$$

we define the Newton polygon (of $\tilde{\mathfrak{I}}=\tilde{\mathfrak{I}}(\lambda)$) at q to be

$$\Delta_q(\lambda):=(\mathbb{R}^2_{\geqslant 0}-\mathrm{Conv}(\mathbb{R}^2_{\geqslant 0}+\Gamma_q))\cap([0,w(\tilde{\mathfrak{I}},q)]\times\mathbb{R}_{\geqslant 0}).$$

We will abbreviate $\Delta_q(\lambda)$ to Δ_q when the choice of the basis $\mathbf{s}$ is understood. Let $|\Delta_q|$ be the area of Δ_q. We state a formula useful for estimating the quantity $e(\mathfrak{I})=e(\tilde{\mathfrak{I}})$.

COROLLARY 2.8. *We have $e(\tilde{\mathfrak{I}})_q=2|\Delta_q|$; hence, $e(\tilde{\mathfrak{I}})=2\sum_{q\in\tilde X}|\Delta_q|$.*

Proof. Since Δ_q is the union of $\Delta_q\cap[0,w(\tilde{\mathfrak{I}},q)]\times[\rho_{\hbar(q)},\infty)$ with $[0,w(\tilde{\mathfrak{I}},q)]\times[0,\rho_{\hbar(q)}]$, by (2.4), (2.12) and Lemma 2.6,

$$e(\tilde{\mathfrak{I}})_q=2\cdot|\Delta_q\cap[0,w(\tilde{\mathfrak{I}},q)]\times[\rho_{\hbar(q)},\infty)|+2\cdot\rho_{\hbar(q)}\cdot w(\tilde{\mathfrak{I}},q)=2\,|\Delta_q|.$$

The second identity follows from Lemma 2.4. □

3. Staircase one-parameter subgroups

We begin with some conventions attached to a fixed 1-PS λ and its diagonalizing basis $\{s_0,\ldots,s_m\}$. For simplicity, we define

$$\mathbb{I}=\{0,1,\ldots,m\}.$$

For each $i\in\mathbb{I}$, we introduce subsheaves

$$\mathcal{E}_i=\mathcal{E}(\lambda)_i:=(s_i,s_{i+1},\ldots,s_m)\subset\mathcal{O}_X(1);\tag{3.1}$$

they form a decreasing sequence of subsheaves. Similarly, we introduce $\mathcal{O}_{\tilde X}$-submodules

$$\tilde{\mathcal{E}}_i=\tilde{\mathcal{E}}(\lambda)_i:=(\tilde s_i,\tilde s_{i+1},\ldots,\tilde s_m)\subset\mathcal{O}_{\tilde X}(1).$$

Definition 3.1. We call $i \in \mathbb{I}$ a *base index* if $i = \hbar_\alpha$ (cf. (2.16)) for some irreducible component X_α. For each X_α, we define $\Lambda_\alpha(\lambda) = \{q \in X_\alpha \mid s_{\hbar_\alpha}(q) = 0\}$; define $\Lambda(\lambda) = \bigcup_{\alpha=1}^r \Lambda_\alpha(\lambda)$; define $\tilde{\Lambda}_\alpha(\lambda) = \{p \in \tilde{X}_\alpha \mid \tilde{s}_{\hbar_\alpha}(p) = 0\}$, and define $\tilde{\Lambda} = \tilde{\Lambda}(\lambda) = \bigcup_{\alpha=1}^m \tilde{\Lambda}_\alpha(\lambda)$.

In the following, for any sheaf of $\mathcal{O}_{\tilde{X}}$-modules $\mathcal{F}$ and $p \in \tilde{X}$, we define $\mathcal{F}_p := \mathcal{F} \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_{\tilde{X},p}$, the localization of $\mathcal{F}$ at p. We remark that for any $p \in \tilde{X}_\alpha$, $\hbar(p) = \hbar_\alpha$ is the largest index i so that $(\tilde{\mathcal{E}}_i)_p \neq 0$.

Definition 3.2. For a closed point $p \in \tilde{X}_\alpha \subset \tilde{X}$, we define

$$\delta(\tilde{s}_i, p) = \mathrm{length}(\tilde{\mathcal{E}}_i/\tilde{\mathcal{E}}_{i+1})_p \quad \text{when } i \leqslant \hbar_\alpha - 1 = \hbar(p) - 1; \quad \delta(\tilde{s}_i, p) = 0 \text{ otherwise.}$$

We define the *increments* of $\tilde{s}_i$ along $\tilde{X}_\alpha$ and $\tilde{X}$ be the 0-cycles

$$\mathrm{inc}_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p)p \quad \text{and} \quad \mathrm{inc}(\tilde{s}_i) = \sum_\alpha \mathrm{inc}_\alpha(\tilde{s}_i);$$

we define their degrees to be $\delta_\alpha(\tilde{s}_i) = \sum_{p \in \tilde{X}_\alpha} \delta(\tilde{s}_i, p)$ and $\delta(\tilde{s}_i) = \sum_\alpha \delta_\alpha(\tilde{s}_i)$. We define the width of $\tilde{\mathcal{E}}_i$ at $p \in \tilde{X}_\alpha$ and at $\tilde{X}_\alpha$ for $i \leqslant \hbar_\alpha$ to be

$$w(\tilde{\mathcal{E}}_i, p) := \mathrm{length}(\mathcal{O}_{\tilde{X}}(1)/\tilde{\mathcal{E}}_i)_p \quad \text{and} \quad w_\alpha(\tilde{\mathcal{E}}_i) := \sum_{p \in \tilde{X}_\alpha} w(\tilde{\mathcal{E}}_i, p). \tag{3.2}$$

We remark that for $p \in \tilde{X}_\alpha$, $i + 1 \leqslant \hbar(p)$ is equivalent to $(\tilde{\mathcal{E}}_{i+1})_p \neq 0$.

Definition 3.3. For any irreducible component $X_\alpha \subset X$ we introduce

$$\mathbb{I}_\alpha = \mathbb{I}_\alpha(\lambda) = \{i \in \mathbb{I} \mid \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \neq \emptyset \text{ or } i = \hbar_\alpha\}; \tag{3.3}$$

for $m_\alpha + 1 = |\mathbb{I}_\alpha|$, the order of $\mathbb{I}_\alpha$, we introduce a reindexing map

$$\mathrm{ind}_\alpha : \mathbb{I}_\alpha \longrightarrow [0, m_\alpha] \cap \mathbb{Z} \quad \text{order preserving and bijective.} \tag{3.4}$$

Similarly, for $p \in \tilde{X}$, we introduce

$$\mathbb{I}_p = \{i \in \mathbb{I} \mid p \in \mathrm{inc}(\tilde{s}_i)\}.$$

For $m_p + 1 = |\mathbb{I}_p|$, we similarly define

$$\mathrm{ind}_p : \mathbb{I}_p \longrightarrow [0, m_p] \cap \mathbb{Z} \quad \text{order preserving and bijective.}$$

Definition 3.4. We call a 1-PS λ a *semi-staircase at index* i if

$$\mathcal{E}_i \supsetneq \mathcal{E}_{i+1} \supsetneq \cdots \supsetneq \mathcal{E}_m.$$

We call λ a *semi-staircase* when λ is a semi-staircase after index 1.

Proposition 3.5. *Given a 1-PS λ, there is a semi-staircase 1-PS λ' with $\rho'_i = \rho_i$ for all i so that $\omega(\lambda) \geqslant \omega(\lambda')$.*

Proof. Suppose λ is a semi-staircase at index i but *not* at $i - 1$; then

$$\mathcal{E}_0 \supseteq \cdots \supseteq \mathcal{E}_{i-2} \supseteq \mathcal{E}_{i-1} = \mathcal{E}_i \supsetneq \mathcal{E}_{i+1} \supsetneq \cdots \supsetneq \mathcal{E}_m. \tag{3.5}$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Therefore, there is a point $p \in X$ such that, if we denote by $\hat{s}_j \in \hat{\mathcal{O}}_{X,p}(1)$ the formal germ of s_j at p, then as $\hat{\mathcal{O}}_{X,p}$-modules,

$$\hat{\mathcal{O}}_{X,p}(1) \supset (\hat{s}_{i-1}, \ldots, \hat{s}_m) = (\hat{s}_i, \ldots, \hat{s}_m) \supsetneq (\hat{s}_{i+1}, \ldots, \hat{s}_m). \tag{3.6}$$

By the middle equality, we can find $\hat{c}_j \in \hat{\mathcal{O}}_{X,p}$ such that $\hat{s}_{i-1} = \sum_{j=i}^m \hat{c}_j \hat{s}_j$. We now construct a new basis $\mathbf{s}'$. Let $c = \hat{c}_i(p) \in \mathbb{k}$. We define

$$s'_j = s_j \quad \text{for } j \neq i, i-1; \quad s'_i = s_{i-1} - cs_i; \quad s'_{i-1} = s_i. \tag{3.7}$$

Clearly, $\mathbf{s}' = \{s'_i\}$ is a basis of $H^0(\mathcal{O}_X(1))$. Let $\mathcal{E}'_j$ be the $\mathcal{E}_i$ in (3.1) with s_i replaced by s'_i. For $j \neq i$, because the linear spans of $\{s_j, \ldots, s_m\}$ and of $\{s'_j, \ldots, s'_m\}$ are identical, we have $\mathcal{E}_j = \mathcal{E}'_j$. For i, we claim that $\mathcal{E}'_i \subsetneq \mathcal{E}_i$. The inclusion $\mathcal{E}'_i \subset \mathcal{E}_i$ follows from $\mathcal{E}'_i \subset \mathcal{E}_{i-1} = \mathcal{E}_i$. For the inequality, we claim that

$$(\hat{s}_{i-1} - c\hat{s}_i, \hat{s}_{i+1}, \ldots, \hat{s}_m) \neq (\hat{s}_i, \hat{s}_{i+1}, \ldots, \hat{s}_m).$$

Suppose instead the equality holds, then there are constants $a_j \in \mathbb{k}$ such that

$$\hat{s}_i = a_i(\hat{s}_{i-1} - c\hat{s}_i) + \sum_{j=i+1}^m a_j \hat{s}_j = \biggl(a_i(\hat{s}_{i-1} - \hat{c}_i\hat{s}_i) + \sum_{j=i+1}^m a_j \hat{s}_j\biggr) + a_i(\hat{c}_i - c)\hat{s}_i.$$

Combined with $\hat{s}_{i-1} = \sum_{j=i}^m \hat{c}_j \hat{s}_j$, we conclude that $\hat{s}_i \in (\hat{s}_{i+1}, \ldots, \hat{s}_m) + \hat{s}_i\mathfrak{m}$, where $\mathfrak{m} \subset \hat{\mathcal{O}}_{X,p}$ is the maximal ideal. By Nakayama's lemma, $\hat{s}_i \in (\hat{s}_{i+1}, \ldots, \hat{s}_m)$, contradicting (3.6). This proves the claim (cf. Figure 2).

Finally, we claim that if we define λ' be the 1-PS with diagonalizing basis $\mathbf{s}'$ and associated weights $\rho'_i = \rho_i$, then $\omega(\lambda') \leqslant \omega(\lambda)$. By Mumford's formula (cf. Proposition 2.1), this is equivalent to $e(\mathcal{I}(\lambda')) \geqslant e(\mathcal{I}(\lambda))$. By our construction, $\mathcal{E}'_i \subseteq \mathcal{E}_i$ for all $i \in \mathbb{I}$; hence since $\rho_{i-1} \geqslant \rho_i$, $\mathcal{I}(\lambda') \subset \mathcal{I}(\lambda)$. Thus $\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda')^k$ surjects onto $\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda)^k$. This proves $e(\mathcal{I}(\lambda')) \geqslant e(\mathcal{I}(\lambda))$.

In conclusion, for any λ that is not a semi-staircase (cf. the black part in Figure 1), we have constructed a new λ' whose associated filtration of subsheaves $\mathcal{E}'_j$ satisfying $\mathcal{E}'_j = \mathcal{E}_j$ for $j \neq i, i-1$, and

$$\mathcal{E}'_0 \supseteq \cdots \supseteq \mathcal{E}'_{i-2} \supseteq \mathcal{E}'_{i-1} \supsetneq \mathcal{E}'_i \supseteq \mathcal{E}'_{i+1} \supsetneq \cdots \supsetneq \mathcal{E}'_m.$$

If $\mathcal{E}'_i = \mathcal{E}'_{i+1}$ (cf. the grey part (blue online) in Figure 1), we repeat this process at $i+1$. Since we always have $\mathcal{E}_{m-1} \supsetneq \mathcal{E}_m$, after finitely many steps, we obtain a λ' that is a semi-staircase at $i-1$. An induction on i proves the proposition. □

Definition 3.6. We call a semi-staircase 1-PS λ a *staircase* if for any $p \in \tilde{\Lambda}$, $v(\tilde{s}_i, p) \leqslant v(\tilde{s}_{i+1}, p)$ for all i (cf. Definition 2.3).

Proposition 3.7. *Given a 1-PS λ, there is a staircase 1-PS λ' with $\rho'_i = \rho_i$ for all i so that $\omega(\lambda) \geqslant \omega(\lambda')$.*

Proof. By Proposition 2.1, the λ-weight $\omega(\lambda)$ (of Chow(X)) depends only the sheaf $\mathcal{I}(\lambda)$ and the weights $\{\rho_i\}$. Thus, for any 1-PS λ' with $\mathcal{I}(\lambda) = \mathcal{I}(\lambda')$ and having weights $\{\rho'_i\}$ identical to those of λ, we have $\omega(\lambda) = \omega(\lambda')$.

Given any 1-PS, we let λ be the corresponding semi-staircase constructed in Proposition 3.5. Let $\tilde{\Lambda}$ and $\{s_i\}$ be the associated objects of λ. Since $\tilde{\Lambda}$ is a finite set, if we replace s_i by $s'_i = s_i + \sum_{j>i} c_{ij}s_j$ for a general choice of $c_{ij} \in \mathbb{k}$, the new 1-PS with the same $\{\rho_i\}$ but new basis $\{s'_i\}$ will be the desired *staircase* 1-PS (cf. Figure 3). □

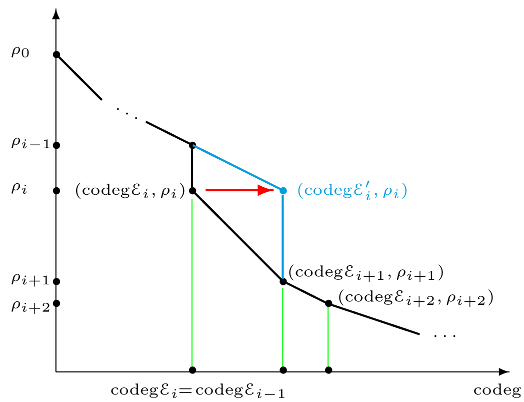

FIGURE 1. (Color online) The figure shows how the lattice points, from which the Newton polytope is constructed, vary in the process of constructing a semi-staircase. Here $\operatorname{codeg}(\mathcal{E}_i) = \operatorname{length}(\mathcal{O}_Y(1)/\mathcal{E}_i|_Y) + \deg \mathcal{O}_{Y^\complement}(1)$, with $Y = \operatorname{Supp}(\mathcal{E}_i)$ and $Y^\complement = \overline{X \backslash Y}$. In particular, one notices that it is possible that after one step a semi-staircase at index i becomes a semi-staircase at index $i+1$ instead of $i-1$.

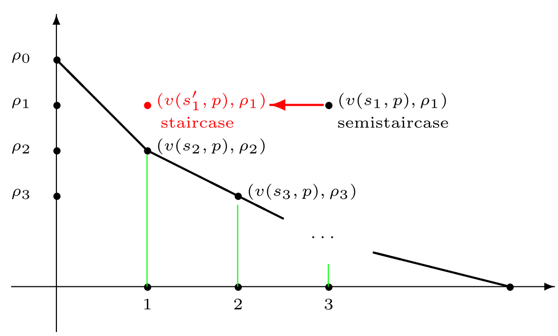

FIGURE 2. (Color online) The figure shows how vanishing order $v(s_1, p)$ varies under the general perturbation of the section s_1 when one creates a staircase from a semi-staircase.

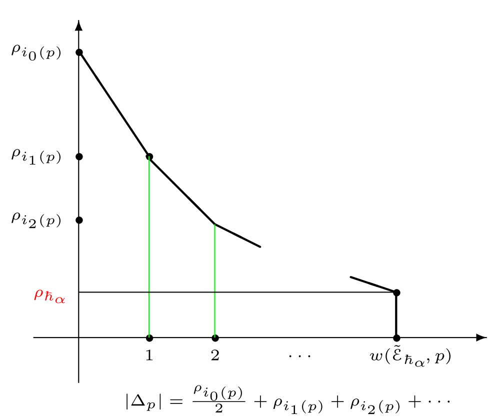

FIGURE 3. (Color online) The shape for a typical Newton polygon and its area.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Lemma 3.8. *Suppose λ is a staircase 1-PS; then for any $p \in \tilde{X}_\alpha$ and $i \leqslant \hbar_\alpha$, $w(\tilde{\mathcal{E}}_i, p) = v(\tilde{s}_i, p)$, and $\delta(\tilde{s}_{i-1}, p) = v(\tilde{s}_i, p) - v(\tilde{s}_{i-1}, p)$.*

Proof. As $i \leqslant \hbar_\alpha$, both $\tilde{s}_i$ and $\tilde{s}_{i-1}$ are restricted non-trivially to $\tilde{X}_\alpha$. The identity is a direct consequence of the definition of staircase 1-PS. □

As we will see, if λ is a *staircase* 1-PS then for most of i, $\delta(\tilde{s}_i) = 1$. For those i with $\delta(\tilde{s}_i) > 1$, we will give a detailed characterization (cf. Proposition 3.9). To this purpose, for any subcurve $Y \subset X$, we let $N_Y = X_{\text{node}} \cap Y$ be the set of *nodes* of X in Y. We let (recall $L_Y := Y \cap Y^\complement$ cf. (1.3))

$$\tilde{N}_Y := \pi^{-1}(N_Y) \cap \tilde{Y} \quad \text{and} \quad \tilde{L}_Y := \pi^{-1}(L_Y) \cap \tilde{Y} \subset \tilde{N}_Y. \tag{3.8}$$

As α is reserved for the index of the components X_α, we abbreviate

$$N_\alpha := N_{X_\alpha}, \quad \tilde{N}_\alpha := \tilde{N}_{X_\alpha}, \quad L_\alpha := L_{X_\alpha}, \quad \tilde{L}_\alpha := \tilde{L}_{X_\alpha}, \quad \ell_\alpha := |L_\alpha|. \tag{3.9}$$

Proposition 3.9. *Suppose λ is a* staircase *1-PS. Let $i \in \mathbb{I}_\alpha$ be a non-base index (cf. Definition 3.1) and $p \in \operatorname{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$. Suppose $\delta(\tilde{s}_i) \geqslant 2$ and that either $\deg X_\alpha = 1$ or*

$$w_\alpha(\tilde{\mathcal{E}}_i) + 1 \leqslant \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha, \tag{3.10}$$

then $q = \pi(p) \in X$ is a node of X, $\operatorname{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$. In this case, let $\{p, p'\} = \pi^{-1}(q)$ and let $\tilde{X}_\beta$ be a component satisfying $p' \in \operatorname{inc}(\tilde{s}_i) \cap \tilde{X}_\beta$ (possibly $\tilde{X}_\alpha = \tilde{X}_\beta$); assuming $\deg X_\beta > 1$ and $w_\beta(\tilde{\mathcal{E}}_i) + 1 \leqslant \deg X_\beta - 2g(X_\beta) - \ell_\beta$, then $\operatorname{inc}(\tilde{s}_i) = p + p'$.

Proof. We adopt the following convention. Since $\tilde{X}_\alpha$ is smooth, we can view a zero-subscheme of $\tilde{X}_\alpha$ as a divisor as well. This way, the union of two effective divisors is the union as zero-subschemes, and the sum is as sum of divisors. For example, $(\sum n_p p) \cup (\sum n'_p p) = \sum \max\{n_p, n'_p\} p$ and $(\sum n_p p) + (\sum n'_p p) = \sum (n_p + n'_p) p$.

We will prove each part of the statement by repeatedly applying the following strategy. Suppose i satisfies (3.10) and $\delta(\tilde{s}_i) \geqslant 2$; we will construct a section $\zeta \in H^0(\mathcal{O}_X(1))$ so that the $\mathcal{O}_X$-modules $\mathcal{F}_j = (\zeta, s_j, \ldots, s_m)$ fit into a strict filtration

$$\mathcal{F}_0 \supsetneq \cdots \supsetneq \mathcal{F}_i \supsetneq \mathcal{F}_{i+1} \supsetneq \mathcal{E}_{i+1} \supsetneq \cdots \supsetneq \mathcal{E}_m \neq 0. \tag{3.11}$$

Since $\mathcal{E}_j$ and $\mathcal{F}_j$ are generated by global sections of $H^0(\mathcal{O}_X(1))$, this implies $h^0(\mathcal{O}_X(1)) \geqslant m + 2$, a contradiction.

We first assume $\deg X_\alpha > 1$. Then $w_i(\tilde{\mathcal{E}}_i)$ satisfies (3.10). We recall an easy consequence of a vanishing result. Let $B \subset \tilde{X}_\alpha$ be a closed zero-subscheme satisfying

$$\deg B \leqslant \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha + 1; \tag{3.12}$$

let $\tilde{N}_\alpha$ be as defined in (3.9). We claim that the γ in the exact sequence

$$H^0(\mathcal{O}_{\tilde{X}_\alpha}(1)) \xrightarrow{\gamma} H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \longrightarrow H^1(\mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B)) \tag{3.13}$$

is surjective. Indeed, this follows from $\deg \tilde{N}_\alpha = 2g(X_\alpha) - 2g(\tilde{X}_\alpha) + \ell_\alpha$ and (3.12), which gives $\deg \mathcal{O}_{\tilde{X}_\alpha}(1)(-\tilde{N}_\alpha \cup B) \geqslant 2g(\tilde{X}_\alpha) - 1$, and thus the last term in (3.13) vanishes.

The section ζ mentioned before (3.11) will be chosen by picking an appropriate B and $v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1))$ so that any element $\tilde{\zeta}_\alpha \in \gamma^{-1}(v)$ descends to a section in $H^0(\mathcal{O}_{X_\alpha}(1))$ which glues with $s_{i+1}|_{X_\alpha^\complement}$ to form the desired section ζ.

We let

$$\tilde{Z}_{\alpha,j} := (\tilde{s}_j = \cdots = \tilde{s}_m = 0) \cap \tilde{X}_\alpha \subset \tilde{X}_\alpha. \tag{3.14}$$

Since $p \in \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, $\delta_\alpha(\tilde{s}_i) \geqslant 1$. In the case $\delta_\alpha(\tilde{s}_i) = 1$, we choose $B = \tilde{Z}_{\alpha,i} + p$, which is a subscheme of $\tilde{Z}_{\alpha,i+1}$. In the case $\delta_\alpha(\tilde{s}_i) \geqslant 2$ and $\delta(\tilde{s}_i, p) = 1$, there exists a $p' \neq p \in \tilde{X}_\alpha$ such that $p + p' \leqslant \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$, (which is equivalent to $\tilde{Z}_{\alpha,i} + p + p' \subset \tilde{Z}_{\alpha,i+1}$). In the case $\delta(\tilde{s}_i, p) \geqslant 2$, we choose $p' = p$. Combined, we let $B = \tilde{Z}_{\alpha,i} + p + p'$.

We then let

$$v_1 = \tilde{s}_{i+1}|_{\tilde{N}_\alpha} \in H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \quad \text{and} \quad v_2 \neq 0 \in H^0(\mathcal{O}_B(1)) \quad \text{s.t. } v_2|_{B-p} = 0.$$

We claim that when $p \notin \tilde{N}_\alpha$, or $\mathrm{ind}_p(i) \geqslant 1$, or $\delta(\tilde{s}_i, p) \geqslant 2$, then both $v_1|_{\tilde{N}_\alpha \cap B}$ and $v_2|_{\tilde{N}_\alpha \cap B}$ are zero.

Indeed, since $\tilde{N}_\alpha \cap B \subset \tilde{Z}_{\alpha,i+1}$ and $\tilde{s}_{i+1}|_{\tilde{Z}_{\alpha,i+1}} = 0$, we have $v_1|_{\tilde{N}_\alpha \cap B} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha \cap B} = 0$. For v_2, we prove case by case. Suppose $p \notin \tilde{N}_\alpha$, then $\tilde{N}_\alpha \cap B = \tilde{N}_\alpha \cap (B - p)$; therefore, since $v_2|_{B-p} = 0$, $v_2|_{\tilde{N}_\alpha \cap B} = 0$. Now suppose $p \in \tilde{N}_\alpha$. Since $v_2|_{B-p} = 0$, $v_2(\bar{p}) = 0$ for all $\bar{p} \in (\tilde{N}_\alpha \cap B) - \{p\}$. It remains to show that $v_2(p) = 0$. We write $B = \sum_{k=0}^{l} n_k p_k$, p_k distinct, as an effective divisor. Since $p \in B$, we can arrange $p_0 = p$. In the case $\mathrm{ind}_p(i) \geqslant 1$, we have $n_0 \geqslant 2$; in the case $\delta(\tilde{s}_i, p) \geqslant 2$, since $p' = p$, we still have $n_0 \geqslant 2$. Thus $p \in B - p$ and $v_2(p) = 0$. This proves that v_1 and v_2 have identical images in $H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1))$. Consequently, (v_1, v_2) lifts to a section $v \in H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1))$ using the exact sequence

$$H^0(\mathcal{O}_{\tilde{N}_\alpha \cup B}(1)) \longrightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha}(1)) \oplus H^0(\mathcal{O}_B(1)) \longrightarrow H^0(\mathcal{O}_{\tilde{N}_\alpha \cap B}(1)).$$

Since $\deg B \leqslant w_\alpha(\tilde{\mathcal{E}}_i) + 2$ and i satisfies (3.10), (by the assumption that $\deg X_\alpha > 1$), $\deg B$ satisfies the inequality (3.12). Therefore, the γ in (3.13) is surjective. We let $\tilde{\zeta}_\alpha \in \gamma^{-1}(v) \subset H^0(\mathcal{O}_{\tilde{X}_\alpha}(1))$ be any lift. Because it is a lift of v_1, $\tilde{\zeta}_\alpha|_{\tilde{N}_\alpha} = \tilde{s}_{i+1}|_{\tilde{N}_\alpha}$. This implies that $\tilde{\zeta}_\alpha$ descends to a section $\zeta_\alpha \in H^0(\mathcal{O}_{X_\alpha}(1))$, and the descent ζ_α glues with $s_{i+1}|_{X_\alpha^\complement}$ to form a new section $\zeta \in H^0(\mathcal{O}_X(1))$.

We now prove the first part of the proposition. We let $Z_{\alpha,j} \subset X_\alpha$ be the subscheme $Z_{\alpha,j} = (s_j = \cdots = s_m = 0) \cap X_\alpha$. We decompose $Z_{\alpha,j}$ into the disjoint union $Z_{\alpha,j} = R_j \cup R'_j$ so that R_j is supported at $q = \pi(p)$ and R'_j is disjoint from q. We let $\bar{Z}_\alpha = (\zeta = s_{i+1} = \cdots = s_m = 0) \cap X_\alpha$ and decompose $\bar{Z}_\alpha = \bar{R} \cup \bar{R}'$ accordingly.

Suppose q is a smooth point of X. Then R_j and $\bar{R}$ are divisors and can be written as $R_j = n_j q$ and $\bar{R} = \bar{n} q$. In the case $\delta_\alpha(\tilde{s}_i) = 1$, the choice of B ensures that $n_i = \bar{n} = n_{i+1} - 1$ and $R'_i \subset \bar{R}' \subsetneq R'_{i+1}$. Thus,

$$(s_i, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supset (\zeta, s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supsetneq (s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha}.$$

Further, since $\delta(\tilde{s}_i) \geqslant 2$ and $\zeta|_{X_\alpha^\complement} = s_{i+1}|_{X_\alpha^\complement}$, we have

$$(s_i, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\complement} \supsetneq (\zeta, s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\complement} \supset (s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha^\complement}.$$

Thus, we have

$$\mathcal{E}_i \supsetneq \mathcal{F}_{i+1} \supsetneq \mathcal{E}_{i+1}. \tag{3.15}$$

In the case $\delta_\alpha(\tilde{s}_i) \geqslant 2$, the choice of B ensures that $R_i \subsetneq \bar{R} \subsetneq R_{i+1}$. Thus,

$$(s_i, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supsetneq (\zeta, s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha} \supsetneq (s_{i+1}, \ldots, s_m) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_\alpha}.$$

This implies (3.15) as well. In summary, by the argument at the beginning of the proof, (3.15) leads to a contradiction which proves that q must be a node of X.

It remains to study when q is a node of X. A careful case by case study shows that when either $\mathrm{ind}_p(i) \geqslant 1$ or $\delta(\tilde{s}_i, p) \geqslant 2$, then $Z_{\alpha,i} \subsetneq \bar{Z}_\alpha \subsetneq Z_{\alpha,i+1}$. Thus (3.15) holds, which leads to a contradiction. This proves that when q is a node, $\mathrm{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$.

We complete the proof of the first part by looking at the case $\deg X_\alpha = 1$. In this case $\mathrm{ind}_p(i) = 0$ and $\delta(\tilde{s}_i, p) = 1$, since otherwise $\deg X_\alpha = 1$ implies that $i = \hbar_\alpha$, contradicting the assumption that i is not a base index. We next show that $p \in L_\alpha$. But this is parallel to the proof of the case $\deg X_\alpha > 1$ by letting $B = p$ because $\delta_\alpha(\tilde{s}_i) = 1$. This completes the proof of the first part.

We now prove the second part. Let $\pi^{-1}(q) = \{p, p'\}$ with $p' \in \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\beta$ so that the assumption on X_β holds. Then by the first part of the proposition, we have $\mathrm{ind}_p(i) = \mathrm{ind}_{p'}(i) = 0$; hence $s_i(q) \neq 0$. Thus, for $Z_j = (s_j = \cdots = s_m = 0) \subset X$, we have $p \notin Z_i$ and $Z_{i+1} = p \cup S$, where S is a zero-subscheme disjoint from p. Since $Z_i \subsetneq Z_{i+1}$ and $p \notin Z_i$, we have $Z_i \subset S$. In the case $Z_i = S$, the second part of the proposition holds. Suppose $Z_i \subsetneq S$; then, repeating the proof of the first part of the proposition, we can find a section $\zeta \in H^0(\mathcal{O}_X(1))$ such that $p \notin (\zeta = 0)$ and $S \subset (\zeta = 0)$. This way, we will have (3.15) again, which leads to a contradiction. This proves the second part of the proposition. □

The proposition above motivates the following definition.

DEFINITION 3.10. For $\deg X_\alpha > 1$, we define the *primary* indices of X_α to be

$$\mathbb{I}_\alpha^{\mathrm{pri}} = \{i \in \mathbb{I}_\alpha \mid w_\alpha(\tilde{\mathcal{E}}_{i+1}) \leqslant \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1\};$$

for $\deg X_\alpha = 1$, we define $\mathbb{I}_\alpha^{\mathrm{pri}} = \mathrm{ind}_\alpha^{-1}(0) \subset \mathbb{I}_\alpha$. We say $i \in \mathbb{I}_\alpha$ is *primary* at $p \in \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$ if $i \in \mathbb{I}_\alpha^{\mathrm{pri}}$; otherwise we say it is *secondary*. We define $\bar{\jmath}_\alpha := \max\{i \mid i \in \mathbb{I}_\alpha^{\mathrm{pri}}\}$.

Note that, in the proof above, the assumption $\delta(\tilde{s}_i) \geqslant 2$ is used only to show that (3.11) is strict. If $i = \hbar_\alpha$ for some α, then $\mathrm{length}(\mathcal{E}_i/\mathcal{E}_{i+1}) = \infty$. This time we choose ζ so that $\mathcal{E}_i/\mathcal{F}_{i+1}$ is finite. Since $\mathcal{E}_i/\mathcal{E}_{i+1}$ is infinite, (3.11) remains strict. Hence we have the following proposition.

PROPOSITION 3.11. *Let i be a* base index *(cf. Definition 3.1), and let $p \in \mathrm{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha$. Suppose either $\delta(\tilde{s}_i) \geqslant 1$ and $\deg X_\alpha = 1$ or $w_\alpha(\tilde{\mathcal{E}}_i)$ satisfies the inequality (3.10). Then $\mathrm{ind}_p(i) = 0$, $\delta(\tilde{s}_i, p) = 1$, and $q = \pi(p) \in X_\alpha$ is a linking node of X_α. Further, let $\{p, p'\} = \pi^{-1}(q)$; then i must be secondary at p' (cf. Definition 3.10), and there is a component $\tilde{X}_\beta$ so that $p' \in \tilde{X}_\beta$ and $i = \hbar_\beta$.*

Proof. The proof is parallel to the proof of the previous proposition. We will omit it here. □

COROLLARY 3.12. *Defining $w_\alpha^{\mathrm{pri}} := w_\alpha(\tilde{\mathcal{E}}_{\bar{\jmath}_\alpha+1})$, suppose $X_\alpha \subsetneq X$; then*

$$0 \leqslant \deg X_\alpha - w_\alpha^{\mathrm{pri}} \leqslant 2(g(X_\alpha) + \ell_\alpha + 1). \tag{3.16}$$

Proof. The first inequality is trivial. We now prove the second one. If $\deg X_\alpha = 1$ we obtain $\deg X_\alpha - w_\alpha^{\mathrm{pri}} = 0$, from which the second inequality follows. So from now on we assume $\deg X_\alpha > 1$. We let $\bar{\imath} \in \mathbb{I}_\alpha$ be the index succeeding $\bar{\jmath}_\alpha$; namely, $\bar{\imath}$ is the smallest index $> \bar{\jmath}_\alpha$ so that $\delta_\alpha(\tilde{s}_{\bar{\imath}}) \geqslant 1$. In particular, this implies that

$$\delta_\alpha(\tilde{s}_{\bar{\jmath}_\alpha}) = \cdots = \delta_\alpha(\tilde{s}_{\bar{\imath}-1}) = 0. \tag{3.17}$$

Since $\bar{i} \notin \mathbb{I}_\alpha^{\text{pri}}$,

$$w_\alpha^{\text{pri}} = w_\alpha(\tilde{\mathcal{E}}_{\bar{\jmath}_\alpha+1}) = w_\alpha(\tilde{\mathcal{E}}_{\bar{i}+1}) - \delta_\alpha(\tilde{s}_{\bar{i}}) > \deg X_\alpha - 2g(X_\alpha) - \ell_\alpha - 1 - \delta_\alpha(\tilde{s}_{\bar{i}}). \tag{3.18}$$

Thus, when $\delta_\alpha(\tilde{s}_{\bar{i}}) \leqslant 2$, the second inequality follows from $\ell_\alpha \geqslant 1$ (since $X_\alpha \subsetneq X$).

Suppose $\delta_\alpha(\tilde{s}_{\bar{i}}) > 2$. By our assumption, $\bar{i}$ is the index in $\mathbb{I}_\alpha$ immediately succeeding $\bar{\jmath}_\alpha$, and thus we have $w_\alpha(\tilde{\mathcal{E}}_{\bar{i}}) = w_\alpha(\tilde{\mathcal{E}}_{\bar{\jmath}_\alpha+1})$ because of (3.17). By Definition 3.10, $w_\alpha(\tilde{\mathcal{E}}_{\bar{i}})$ satisfies (3.10). So we can apply Proposition 3.9 to the index $\bar{i}$ to conclude that every $p \in \operatorname{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha$ lies in $\tilde{N}_\alpha$ and has $\delta(\tilde{s}_{\bar{i}}, p) = 1$.

We claim that $\operatorname{inc}(\tilde{s}_i) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha$. Indeed, let $p \in \operatorname{inc}(\tilde{s}_{\bar{i}}) \cap (\tilde{N}_\alpha \backslash \tilde{L}_\alpha)$; then the second part of Proposition 3.9 implies that $\operatorname{inc}(\tilde{s}_{\bar{i}}) = p + p'$ and $\delta(\tilde{s}_{\bar{i}}) = 2$, contradicting the assumption $\delta_\alpha(\tilde{s}_{\bar{i}}) > 2$. This proves that $\operatorname{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha \subset \tilde{L}_\alpha$. Adding that $\delta(\tilde{s}_{\bar{i}}, p) = 1$ for $p \in \operatorname{inc}(\tilde{s}_{\bar{i}}) \cap \tilde{X}_\alpha$, we conclude that $\delta_\alpha(\tilde{s}_{\bar{i}}) \leqslant \ell_\alpha$. These and (3.18) prove the second inequality in (3.16). □

4. Main estimate for irreducible curves

Throughout this section, we fix a staircase 1-PS λ and an irreducible X_α. We will derive an estimate of $e(\tilde{\mathfrak{I}}_\alpha(\lambda))$ for the $X_\alpha \subset X$.

We let g_α be the genus of X_α; we define the set of *special points*

$$\tilde{S}_\alpha = (\pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha) \cup \tilde{N}_\alpha \subset \tilde{X}_\alpha, \tag{4.1}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \subset X$ is the set of weighted points. We continue to write $\bar{\rho}_i = \rho_i - \rho_{\hbar_\alpha}$. For each $p \in \tilde{\Lambda}_\alpha$, we define the *initial index*

$$i_0(p) := \min\{i \mid i \in \mathbb{I}_p\}. \tag{4.2}$$

Given a fixed $\epsilon > 0$, we define

$$E_\alpha^\epsilon(\rho) := \left(2 + \frac{2\epsilon}{\deg X_\alpha}\right) \sum_{i \in \mathbb{I}_\alpha^{\text{pri}}} \delta_\alpha(\tilde{s}_i)\bar{\rho}_i - \left(1 + \frac{2\epsilon}{\deg X_\alpha}\right) \sum_{q \in \tilde{S}_\alpha \cap \tilde{\Lambda}_\alpha} \bar{\rho}_{i_0(q)} + 2\deg X_\alpha \cdot \rho_{\hbar_\alpha} \tag{4.3}$$

for $\deg X_\alpha > 1$; for $\deg X_\alpha = 1$, we define

$$E_\alpha^\epsilon(\rho) := \delta_\alpha(\tilde{s}_{i_0})\bar{\rho}_{i_0} + 2 \cdot \rho_{\hbar_\alpha}; \quad i_0 = \operatorname{ind}_\alpha^{-1}(0). \tag{4.4}$$

It is clear that in both cases $E_\alpha^\epsilon(\rho)$ is linear in $\rho \in \mathbb{R}_+^{m+1}$. Our main result of this section is the following theorem.

Theorem 4.1. *For any $1 \geqslant \epsilon > 0$ there is a constant $M_1 = M_1(g_\alpha, \ell_\alpha, n, \epsilon)$, which is a rational function of g_α, ℓ_α, n and ϵ, such that either when $\deg X_\alpha \geqslant M_1$ or when $\deg X_\alpha = 1$ we have*

$$e(\tilde{\mathfrak{I}}_\alpha(\lambda)) \leqslant E_\alpha^\epsilon(\rho).$$

Note that the theorem implies that we can bound $e(\tilde{\mathfrak{I}}(\lambda))$ in terms of the primary ρ_i only, with an additional margin related to the markings and nodes. This extra margin will be crucial to study the stability of curves with nodes and markings.

We begin with a useful bound on the area of Δ_p.

Lemma 4.2. *Let 1-PS λ be a staircase. Then for any $p \in \tilde{\Lambda}_\alpha$ we have*

$$|\Delta_p| - \rho_{\hbar_\alpha} \cdot w(\tilde{\mathcal{E}}_{\hbar_\alpha}, p) \leqslant \sum_{i \in \mathbb{I}_p} \delta(\tilde{s}_i, p)\bar{\rho}_i - \frac{\bar{\rho}_{i_0(p)}}{2}. \tag{4.5}$$

Proof. Let $0 \leqslant l \leqslant k \leqslant \hbar_\alpha$; let $i_{\min} := \min(\mathbb{I}_p \cap [l,k])$ and $i_{\max} := \max(\mathbb{I}_p \cap [l,k])$; we prove

$$\begin{aligned} &|\Delta_p \cap ([w(\tilde{\mathcal{E}}_l,p), w(\tilde{\mathcal{E}}_k,p)] \times \mathbb{R})| - \rho_{\hbar_\alpha} \cdot (w(\tilde{\mathcal{E}}_k,p) - w(\tilde{\mathcal{E}}_l,p)) \\ &\quad \leqslant \sum_{i \in \mathbb{I}_p \cap [l,k]} \delta(\tilde{s}_i,p)\bar{\rho}_i - \frac{(\bar{\rho}_{i_{\min}(p)} + \bar{\rho}_{i_{\max}(p)})}{2}. \end{aligned} \tag{4.6}$$

Note that by letting $l = 0$ and $k = \hbar_\alpha$, we obtain the lemma.

We prove (4.6). As it is invariant when varying $\rho_{\hbar_\alpha}$, without loss of generality we assume $\rho_{\hbar_\alpha} = 0$; hence $\bar{\rho}_i = \rho_i$. Let $\Gamma_p := \{(w(\tilde{\mathcal{E}}_i,p), \rho_i)\}_{0 \leqslant i \leqslant m, w(\tilde{\mathcal{E}}_i,p) \neq \infty}$; it follows from Definitions 2.7 and 3.6 that

$$\Delta_p = (\mathbb{R}^2_+ - \mathrm{Conv}(\mathbb{R}^2_+ + \Gamma_p)) \cap ([0, w(\tilde{\mathcal{I}},p)] \times \mathbb{R}). \tag{4.7}$$

Fixing an indexing

$$\mathbb{I}_p = \{i_0(p), \ldots, i_d(p)\} \subset \mathbb{I}, \quad i_j(p) \text{ increasing and } d+1 = |\mathbb{I}_p|, \tag{4.8}$$

we let $\mathbb{T}$ be the continuous piecewise linear function on $[0, w(\tilde{\mathcal{I}},p)]$ defined by linearly interpolating the points

$$\{(0, \rho_{i_0}), \ldots, (w(\tilde{\mathcal{E}}_{i_k},p), \rho_{i_k}), \ldots, (w(\tilde{\mathcal{E}}_{i_d},p), \rho_{\hbar_\alpha})\} \subset \mathbb{R}^2,$$

and we let $\Delta_{\mathbb{T}}$ be the polygon bounded on two sides by $x = 0$ and $x = w(\tilde{\mathcal{E}}_k,p)$, from below by $y = 0$ and from above by the graph of $y = \mathbb{T}$. By the convexity of Δ_p, we have

$$\Delta_p \cap ([w(\tilde{\mathcal{E}}_l,p), w(\tilde{\mathcal{E}}_k,p)] \times \mathbb{R}) \subset \Delta_{\mathbb{T}} \cap ([w(\tilde{\mathcal{E}}_l,p), w(\tilde{\mathcal{E}}_k,p)] \times \mathbb{R}) \subset \mathbb{R}^2.$$

By Lemma 3.8, $w(\tilde{\mathcal{E}}_i,p) = \sum_{j=0}^{i-1} \delta(\tilde{s}_j,p)$; hence

$$\begin{aligned} &|\Delta_p \cap ([w(\tilde{\mathcal{E}}_l,p), w(\tilde{\mathcal{E}}_k,p)] \times \mathbb{R})| \leqslant |\Delta_{\mathbb{T}} \cap ([w(\tilde{\mathcal{E}}_l,p), w(\tilde{\mathcal{E}}_k,p)] \times \mathbb{R})| \\ &\quad = \sum_{i \in \mathbb{I}_p \cap [l,k]} \delta(\tilde{s}_i,p)\rho_i - \frac{1}{2}(\rho_{i_{\min}(p)} + \rho_{i_{\max}(p)}). \end{aligned}$$

This proves (4.6), and the lemma. □

The idea of the proof of the theorem is as follows: when $|\tilde{\Lambda}_\alpha|$ (cf. Definition 3.1) is large, applying Lemma 4.2, we gain a sizable multiple of $\frac{1}{2}\rho_{i_0(p)}$ (cf. (4.5) and Figure 5) in the estimate of Δ_p; these extra gains will take care of the contributions from the non-primary ρ_i. When $|\tilde{\Lambda}_\alpha|$ is small, one large Δ_p (cf. Figure 4) is sufficient to cancel the contribution from the non-primary ρ_i.

We need a few more notions. For any $p \in \tilde{\Lambda}_\alpha$, we let $\mathbb{I}_p^{\mathrm{pri}} := \mathbb{I}_\alpha^{\mathrm{pri}} \cap \mathbb{I}_p$ and define

$$\bar{\jmath}_p := \max\{i \in \mathbb{I}_p^{\mathrm{pri}}\}, \quad w^{\mathrm{pri}}(p) := w(\tilde{\mathcal{E}}_{\bar{\jmath}_\alpha+1},p) \quad \text{and} \quad w(p) := w(\tilde{\mathcal{I}},p) \text{ (cf. (2.9))}. \tag{4.9}$$

Note that $w(p)$ is the base-width of the Newton polygon Δ_p. Using $\bar{\jmath}_p$, we truncate the Newton polygon Δ_p by intersecting it with the strip $[0, w^{\mathrm{pri}}(p)] \times \mathbb{R}$:

$$\Delta_p^{\mathrm{pri}} := \Delta_p \cap [0, w^{\mathrm{pri}}(p)] \times \mathbb{R}.$$

Our next lemma says that if one Δ_p is big enough, the contribution from the non-primary ρ_i can be absorbed by the difference between $E_\alpha^\epsilon(\rho)$ and $e(\tilde{\mathcal{I}}_\alpha(\rho))$. Recall that w_α^{pri} is defined in Corollary 3.12.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

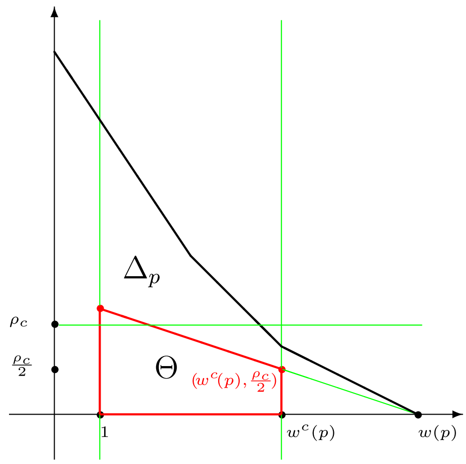

FIGURE 4. (Color online) A *big* Newton polygon at point p.

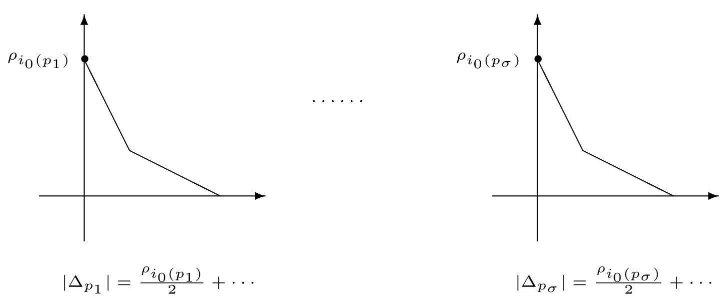

FIGURE 5. Newton polygons supported at many points gain us a lot of $\rho_{i_0(p)}/2$.

LEMMA 4.3. *For any* $1 > \epsilon > 0$, *there is an* $M = M(g_\alpha, \ell_\alpha, \epsilon)$ *such that whenever* $w(p) \geqslant M$ *(cf. (4.9)),*

$$|\Delta_p^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\bar{\rho}_{\bar{\jmath}_\alpha} \leqslant \left(1 + \frac{\epsilon}{w(p)}\right) \sum_{i \in \mathbb{I}_p^{\mathrm{pri}}} \delta(\tilde{s}_i, p)\bar{\rho}_i + \rho_{\hbar_\alpha} \cdot w^{\mathrm{pri}}(p) - \left(\frac{1}{2} + \frac{\epsilon}{w(p)}\right)\bar{\rho}_{i_0(p)}.$$

Proof. We make a simple simplification. Similarly to the proof of Lemma 4.2, we assume $\rho_{\hbar_\alpha} = 0$; hence $\bar{\rho}_i = \rho_i$. Our proof is based on studying the proximity of $\partial^+\Delta_p$ ($\partial^+\Delta_p$ is the boundary component of Δ_p lying in the (open) first quadrant) with the lattice points $(w(\tilde{\mathcal{E}}_i, p), \rho_i)$ (cf. (3.2)). In the case where they differ slightly, the term $(\epsilon/w(p)) \sum_{i \in \mathbb{I}_p^{\mathrm{pri}}} \delta(\tilde{s}_i, p)\bar{\rho}_i$ is sufficient to absorb the term $2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha}$ in the inequality (note $\bar{\rho}_i = \rho_i$, by assumption). Otherwise, the difference between $\sum_{i \in \mathbb{I}_p^{\mathrm{pri}} \cap [c, \bar{\jmath}_p]} \delta(\tilde{s}_i, p)\bar{\rho}_i$ (for some c that will be specified below) and $|\Delta_p|$ is sufficient to imply the desired estimate.

We assume $M > 4$; then $w(p) - \sqrt{w(p)} \geqslant 2$ whenever $w(p) \geqslant M$. We introduce

$$c = \max\left\{ i \in \mathbb{I}_p^{\mathrm{pri}} \;\middle|\; \left(w(\tilde{\mathcal{E}}_i, p), \frac{\rho_i}{2}\right) \in \Delta_p \subset \mathbb{R}^2 \right\}$$

and let $w^c(p) := w(\tilde{\mathcal{E}}_c, p)$ and $\Delta_p^{\leqslant c} = \Delta_p \cap [0, w^c(p)] \times \mathbb{R}$.

We divide our study into two cases. The first is when $w(p) - w^c(p) \leqslant \sqrt{w(p)}$, which implies $w^c(p) - 1 \geqslant w(p) - \sqrt{w(p)} - 1 \geqslant (w(p) - \sqrt{w(p)})/2$. We let Θ be the trapezoid that is bounded on two sides by $x = 1$ and $x = w^c(p)$, from below by $y = 0$ and from above by the line passing through $(w(p), 0)$ and $(w^c(p), \rho_c/2)$. Since the length of its two vertical edges are $\rho_c/2$ and $((w(p)-1)/(w(p)-w^c(p)))\cdot(\rho_c/2)$, a simple estimate gives

$$|\Theta| \geqslant \left(\frac{\sqrt{w(p)}}{2}+1\right)\cdot\frac{w(p)-\sqrt{w(p)}}{2}\cdot\frac{\rho_c}{4} \geqslant \frac{w(p)^{3/2}\cdot\rho_c}{32}.$$

Since the piecewise linear $\partial^+\Delta_p$ is convex, Θ lies inside Δ_p, and hence

$$|\Delta_p| - \frac{\rho_{i_0(p)}}{2} > |\Theta| > \frac{w(p)^{3/2}}{32}\rho_c.$$

By the definition of Δ_p^{pri}, the difference between the base-width of Δ_p^{pri} and that of Δ_p is bounded by $w(p) - w^{\mathrm{pri}}(p)$; therefore by Lemma 4.2 we have

$$|\Delta_p^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \geqslant |\Delta_p^{\mathrm{pri}}| + (w(p)-w^{\mathrm{pri}}(p))\rho_{\bar{\jmath}_\alpha} \geqslant |\Delta_p| \geqslant \frac{w(p)^{3/2}}{32}\rho_c + \frac{\rho_{i_0(p)}}{2}.$$

Since $\rho_{\bar{\jmath}_\alpha} \leqslant \rho_c$, this implies

$$|\Delta_p^{\mathrm{pri}}| - \frac{\rho_{i_0(p)}}{2} > \left(\frac{w(p)^{3/2}}{32} - 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\right)\rho_c. \tag{4.10}$$

We now choose M so that $M^{3/2} \geqslant 2^8(g_\alpha + \ell_\alpha + 1)$. By Corollary 3.12, we have $\deg X_\alpha - w_\alpha^{\mathrm{pri}} \leqslant 2(g_\alpha + \ell_\alpha + 1)$. Therefore, when $w(p) \geqslant M$, we have

$$2(\deg X_\alpha - w_\alpha^{\mathrm{pri}}) \leqslant 4(g_\alpha + \ell_\alpha + 1) \leqslant \frac{w(p)^{3/2}}{64}.$$

Plugging this into (4.10), we obtain $\rho_c \leqslant (2^6/w(p)^{3/2})(|\Delta_p^{\mathrm{pri}}| - \rho_{i_0(p)}/2)$. Hence

$$2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \leqslant 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_c \leqslant \frac{2^6(\deg X_\alpha - w_\alpha^{\mathrm{pri}})}{w(p)^{3/2}}\left(|\Delta_p^{\mathrm{pri}}| - \frac{\rho_{i_0(p)}}{2}\right).$$

So if we further assume $M \geqslant 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2$, then whenever $w(p) \geqslant M$ we have $2^6(\deg X_\alpha - w_\alpha^{\mathrm{pri}})w(p)^{-3/2} \leqslant \epsilon/w(p)$; thus

$$\begin{aligned}|\Delta_p^{\mathrm{pri}}| - \frac{1}{2}\rho_{i_0(p)} + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} &\leqslant \left(1+\frac{\epsilon}{w(p)}\right)\left(|\Delta_p^{\mathrm{pri}}| - \frac{\rho_{i_0(p)}}{2}\right)\\ &\leqslant \left(1+\frac{\epsilon}{w(p)}\right)\left(\sum_{i\in\mathbb{I}_p^{\mathrm{pri}}}\delta(\tilde{s}_i,p)\rho_i - \rho_{i_0(p)}\right),\end{aligned}$$

where the last inequality follows from Lemma 4.2. Thus this case is settled.

The other case is when $w(p) - w^c(p) > \sqrt{w(p)}$. By the definition of c, for $j \in \mathbb{J} := \mathbb{I}_p \cap (c, \bar{\jmath}_\alpha]$, $(w(\tilde{\mathcal{E}}_j, p), \rho_j/2) \notin \Delta_p$. Since $\partial^+\Delta_p$ is convex, by Lemma 4.2, we have

$$\sum_{i\in\mathbb{J}}\delta(\tilde{s}_i,p)\rho_i - |\Delta_p^{\mathrm{pri}}\backslash\Delta_p^{\leqslant c}| \geqslant \sum_{i\in\mathbb{J}}\delta(\tilde{s}_i,p)\rho_i/2.$$

Since $\deg X_\alpha - w_\alpha^{\mathrm{pri}} \geqslant w(p) - w^{\mathrm{pri}}(p)$ and $w(p) - w^c(p) > \sqrt{w(p)}$ by our assumption, we have

$$\sum_{i\in\mathbb{J}}\delta(\tilde{s}_i,p) = w(p) - w^c(p) - (w(p) - w^{\mathrm{pri}}(p)) > \sqrt{w(p)} - (\deg X_\alpha - w_\alpha^{\mathrm{pri}}).$$

We choose

$$M \geqslant 10^2(g_\alpha + \ell_\alpha + 1)^2 \geqslant 5^2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})^2$$

and require $w(p) \geqslant M$; then $\sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p) \geqslant 4(\deg X_\alpha - w_\alpha^{\mathrm{pri}})$. This implies

$$\sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p)\rho_i - |\Delta_p^{\mathrm{pri}}\backslash\Delta_p^{\leqslant c}| \geqslant \sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p)\rho_i/2 \geqslant 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha},$$

and, combining this with Lemma 4.2, we obtain

$$\begin{aligned}
&|\Delta_p^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \\
&\quad\leqslant |\Delta_p^{\leqslant c}| + |\Delta_p^{\mathrm{pri}}\backslash\Delta_p^{\leqslant c}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \\
&\quad\leqslant |\Delta_p^{\leqslant c}| + |\Delta_p^{\mathrm{pri}}\backslash\Delta_p^{\leqslant c}| - \sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p)\rho_i + \sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p)\rho_i + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \\
&\quad\leqslant |\Delta_p^{\leqslant c}| + \sum_{i\in\mathbb{J}} \delta(\tilde{s}_i,p)\rho_i \leqslant \sum_{i\in\mathbb{I}_p^{\mathrm{pri}}} \delta(\tilde{s}_i,p)\rho_i - \frac{\rho_{i_0(p)}}{2} \\
&\quad< \left(1+\frac{\epsilon}{w(p)}\right)\left(\sum_{i\in\mathbb{I}_p^{\mathrm{pri}}} \delta(\tilde{s}_i,p)\rho_i - \frac{\rho_{i_0(p)}}{2}\right) + \frac{\rho_{i_0(p)}}{2}.
\end{aligned}$$

In the end, since $\epsilon < 1$, we choose $M := 2^{14}(g_\alpha + \ell_\alpha + 1)^2/\epsilon^2$. Then for $w(p) > M$, (4.3) holds. This proves the lemma. □

Proof of Theorem 4.1. First, for the same reason, we can assume $\rho_{\hbar_\alpha} = 0$ and $\bar{\rho}_i = \rho_i$. Also, when $\deg X_\alpha = 1$, then the statement is a direct consequence of Lemma 4.2. So from now on we assume that $\deg X_\alpha \geqslant M_1 \geqslant 2$. Let $1 > \epsilon > 0$ be any constant. Since $\epsilon < 1$, we have $\epsilon/(\deg X_\alpha) \leqslant 1/2$. We define σ to be the number of Newton polytopes supported on $\tilde{X}_\alpha$. We divide our study into two cases.

The first case is when $\sigma > 10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|$. Since Corollary 3.12 implies

$$|\{p \in \tilde{\Lambda}_\alpha \cap \tilde{S}_\alpha \mid i_0(p) > \bar{\jmath}_\alpha\}| \leqslant \sum_{i\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i) \leqslant (\deg X_\alpha - w_\alpha^{\mathrm{pri}}) \leqslant 2(g_\alpha + \ell_\alpha + 1),$$

the number of $p \in \tilde{\Lambda}_\alpha\backslash\tilde{S}_\alpha$ satisfying $\rho_{i_0(p)} \geqslant \rho_{\bar{\jmath}_\alpha}$ is at least $8(\tilde{g}_\alpha + \ell_\alpha + 1)$. By Lemma 4.2, for each $p \in \tilde{\Lambda}_\alpha$ we gain an extra $\rho_{i_0(p)}/2$ on the right-hand side in the estimate Δ_p in terms of $\{\rho_i\}_{i=0}^m$. This implies

$$\sum_{i\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i \leqslant (\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} \leqslant 2(g_\alpha + \ell_\alpha + 1)\rho_{\bar{\jmath}_\alpha} \leqslant \frac{1}{4} \sum_{p\in\tilde{\Lambda}_\alpha\backslash\tilde{S}_\alpha} \rho_{i_0(p)}. \tag{4.11}$$

So we obtain, using Lemma 4.2 and summing over $p \in \tilde{\Lambda}_\alpha$,

$$\begin{aligned}
\sum_{p\in\tilde{\Lambda}_\alpha} |\Delta_p| &\leqslant \sum_{i\in\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i + \sum_{i\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i - \frac{1}{2}\sum_{p\in\tilde{\Lambda}_\alpha} \rho_{i_0(p)} \\
&= \left(\sum_{i\in\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i - \frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p\in\tilde{\Lambda}_\alpha\cap\tilde{S}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}}} \rho_{i_0(p)} - \frac{1}{2}\sum_{p\in\tilde{\Lambda}_\alpha\cap\tilde{S}_\alpha} \rho_{i_0(p)}\right) \\
&\quad + \left(\sum_{i\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_i)\rho_i + \frac{\epsilon}{\deg X_\alpha} \sum_{\substack{p\in\tilde{\Lambda}_\alpha\cap\tilde{S}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha\backslash\mathbb{I}_\alpha^{\mathrm{pri}}}} \rho_{i_0(p)} - \frac{1}{2}\sum_{p\in\tilde{\Lambda}_\alpha\backslash\tilde{S}_\alpha} \rho_{i_0(p)}\right).
\end{aligned} \tag{4.12}$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Using (4.11) and

$$\frac{\epsilon}{\deg X_\alpha}\sum_{\substack{p\in\tilde\Lambda_\alpha\cap\tilde S_\alpha\\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\rm pri}}}\rho_{i_0(p)}\leqslant\sum_{i\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\rm pri}}\delta_\alpha(\tilde s_i)\rho_i\leqslant\frac{1}{4}\sum_{p\in\tilde\Lambda_\alpha\setminus\tilde S_\alpha}\rho_{i_0(p)},$$

the sum in the line of (4.12) is non-positive. Therefore, for any $0<\epsilon<1$ we have

$$\begin{aligned}\sum_{p\in\tilde\Lambda_\alpha}|\Delta_p|&\leqslant\sum_{i\in\mathbb{I}_\alpha^{\rm pri}}\delta_\alpha(\tilde s_i)\rho_i-\frac{\epsilon}{\deg X_\alpha}\sum_{\substack{p\in\tilde\Lambda_\alpha\cap\tilde S_\alpha\\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\rm pri}}}\rho_{i_0(p)}-\frac{1}{2}\sum_{p\in\tilde\Lambda_\alpha\cap\tilde S_\alpha}\rho_{i_0(p)}\\&\leqslant\left(1+\frac{\epsilon}{\deg X_\alpha}\right)\sum_{i\in\mathbb{I}_\alpha^{\rm pri}}\delta_\alpha(\tilde s_i)\rho_i-\left(\frac{1}{2}+\frac{\epsilon}{\deg X_\alpha}\right)\sum_{p\in\tilde\Lambda_\alpha\cap\tilde S_\alpha}\rho_{i_0(p)}=\frac{E^\epsilon_\alpha(\rho)}{2},\end{aligned}$$

since

$$\frac{\epsilon}{\deg X_\alpha}\sum_{\substack{p\in\tilde\Lambda_\alpha\cap\tilde S_\alpha\\ i_0(p)\in\mathbb{I}_\alpha^{\rm pri}}}\rho_{i_0(p)}\leqslant\frac{\epsilon}{\deg X_\alpha}\sum_{i\in\mathbb{I}_\alpha^{\rm pri}}\delta_\alpha(\tilde s_i)\rho_i.$$

This verifies the theorem in this case.

The other case is when $\sigma\leqslant 10(g_\alpha+\ell_\alpha+1)+|\tilde S_\alpha|$. By the pigeonhole principle, there exists at least one $p_0\in\tilde\Lambda_\alpha$ such that

$$w(\tilde{\mathfrak{I}},p_0)\geqslant\frac{\deg X_\alpha}{\sigma}\geqslant\frac{\deg X_\alpha}{10(g_\alpha+\ell_\alpha+1)+|\tilde S_\alpha|}. \tag{4.13}$$

By Corollary 2.8, we have

$$\frac{e_{X_\alpha}(\mathfrak{I}(\lambda))}{2}=\sum_{p\in\tilde\Lambda_\alpha}|\Delta_p|.$$

Our assumption $\epsilon\leqslant 1, 1/\mathrm{deg}X\leqslant 1/2$ and Corollary 3.12 imply

$$\left(\frac{1}{2}+\frac{\epsilon}{\deg X_\alpha}\right)\sum_{\substack{p\in\tilde S_\alpha\cap\tilde\Lambda_\alpha\\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\rm pri}}}\rho_{i_0(p)}\leqslant\sum_{\substack{p\in\tilde S_\alpha\cap\tilde\Lambda_\alpha\\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\rm pri}}}\rho_{i_0(p)}\leqslant(\deg X_\alpha-w_\alpha^{\rm pri})\rho_{\bar{\jmath}_\alpha}. \tag{4.14}$$

So we obtain

$$\begin{aligned}&\frac{e_{X_\alpha}(\mathfrak{I}(\lambda))}{2}-\sum_{p\in\tilde S_\alpha\cap\tilde\Lambda_\alpha}\frac{\rho_{i_0(p)}}{2}\\&\quad=|\Delta_{p_0}^{\rm pri}|+|\Delta_{p_0}\setminus\Delta_{p_0}^{\rm pri}|+\sum_{p_0\neq p\in\tilde\Lambda_\alpha}(|\Delta_p^{\rm pri}|+|\Delta_p\setminus\Delta_p^{\rm pri}|)-\sum_{p\in\tilde S_\alpha\cap\tilde\Lambda_\alpha}\frac{\rho_{i_0(p)}}{2}.\end{aligned}$$

By Lemma 4.2 and the first inequality of (4.11), we have

$$|\Delta_{p_0}\setminus\Delta_{p_0}^{\rm pri}|+\sum_{p_0\neq p\in\tilde\Lambda_\alpha}|\Delta_p\setminus\Delta_p^{\rm pri}|=\sum_{p\in\tilde\Lambda_\alpha}|\Delta_p\setminus\Delta_p^{\rm pri}|\leqslant(\deg X_\alpha-w_\alpha^{\rm pri})\rho_{\bar{\jmath}_\alpha}.$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

So

$$\begin{aligned}
&\frac{e_{X_\alpha}(\mathfrak{I}(\lambda))}{2} - \sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\
&\leqslant |\Delta_{p_0}^{\mathrm{pri}}| + (\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} + \sum_{p_0\neq p\in\tilde{\Lambda}_\alpha} |\Delta_p^{\mathrm{pri}}| - \sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\
&\leqslant |\Delta_{p_0}^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} - \frac{\rho_{i_0(p_0)}}{2}|\{p_0\}\cap\tilde{S}_\alpha| + \sum_{p_0\neq p\in\tilde{\Lambda}_\alpha} |\Delta_p^{\mathrm{pri}}| \\
&\quad - \sum_{p_0\neq p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(\frac{1}{2} + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\mathrm{pri}}}} \rho_{i_0(p)} \qquad (4.15)
\end{aligned}$$

where we have used (4.14) in (4.15). By definition, $|\tilde{S}_\alpha| \leqslant n + \ell_\alpha + g_\alpha$. Let

$$\epsilon_0 = \frac{\epsilon}{11(g_\alpha + \ell_\alpha + 1) + n} \leqslant \frac{\epsilon}{10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|}.$$

By (4.13) we obtain

$$\frac{\epsilon_0}{w(\tilde{\mathfrak{I}}, p_0)} \leqslant \frac{\epsilon}{w(\tilde{\mathfrak{I}}, p_0)(10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|)} \leqslant \frac{\epsilon}{\deg X_\alpha}. \qquad (4.16)$$

If we let $M = M(g_\alpha, \ell_\alpha, \epsilon_0)$ be the constant fixed in Lemma 4.3 for $\epsilon = \epsilon_0$ and choose

$$M_1(g_\alpha, \ell_\alpha, n, \epsilon) := (11(g_\alpha + \ell_\alpha + 1) + n)M \geqslant (10(g_\alpha + \ell_\alpha + 1) + |\tilde{S}_\alpha|)M,$$

then $\deg X_\alpha \geqslant M_1$ implies $w(\tilde{\mathfrak{I}}, p_0) > M$. In particular, we have $i_0(p_0) \in \mathbb{I}_\alpha^{\mathrm{pri}}$. The whole term after (4.15) is

$$\begin{aligned}
&= |\Delta_{p_0}^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha} - \frac{\rho_{i_0(p_0)}}{2}|\{p_0\}\cap\tilde{S}_\alpha| + \sum_{p_0\neq p\in\tilde{\Lambda}_\alpha} |\Delta_p^{\mathrm{pri}}| \\
&\quad - \sum_{\substack{p_0\neq p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha^{\mathrm{pri}}}} \frac{\rho_{i_0(p)}}{2} - \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\mathrm{pri}}}} \rho_{i_0(p)}.
\end{aligned}$$

Applying Lemma 4.2 to the term $|\Delta_{p_0}^{\mathrm{pri}}| + 2(\deg X_\alpha - w_\alpha^{\mathrm{pri}})\rho_{\bar{\jmath}_\alpha}$, Lemma 4.3 to the term $\sum_{p_0\neq p\in\tilde{\Lambda}_\alpha} |\Delta_p^{\mathrm{pri}}| - \sum_{p_0\neq p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha, i_0(p)\in\mathbb{I}_\alpha^{\mathrm{pri}}} (\rho_{i_0(p)}/2)$ in the above identity and using (4.16), we obtain

$$\begin{aligned}
&\frac{e_{X_\alpha}(\mathfrak{I}(\lambda))}{2} - \sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha} \frac{\rho_{i_0(p)}}{2} \\
&\leqslant \left(1 + \frac{\epsilon_0}{w(\tilde{\mathfrak{I}}, p_0)}\right)\left(\sum_{i\in\mathbb{I}_{p_0}^{\mathrm{pri}}} \delta(\tilde{s}_i, p_0)\rho_i - \rho_{i_0(p_0)}\right) + \frac{\rho_{i_0(p_0)}}{2}(1 - |\{p_0\}\cap\tilde{S}_\alpha|) \\
&\quad + \left(\sum_{p_0\neq p\in\tilde{\Lambda}_\alpha} \sum_{i\in\mathbb{I}_p^{\mathrm{pri}}} \delta(\tilde{s}_i, p)\rho_i - \sum_{\substack{p_0\neq p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha^{\mathrm{pri}}}} \rho_{i_0(p)}\right) - \left(1 + \frac{\epsilon}{\deg X_\alpha}\right) \sum_{\substack{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha \\ i_0(p)\in\mathbb{I}_\alpha\setminus\mathbb{I}_\alpha^{\mathrm{pri}}}} \frac{\rho_{i_0(p)}}{2}
\end{aligned}$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

$$\leqslant \left(1+\frac{\epsilon}{\deg X_\alpha}\right)\left(\sum_{p\in\tilde{\Lambda}_\alpha}\sum_{i\in\mathbb{I}_p^{\mathrm{pri}}}\delta(\tilde{s}_i,p)\rho_i-\sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha}\rho_{i_0(p)}\right)$$
$$=\left(1+\frac{\epsilon}{\deg X_\alpha}\right)\left(\sum_{i\in\mathbb{I}_\alpha^{\mathrm{pri}}}\delta_\alpha(\tilde{s}_i)\rho_i-\sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha}\rho_{i_0(p)}\right)$$
$$=\frac{E_\alpha^\epsilon(\rho)}{2}-\sum_{p\in\tilde{S}_\alpha\cap\tilde{\Lambda}_\alpha}\frac{\rho_{i_0(p)}}{2}.$$

This proves the theorem. □

5. Stability of weighted pointed nodal curve

In this section we will prove Theorems 1.5 and 1.6. For any subcurve $Y\subset X$, we continue to write $a_Y=\sum_{x_i\in Y}a_i$.

Lemma 5.1. *Let $(X,\mathcal{O}_X(1),\mathbf{x},\mathbf{a})$ be a polarized weighted pointed nodal curve. Suppose $(X,\mathcal{O}_X(1))$ is non-special. Then it satisfies (1.6) for all subcurves $Y\subsetneq X$ if and only if it satisfies (1.4) for all subcurves $Y\subsetneq X$.*

Proof. Let $Y\subsetneq X$ be a subcurve. Since $(X,\mathcal{O}_X(1))$ is non-special, we have vanishing $h^1(\mathcal{O}_Y(1))=h^1(\mathcal{O}_{Y^\complement}(1))=0$. Following the proof of [Cap94, Proposition 3.1], we see that (1.4) holding for $Y\subsetneq X$ is equivalent to

$$\left(\deg Y+\frac{a_Y}{2}\right)-\frac{\deg_Y\omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg\omega_X(\mathbf{a}\cdot\mathbf{x})}\left(\deg X+\frac{a_X}{2}\right)\geqslant-\frac{\ell_Y}{2},\tag{5.1}$$

and (1.4) holding for $Y^\complement\subsetneq X$ is equivalent to

$$\left(\deg Y+\frac{a_Y}{2}\right)-\frac{\deg_Y\omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg\omega_X(\mathbf{a}\cdot\mathbf{x})}\left(\deg X+\frac{a_X}{2}\right)\leqslant\frac{\ell_Y}{2}.\tag{5.2}$$

So (1.4) holding for any subcurve $Y\subsetneq X$ implies (1.6) holding for any subcurve $Y\subsetneq X$.

The other direction is trivial, since (1.6) is equivalent to both (5.1) and (5.2). This proves the lemma. □

Lemma 5.2. *Given g, n and $\mathbf{a}\in\mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g}>0$ (cf. (1.5)), there are positive constants $M_2=M_2(g,n,\mathbf{a})$ and $C=C(g,n,\mathbf{a})$ such that for any genus g polarized weighted pointed nodal curve $(X,\mathcal{O}_X(1),\mathbf{x},\mathbf{a})$ satisfying (1.6) and $\deg X>M_2$, any connected subcurve $Y\subset X$ either has $\deg Y\geqslant C\deg X$ and $\deg_Y\omega_X>0$ or is an exceptional component.*

Proof. Suppose $\deg_Y\omega_X=2g(Y)-2+\ell_Y\geqslant 1$; since $a_i\geqslant 0$, (1.6) implies

$$\frac{\deg Y+a_Y/2}{g(Y)-1+a_Y/2+\ell_Y/2}$$
$$\geqslant\frac{\deg X+a_X/2}{g-1+a_X/2}-\frac{\ell_Y/2}{g(Y)-1+a_Y/2+\ell_Y/2}\geqslant\frac{\deg X+a_X/2}{g-1+a_X/2}-3.$$

This inequality implies

$$\deg Y\geqslant\left(\frac{\deg X}{2\chi_{\mathbf{a},g}}-6\right)-\frac{n}{2}.$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Therefore, by choosing $C = 1/4\chi_{\mathbf{a},g}$ and $M_2 \geqslant 4\chi_{\mathbf{a},g}(6+n/2)$, we obtain

$$\deg Y \geqslant (4\chi_{\mathbf{a},g})^{-1} \deg X = C \cdot \deg X$$

provided $\deg X > M_2$.

Now suppose $\deg_Y \omega_X = 2g(Y) - 2 + \ell_Y \leqslant 0$. Since X is connected, $\ell_Y \geqslant 1$ and $g(Y) \geqslant 0$. Thus $g(Y) = 0$ and $\ell_Y \leqslant 2$. In this case, (1.6) becomes

$$\left|\deg Y + \frac{a_Y}{2} - \frac{\deg X + a_X/2}{g-1+a_X/2} \cdot \left(-1 + \frac{a_Y}{2} + \frac{\ell_Y}{2}\right)\right| \leqslant 1. \tag{5.3}$$

Let $A := -1 + a_Y/2 + \ell_Y/2$. In the case $A \leqslant 0$, we have $\deg Y = 1$, $a_Y = 0$ and $\ell_Y = 2$. Because $\mathcal{O}_X(1)$ is ample, Y must be irreducible and thus isomorphic to $\mathbb{P}^1$. Thus $Y \subsetneq X$ is an exceptional component.

In the case $A > 0$, we let

$$A_0 = \min_{I \subset \{1,\dots,n\}, k \geqslant 0} \left\{ \sum_{i \in I} a_i/2 + k/2 \;\middle|\; \sum_{i\in I} a_i/2 + k/2 > 0 \right\}, \tag{5.4}$$

which is positive by the finiteness of $\{a_i\}$. Then $A \geqslant A_0$ and

$$\deg Y > \frac{\deg X + a_X/2}{\chi_{\mathbf{a},g}} \cdot 2A - 1 - \frac{a_Y}{2} > \frac{\deg X}{\chi_{\mathbf{a},g}} A_0,$$

when $\deg X \geqslant M_2 \geqslant \chi_{\mathbf{a},g}(1 - a_Y/2)/A_0$. Combined, we have proved the lemma by choosing $M_2 := \max\{\chi_{\mathbf{a},g}(1-a_Y/2)/A_0, 4\chi_{\mathbf{a},g}(6+n/2)\}$ and $C = \min\{1/4\chi_{\mathbf{a},g}, A_0/\chi_{\mathbf{a},g}\}$. □

Corollary 5.3. *Let the situation and the constant C be as in Lemma 5.2. Then for any genus g polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ (with $\mathcal{O}_X(1)$ only assumed to be ample) satisfying $\deg X > M_3 = M_3(g,n,\mathbf{a}) := \max\{M_2, (9g+n)/C\}$ and the inequality (1.6), we have that $\mathcal{O}_X(1)$ is very ample, $h^1(\mathcal{O}_X(1)) = 0$, and the number of nodes of X is bounded from above by $6(g+n)$.*

Proof. First, we notice that $h^1(\mathcal{O}_X(1)) = 0$ if $h^1(\mathcal{O}_Y(1)(-L_Y)) = 0$ for any irreducible component $Y \subset X$. In the case $Y \cong \mathbb{P}^1$ with $\ell_Y = 2$, this follows from the fact that $\mathcal{O}_Y(1)$ is ample. Otherwise, Y is not an exceptional component, and then, by the previous lemma, $\deg \mathcal{O}_Y(1) \geqslant C \deg X \geqslant CM_3$. As X has genus g, we have $g(Y) \leqslant g$ and $\ell_Y \leqslant g+n+1$. Therefore, by our choice of M_3 we obtain

$$\deg Y - \ell_Y \geqslant 8g(Y) - 1 \tag{5.5}$$

from which we deduce that $h^1(\mathcal{O}_Y(1)(-L_Y)) = 0$ for all irreducible components of X and $\mathcal{O}_X(1)|_Y$ is very ample. This proves the non-speciality of $(X, \mathcal{O}_X(1))$.

By (1.6), we conclude that any chain of exceptional components consists of a single component. Thus the number of nodes of X is no more than twice the number of the nodes of the stabilization (cf. (6.6)) of $(X, \mathbf{x})$ which is less than $3g - 3 + n < 3(g+n)$,[1] and the stated bound follows.

[1] One can see this by induction. Notice that adding one marked point will introduce at most one node when $\mathbb{P}^1$ with 3 marked points is bubbled off. On the other hand, increasing the genus by 1 will increase the nodes at most by 3 when a nodal rational curve is connected to the main component through a $\mathbb{P}^1$ with 3 marked points.

Finally, we prove that $\mathcal{O}_X(1)$ is very ample. First, one notices that for each irreducible component $Y \subset X$, the inequality (5.5) implies the *very ampleness* of $\mathcal{O}_X(1)|_Y$ and the exact sequence

$$H^0(\mathcal{O}_X(1)|_Y(-L_Y)) \to H^0(\mathcal{O}_X(1)|_Y) \to H^0(\mathcal{O}_X(1)|_{L_Y}) \to H^1(\mathcal{O}_X(1)|_Y(-L_Y)) = 0 \qquad (5.6)$$

from which we conclude that for any non-exceptional irreducible component $X_\alpha \subset X$, there is a section $s \in H^0(\mathcal{O}_X(1)|_{X_\alpha})$ taking any given boundary value on L_α.

For any two points $p_\alpha \in X$, $\alpha = 1, 2$, we claim that there is a section $s \in H^0(\mathcal{O}_X(1))$ such that $s(p_1) = 0$ and $s(p_2) \neq 0$. Without loss of generality, we assume p_α lies in the irreducible component X_α, $\alpha = 1, 2$. If $X_1 = X_2$ then our claim follows from the very ampleness of $\mathcal{O}_X(1)|_{X_1}$. From now on, we assume $X_1 \neq X_2$. We first consider the case where both X_1 and X_2 are exceptional, which is the most involved case. Since X_1 is an exceptional component, there is a section $0 \neq s_1 \in H^0(\mathcal{O}_X(1)|_{X_1})$ satisfying $s_1(p_1) = 0$. Since X_2 is also exceptional, we have $X_1 \cap X_2 = \emptyset$, by (1.6), which allows us to choose a section $s_2 \in H^0(\mathcal{O}_X(1)|_{X_2})$ with $s_2(p_2) \neq 0$. To construct the global section $s \in H^0(\mathcal{O}_X(1))$, we first let s be s_1 and s_2 on X_1 and X_2, respectively; we let it be the zero section on the exceptional components of X different from X_1 and X_2. We next extend it to non-exceptional components, one at a time.

Suppose we have extended it to a section s_β on a component $X_\beta \subset X$; we then apply (5.6) to construct a section $s_{\beta+1} \in H^0(\mathcal{O}_X(1)|_{X_{\beta+1}})$ satisfying the boundary value prescribed by the previous stage. By continuing this process, we obtain the section s that we want.

The other cases are similar and will be left to the readers. Because of the claim, we deduce that the complete linear system $W^\vee = H^0(\mathcal{O}_X(1))$ provides an embedding of $X \subset \mathbb{P}W$. This completes the proof. □

As a consequence, we have the following.

Proposition 5.4. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g} > 0$, then for any polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ (with $\mathcal{O}_X(1)$ only assumed to be ample) of $\deg X \geqslant M_3$ (the constant in Corollary 5.3), the following two are equivalent:*

1. *$\mathcal{O}_X(1)$ is very ample and $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope semistable (respectively, slope stable);*
2. *$(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ satisfies (1.6) for all subcurves $Y \subsetneq X$ (respectively, and the strict (1.6) holds except when Y or $Y^{\complement}$ is a disjoint union of exceptional components of $(X, \mathcal{O}_X(1), \mathbf{x})$).*

Proof. By the definition of slope semistability (Definition 1.4), (1) implies that $(X, \mathcal{O}_X(1))$ is non-special. On the other hand, by Corollary 5.3, $(X, \mathcal{O}_X(1))$ is non-special and $\mathcal{O}_X(1)$ is very ample if it satisfies (1.6) and $\deg X \geqslant M_3$. Hence in both cases we have $h^1(X, \mathcal{O}_X(1)) = 0$. Applying Lemma 5.1, we conclude that in cases (1) and (2), (1.4) is equivalent to (1.6). This proves the equivalence of (the non-respectively cases of) (1) and (2).

We now prove the case for slope stability. Suppose (1) holds for $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ and the latter is slope stable, but for a subcurve $Y \subsetneq X$, (1.6) is an equality; then either (5.2) or (5.1) is an equality. It follows from the proof of Lemma 5.1 that (1.4) becomes an equality for either Y or $Y^{\complement}$. By the slope stability assumption, either Y or $Y^{\complement}$ is a disjoint union of exceptional components. This proves one direction for the 'respectively' case. The other case is similar and we leave the proof to the readers. □

In order to prove Theorem 1.5 for the stable case, we also need the following lemma.

Lemma 5.5. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g} > 0$, there is a constant $M_4 = M_4(g, n, \mathbf{a})$ such that for any* slope semistable *polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ of $\deg X \geqslant M_4$, a subcurve $Y \subset X$ satisfies $h^0(\mathcal{O}_X(1)|_Y) = h^0(\mathcal{O}_X(1))$ if and only if $Y^{\complement}$ is a disjoint union of exceptional components. In particular, if we assume further that $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is* slope stable, *then $h^0(\mathcal{O}_X(1)|_Y) < h^0(\mathcal{O}_X(1))$ implies that (1.4) is a strict inequality.*

Proof. For any subcurve $Y \subset X$, let $W_Y = \{v \in W \mid s(v) = 0, \forall s \in H^0(\mathcal{O}_X(1) \otimes \mathcal{I}_Y)\} \subset W$ denote the linear subspace spanned by Y. By our slope semistability assumption, the embedding $X \subset \mathbb{P}W$ is given by a complete *non-special* linear system, and hence $\dim W_Y = h^0(\mathcal{O}_X(1)|_Y)$. So to prove the lemma all we need to show is that for $\deg X$ sufficiently large, $\dim W_Y = \dim W$ if and only if $Y^{\complement}$ is a disjoint union of exceptional components.

To achieve that, we notice that for any component $X_\alpha \subset Y^{\complement}$, we have

$$\dim W_{Y \cup X_\alpha} = \dim W_Y + \dim W_{X_\alpha} - \dim W_Y \cap W_{X_\alpha}. \tag{5.7}$$

We claim that there is an $M_4 = M_4(g, n, \mathbf{a})$ such that whenever $\deg X \geqslant M_4$, we have $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$. This is trivially true when X_α is exceptional. If X_α is non-exceptional and $\deg X \geqslant M_2$ (the constant in Lemma 5.2), we have $\deg X_\alpha \geqslant C \deg X$ by the semistability assumption and Lemma 5.2. So as long as $\deg X \geqslant \max\{M', M_2\}$ with M' satisfying

$$CM' > 2g - 2 + \text{ number of nodes in } X \geqslant 2g - 2 + |X_\alpha \cap Y|,$$

where C is given in Lemma 5.2, by the vanishing theorem we have the surjectivity of the restriction maps

$$H^0(\mathcal{O}_X(1)|_Y) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}) \quad \text{and} \quad H^0(\mathcal{O}_X(1)|_{X_\alpha}) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}),$$

from which we deduce the exact sequence

$$0 \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cup Y}) \to H^0(\mathcal{O}_X(1)|_Y) \oplus H^0(\mathcal{O}_X(1)|_{X_\alpha}) \to H^0(\mathcal{O}_X(1)|_{X_\alpha \cap Y}) \to 0.$$

This, together with the assumption of being non-special and (5.7), implies $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$. On the other hand, by Corollary 5.3, the number of nodes in X is bounded by $6(g+n)$ provided $\deg X \geqslant M_3$. So our claim follows if we choose $M_4 \geqslant \max\{(2g-2+6(g+n))/C, M_3\}$.

Now let us define $M_4 = M_4(g, n, \mathbf{a}) := \max\{8(g+n+1)/C, M_3\}$ and assume $\deg X > M_4$. Then for any $X_\alpha \subset Y^{\complement}$ non-exceptional we have

$$\dim W_{X_\alpha} - |X_\alpha \cap Y| \geqslant \deg X_\alpha + 1 - g(X_\alpha) - |X_\alpha \cap Y| \geqslant CM_4 + 1 - g - 6(g+n) > 1.$$

Plugging the above inequality and $\dim W_Y \cap W_{X_\alpha} = |X_\alpha \cap Y|$ into (5.7), we obtain

$$\dim W_{Y \cup X_\alpha} = \dim W_Y + \dim W_{X_\alpha} - \dim W_Y \cap W_{X_\alpha} \geqslant \dim W_Y + 1,$$

from which we deduce that $W_Y = W$ if and only if $X_\alpha \subset Y^{\complement}$ is exceptional and $|X_\alpha \cap Y| = 2$. This proves the lemma. □

Let $\mathbf{s}$ be a diagonalizing basis of λ:

$$\lambda(t) := \mathrm{diag}[t^{\rho_0}, \dots, t^{\rho_m}] \cdot t^{-\rho_{\mathrm{ave}}} \quad \text{with } \rho_0 \geqslant \rho_1 \geqslant \cdots \geqslant \rho_m = 0.$$

The $\mathbf{a}$-λ-weight of $\mathrm{Chow}(X, \mathbf{x})$ is the sum of the contributions from $\mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ and $(\mathbb{P}W)^n$. By Proposition 2.1, the contribution from $\mathrm{Div}^{d,d}[(\mathbb{P}W^\vee)^2]$ is $\omega(\lambda)$.

For the contribution from $(\mathbb{P}W)^n$, we introduce subspaces

$$(W_i =)\ W_i(\lambda) := \{v \in W \mid s_i(v) = \cdots = s_m(v) = 0\} \subset W = H^0(\mathcal{O}_X(1))^\vee. \tag{5.8}$$

They form a strictly increasing filtration of W. Also, for any closed subscheme $\Sigma \subset X$, we denote by

$$W_\Sigma := \{v \in W \mid s(v) = 0 \text{ for all } s \in H^0(\mathcal{O}_X(1) \otimes \mathcal{I}_\Sigma)\} \subset W \tag{5.9}$$

the *linear subspace spanned by* $\Sigma \subset X$. For instance, for a marked point x_i, W_{x_i} is the line in W spanned by $x_i \in \mathbb{P}W$.

By [MFK94, Proposition 4.3], the $\mathbf{a}$-λ-weight of $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{P}W)^n$ is

$$\mu_{\mathbf{a}}(\lambda) := \sum_{j=1}^{n} a_j \left(\frac{\sum_{i=0}^{m} \rho_i}{m+1} + \sum_{i=0}^{m-1} (\rho_{i+1} - \rho_i) \dim(W_{x_j} \cap W_{i+1}(\lambda)) \right). \tag{5.10}$$

(Note that $\mu_{\mathbf{a}}(\lambda)$ implicitly depends on ρ_i, which we fix for the moment.) Therefore, the $\mathbf{a}$-λ-weight $\omega_{\mathbf{a}}(\lambda)$ of $\mathrm{Chow}(X, \mathbf{x}) \in \Xi$ is

$$\omega_{\mathbf{a}}(\lambda) = \omega(\lambda) + \mu_{\mathbf{a}}(\lambda). \tag{5.11}$$

We now argue that, for the staircase λ' constructed from λ by applying Proposition 3.5, we have

$$\omega_{\mathbf{a}}(\lambda) \geqslant \omega_{\mathbf{a}}(\lambda'). \tag{5.12}$$

Indeed, since $\omega(\lambda) \geqslant \omega(\lambda')$, it suffices to show that $\mu_{\mathbf{a}}(\lambda) \geqslant \mu_{\mathbf{a}}(\lambda')$. To see this, we first notice that

$$\dim(W_{x_j} \cap W_{i+1}(\lambda)) = \#(x_i \cap \mathrm{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_{i+1})). \tag{5.13}$$

(Here $\mathcal{E}(\lambda)_i = (s_i, s_{i+1}, \ldots, s_m) \subset \mathcal{O}_X(1)$.) On the other hand, by the proof of Proposition 3.5, we conclude that

$$\mathrm{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda)_i) \subset \mathrm{Supp}(\mathcal{O}_X(1)/\mathcal{E}(\lambda')_i).$$

This together with (5.13) proves

$$\dim(W_{x_j} \cap W_{i+1}(\lambda)) \leqslant \dim(W_{x_j} \cap W_{i+1}(\lambda')).$$

The inequality $\mu_{\mathbf{a}}(\lambda) \geqslant \mu_{\mathbf{a}}(\lambda')$ then follows from the facts $\rho_i \geqslant \rho_{i+1}$ and $\rho_i = \rho_i'$. Therefore, to prove Theorem 1.5, it suffices to show that $\omega_{\mathbf{a}}(\lambda) > 0$ for all staircase 1-PS λ. From now on we assume λ is a *staircase*. For simplicity, we define $W_i = W_i(\lambda)$.

To state the estimate of this section, we define $E_X^\epsilon(\lambda, \rho) := \sum_{\alpha=1}^{r} E_\alpha^\epsilon(\rho)$. Since λ is a staircase 1-PS, $\bigcup_{\alpha=1}^{r} \mathbb{I}_\alpha = \{0, \ldots, m\}$, where $\mathbb{I}_\alpha$ is the index set of the component X_α defined in (3.3). This allows us to define the shifted weights $\{\hat{\rho}_i\}$ by

$$\hat{\rho}_i := \min_{\alpha}\{\rho_i - \rho_{\hbar_\alpha} \mid i \in \mathbb{I}_\alpha\} \geqslant 0. \tag{5.14}$$

We caution that the $\hat{\rho}_i$ are only defined for *staircase* 1-PS, and possibly they are non-monotone.

Proposition 5.6. *Given g, n, $\mathbf{a} \in \mathbb{Q}_+^n$ and $0 < \epsilon < 1$ satisfying $\chi_{\mathbf{a},g} > 0$, suppose $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is a genus g* slope semistable *polarized weighted pointed nodal curve of $\deg X \geqslant M_4$, the constant given in Lemma 5.5. Then for any staircase 1-PS λ, we have*

$$\frac{E_X^\epsilon(\lambda, \rho)}{2} \leqslant \sum_{i=0}^{m} \rho_i - \sum_{q \in \tilde{S}_{\mathrm{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \sum_{\alpha=1}^{r} \left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1 \right) \cdot \rho_{\hbar_\alpha} + \frac{2C^{-1}\epsilon}{m+1} \sum_{i=0}^{m} \hat{\rho}_i \tag{5.15}$$

where $\tilde{S}_{\mathrm{reg}} := \bigcup_{\alpha=1}^{r} (\pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha \cap \tilde{\Lambda})$ and $C > 0$ is the constant given in Lemma 5.2.

Proof. By the definition of $E^{\epsilon}_{\alpha}(\rho)$ (cf. (4.3)), $E^{\epsilon}_{X}(\lambda,\rho)=\sum_{\alpha=1}^{r}E^{\epsilon}_{\alpha}(\rho)$ is linear in $\rho=(\rho_i)$. By linear programming, (5.15) holds on

$$\mathbb{R}^{m+1}_{+}:=\{(\rho_0,\ldots,\rho_m)\in\mathbb{R}^{m+1}\mid \rho_0\geqslant\rho_1\geqslant\cdots\geqslant\rho_m=0\}$$

if and only if it holds on every edge of $\mathbb{R}^{m+1}_{+}$; these edges are spanned by the vectors

$$\rho=(\overbrace{1,\ldots,1}^{m_0},0,\ldots,0),\quad 0<m_0<m. \tag{5.16}$$

We now fix a $0<m_0<m$. By possibly reindexing the irreducible components of X, we can assume that for some $\bar r\leqslant r$, $\hbar_1\leqslant\cdots\leqslant\hbar_{\bar r}<m_0\leqslant\hbar_{\bar r+1}\leqslant\cdots\leqslant\hbar_r$. In other words,

$$\rho_{\hbar_1}=\cdots=\rho_{\hbar_{\bar r}}=1\quad\text{and}\quad\rho_{\hbar_{\bar r+1}}=\cdots=\rho_{\hbar_r}=0. \tag{5.17}$$

We let $Y:=\bigcup_{\alpha>\bar r}X_\alpha$; thus its complement $Y^{\complement}=\bigcup_{\alpha\leqslant\bar r}X_\alpha$.

We claim that $Y^{\complement}$ is the maximal subcurve of X contained in the linear subspace $\mathbb{P}W_{m_0}$ (cf. (5.8)). By definition, for any α, $\hbar_\alpha$ is the largest index $0<i\leqslant m$ for which $s_i|_{X_\alpha}\neq 0$. On the other hand, because $\mathbb{P}W_{m_0}=\{s_{m_0}=\cdots=s_m=0\}$, $X_\alpha\subset\mathbb{P}W_{m_0}$ if and only if $s_i|_{X_\alpha}=0$ for all $i\geqslant m_0$, which is equivalent to $\hbar_\alpha<m_0$. This proves the claim.

Let X_α be a component in $Y^{\complement}$. Since $\rho_{\hbar_\alpha}=1$, $\rho_i=1$ for $i\in\mathbb{I}_\alpha$. Using the explicit expression of $E^{\epsilon}_{\alpha}(\lambda,\rho)$(cf. (4.3)), we obtain $E^{\epsilon}_{\alpha}(\lambda,\rho)=2\deg X_\alpha$. Thus

$$\sum_{\alpha\leqslant\bar r}E^{\epsilon}_{\alpha}(\lambda,\rho)=\sum_{\alpha\leqslant\bar r}2\deg X_\alpha=2\deg Y^{\complement}.$$

We next look at Y. Following (1.3) and (3.8), $\tilde L_Y:=\pi^{-1}(Y\cap Y^{\complement})\cap\tilde Y$. We claim that $\tilde L_Y\subset\tilde\Lambda_Y:=\bigcup_{\alpha>\bar r}\tilde\Lambda_\alpha$. Indeed, for any $\alpha>\bar r$, there is an $i\geqslant m_0$ such that $s_i|_{X_\alpha}\neq 0$. However, for any $\beta\leqslant\bar r$, $i\geqslant m_0$ implies $s_i|_{X_\beta}=0$. Thus $s_i|_{X_\alpha\cap X_\beta}=0$ and consequently $\pi^{-1}(X_\alpha\cap X_\beta)\cap\tilde X_\alpha\subset\tilde\Lambda_\alpha$. Summing over all $\alpha>\bar r$ and $\beta\leqslant\bar r$, we obtain $\tilde L_Y\subset\tilde\Lambda_Y$. As a consequence,

$$\sum_{p\in\tilde L_Y}\rho_{i_0(p)}=\ell_Y. \tag{5.18}$$

To simplify the notation, in the remaining part of this section, we will abbreviate

$$\sum_{p\in\Sigma}\rho_{i_0(p)}:=\sum_{p\in\Sigma\cap\tilde\Lambda}\rho_{i_0(p)},$$

with the understanding that for any closed subset $\Sigma\subset\tilde X$, $\sum_{p\in\Sigma}$ only sums over $p\in\Sigma\cap\tilde\Lambda$.

SUBLEMMA 5.7. *Let the notation be as before. Then*

$$\begin{aligned}&\sum_{\alpha>\bar r}\frac{E^{\epsilon}_{\alpha}(\lambda,\rho)}{2}-\frac{\ell_Y}{2}\\&\quad\leqslant\left(1+\frac{C^{-1}\epsilon}{\deg X}\right)\left(\sum_{\substack{\alpha>\bar r\\ i\in\mathbb{I}^{\mathrm{pri}}_\alpha}}\delta_\alpha(\tilde s_i)\rho_i-\sum_{p\in\tilde L_Y}\rho_{i_0(p)}-\sum_{p\in\tilde N_Y\setminus\tilde L_Y}\frac{\rho_{i_0(p)}}{2}\right)-\sum_{\pi(p)\in\mathbf{x}\cap Y}\frac{\rho_{i_0(p)}}{2}.\end{aligned}$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Proof. We let $X_\alpha \subset Y$ be an irreducible component; then $\alpha > \bar{r}$ and $\rho_{\hbar_\alpha} = 0$. Since by our assumption $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is *slope semistable* and satisfies $\deg X \geqslant M_4 \geqslant M_2$, by Lemma 5.2, there is a positive constant C such that either $\deg X_\alpha > C \deg X$ or $\deg X_\alpha = 1$ satisfying $\mathbf{x} \cap X_\alpha = \emptyset$. If $\deg X_\alpha > C \deg X$, from the definition of $E^\epsilon_\alpha(\rho)$ (cf. (4.3)) and $\rho_{\hbar_\alpha} = 0$, we have

$$\frac{E^\epsilon_\alpha(\lambda,\rho)}{2} \leqslant \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{i \in \mathbb{I}^{\mathrm{pri}}_\alpha} \delta_\alpha(\tilde{s}_i)\rho_i - \left(\frac{1}{2} + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)}. \tag{5.19}$$

If $\deg X_\alpha = 1$, (5.19) still holds, since by (4.4) and Definition 3.10 we have

$$\begin{aligned}\frac{E^\epsilon_\alpha(\lambda,\rho)}{2} &= \delta_\alpha(\tilde{s}_{\mathrm{ind}_\alpha^{-1}(0)}) \cdot \frac{\bar{\rho}_{\mathrm{ind}_\alpha^{-1}(0)}}{2} + \rho_{\hbar_\alpha} = \sum_{i \in \mathbb{I}^{\mathrm{pri}}_\alpha} \delta_\alpha(\tilde{s}_i) \cdot \frac{\rho_i}{2} \\ &\leqslant \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{i \in \mathbb{I}^{\mathrm{pri}}_\alpha} \delta_\alpha(\tilde{s}_i)\rho_i - \left(\frac{1}{2} + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)},\end{aligned}$$

where $\tilde{S}_\alpha \cap \tilde{\Lambda}$ contains only linking nodes since $\mathbf{x} \cap X_\alpha = \emptyset$.

Next, we split

$$-\sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)} = -\sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \rho_{i_0(p)} - \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \rho_{i_0(p)} - \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)}.$$

Then, using $\rho_i \geqslant 0$, we get

$$\begin{aligned}-\left(\frac{1}{2} + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{S}_\alpha} \rho_{i_0(p)} &\leqslant -\sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(\frac{1}{2} + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \rho_{i_0(p)} \\ &\quad - \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)} + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} \\ &\leqslant -\sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \frac{\rho_{i_0(p)}}{2} \\ &\quad - \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)} + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2}.\end{aligned}$$

Putting them together, we obtain

$$\begin{aligned}\frac{E^\epsilon_\alpha(\lambda,\rho)}{2} \leqslant \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right) &\left(\sum_{i \in \mathbb{I}^{\mathrm{pri}}_\alpha} \delta_\alpha(\tilde{s}_i)\rho_i - \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \rho_{i_0(p)}\right. \\ &\left. - \sum_{p \in \tilde{N}_\alpha \setminus \tilde{L}_Y} \frac{\rho_{i_0(p)}}{2}\right) + \sum_{p \in \tilde{L}_Y \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2} - \sum_{p \in \pi^{-1}(\mathbf{x}) \cap \tilde{X}_\alpha} \frac{\rho_{i_0(p)}}{2}.\end{aligned}$$

Summing over α and applying (5.18) proves the lemma. □

The following inequality is crucial for the proof of the proposition.

Lemma 5.8. *For $1 \leqslant k \leqslant m_0$, we have*

$$\sum_{\substack{\alpha > \bar{r} \\ i \in \mathbb{I}^{\mathrm{pri}}_\alpha \cap [0,k)}} \delta_\alpha(\tilde{s}_i)\rho_i - \sum_{\substack{p \in \tilde{L}_Y \\ i_0(p) < k}} \rho_{i_0(p)} - \sum_{\substack{p \in \tilde{N}_Y \setminus \tilde{L}_Y \\ i_0(p) < k}} \frac{\rho_{i_0(p)}}{2} \leqslant \dim W_Y \cap W_k - \dim W_{Y \cap Y^{\complement}} \cap W_k,$$

where $W_{Y \cap Y^{\complement}}$ is the linear subspace in W spanned by $Y \cap Y^{\complement}$.

Proof. We prove the lemma by induction on k. When $k=0$, both sides of the inequality are zero and the inequality follows. Suppose the lemma holds for some $0 \leqslant k < m_0$. Then the lemma holds for $k+1$ if, for the expressions

$$A_{k,1} := \sum_{\substack{\alpha>\bar{r},\\ k\in\mathbb{I}_\alpha^{\rm pri}}} \delta_\alpha(\tilde{s}_k)\rho_k, \quad A_{k,2} := \sum_{\substack{p\in\tilde{L}_Y,\\ i_0(p)=k}} \rho_{i_0(p)}, \quad A_{k,3} := \sum_{\substack{p\in\tilde{N}_Y\setminus\tilde{L}_Y,\\ i_0(p)=k}} \frac{\rho_{i_0(p)}}{2}$$

and

$$B_{k,1} := \dim W_Y \cap W_{k+1} - \dim W_Y \cap W_k, \quad B_{k,2} := \dim W_{Y\cap Y^{\complement}} \cap W_{k+1} - \dim W_{Y\cap Y^{\complement}} \cap W_k,$$

the following inequality holds:

$$A_{k,1} - A_{k,2} - A_{k,3} \leqslant B_{k,1} - B_{k,2}. \tag{5.20}$$

To study the left-hand side of (5.20), we introduce the set

$$R_k = \{p \in \tilde{Y} \mid k \in \mathbb{I}_p^{\rm pri}\}. \tag{5.21}$$

By Propositions 3.9 and 3.11, R_k can take three possibilities, according to

$$\sum_{\alpha>\bar{r},k\in\mathbb{I}_\alpha^{\rm pri}} \delta_\alpha(\tilde{s}_k), \tag{5.22}$$

taking values 0, 1 or $\geqslant 2$. Notice that if $A_{k,1}=0$, then $A_{k,1}-A_{k,2}-A_{k,3} \leqslant 0$. The lemma holds trivially in this case since the right-hand side of (5.20) is non-negative. So, from now on, we will assume that $A_{k,1} \geqslant 1$; in particular, (5.22) is positive.

We first observe that since $\dim W_{k+1} - \dim W_k = 1$, both $B_{k,1}$ and $B_{k,2}$ can only take values 0 or 1. We now investigate the case when $B_{k,2}=1$.

CLAIM 5.9. *Suppose (5.22) is positive and $B_{k,2}=1$. Then there is a $p \in R_k$ (cf. (5.21)) such that $i_0(p)=k$ and*

$$q = \pi(p) \in Y \cap Y^{\complement} \cap (\mathbb{P}W_{k+1} - \mathbb{P}W_k). \tag{5.23}$$

Proof. Suppose (5.22) is positive; then there is a $p \in \mathrm{inc}(\tilde{s}_k) \cap \tilde{X}_\alpha$ with $\alpha > \bar{r}$ and $k \in \mathbb{I}_\alpha^{\rm pri}$. Let $Z_k = \mathbb{P}W_k \cap X$ be as defined after (5.9) and $W_{Z_k+q} \supsetneq W_k$ be as defined in (5.9). Then $W_{k+1} = W_{Z_k+q}$, since $\dim W_{k+1} = \dim W_k + 1$. Suppose $q = \pi(p) \notin Y \cap Y^{\complement}$ and $k \in \mathbb{I}_\alpha^{\rm pri}$; then, by applying the argument parallel to Propositions 3.9 and 3.11, we deduce

$$W_{Z_k+q} + W_{Y\cap Y^{\complement}} \supsetneq W_{Z_k} + W_{Y\cap Y^{\complement}}. \tag{5.24}$$

On the other hand, $B_{k,2}=1$ implies that

$$\begin{aligned}\dim(W_k + W_{Y\cap Y^{\complement}}) &= \dim W_k + \dim W_{Y\cap Y^{\complement}} - \dim W_k \cap W_{Y\cap Y^{\complement}}\\ &= \dim W_{k+1} + \dim W_{Y\cap Y^{\complement}} - \dim W_{k+1} \cap W_{Y\cap Y^{\complement}}\\ &= \dim(W_{k+1} + W_{Y\cap Y^{\complement}}),\end{aligned}$$

which means $W_k + W_{Y\cap Y^{\complement}} = W_{k+1} + W_{Y\cap Y^{\complement}}$, contradicting (5.24). So we must have $q \in Y \cap Y^{\complement}$.

By definition, $q \in \mathbb{P}W_{k+1}$ (cf. (5.8)) implies that $s_i(q)=0$ for $i \geqslant k+1$; $q \notin \mathbb{P}W_k$ implies that not all $s_i(q)$, $k \leqslant i \leqslant m$, are zero. Combined, we have $s_k(q) \neq 0$. This implies $i_0(q)=k$. As an easy consequence, this shows that $B_{k,2}=1$ forces $W_Y \cap W_{k+1} \neq W_Y \cap W_k$, and hence $B_{k,1}=1$. In particular, the right-hand side of (5.20) is non-negative. This proves the claim. □

We complete our proof of Lemma (5.8). When (5.22) takes value 1, then R_k consists of a single point, say $p \in \tilde{Y}$. In the case where $\pi(p) \in Y$ is a smooth point of X, $A_{k,1} = 1$ and $A_{k,2} = A_{k,3} = 0$. We claim that $B_{k,1} = 1$ and $B_{k,2} = 0$. Indeed, if $B_{k,1} = 0$, then $\mathbb{P}W_Y \cap \mathbb{P}W_{k+1} = \mathbb{P}W_Y \cap \mathbb{P}W_k$, which is the same as $Y \cap (s_k = \cdots = s_m = 0) = Y \cap (s_{k+1} = \cdots = s_m = 0)$ as subschemes of Y. But this contradicts $\sum_{\alpha > \bar{r}, k \in \mathbb{I}_\alpha^{\mathrm{pri}}} \delta_\alpha(\tilde{s}_k) = 1$. Thus $B_{k,1} = 1$. On the other hand, if $B_{k,2} = 1$, then Claim 5.9 shows that $R_k \cap \tilde{Y}$ contains an element in $\tilde{L}_Y$, contradicting our assumption that $R_k = \{p\}$ lies over a smooth point of X.

In the case $p \in \tilde{L}_Y$, the previous paragraph shows that $A_{k,1} = B_{k,1} = 1$, $A_{k,3} = 0$. For the values of $A_{k,2}$ and $B_{k,2}$, when $i_0(p) = k$, then both $A_{k,2} = B_{k,2} = 1$; when $i_0(p) \neq k$, then both $A_{k,2} = B_{k,2} = 0$. Therefore, (5.20) holds.

The last case is when $p \in \tilde{N}_Y - \tilde{L}_Y$. In this case, since the point p' in $\tilde{Y} \cap \pi^{-1}(\pi(p))$ other than p is not contained in R_k, either $i_0(p) \neq k$ or $i_0(p) = i_0(p') = k$ and $k \notin \mathbb{I}_{p'}^{\mathrm{pri}}$. In both cases, $A_{k,1} = B_{k,1} = 1$ and $B_{k,2} = 0$; the inequality (5.20) holds.

Lastly, when (5.22) is bigger than 1, by Propositions 3.9 and 3.11, either $R_k = \{p_-, p_+\}$ such that $\pi(p_-) = \pi(p_+)$ is a node of Y, i.e. $p_\pm \in \tilde{N}_Y$, and $i_0(p_-) = i_0(p_+) = k$, or $R_k = \{p_1, \ldots, p_l\}$ such that $i_0(p_i) = k$ and $\{\pi(p_i)\}_{1 \leqslant i \leqslant s}$ are distinct nodes of X. In the case $R_k = \{p_-, p_+\}$, since $p_\pm \in \tilde{N}_Y \backslash \tilde{L}_Y$, $A_{k,1} = 2$, $A_{k,2} = B_{k,2} = 0$ and $A_{k,3} = B_{k,1} = 1$. The inequality (5.20) holds in this case.

The other case is when $R_k = \{p_1, \ldots, p_l\}$. By reindexing, we may assume $p_1, \ldots, p_{l_1}$ are in $\tilde{N}_Y \backslash \tilde{L}_Y$ and $p_{l_1+1}, \ldots, p_l$ are in $\tilde{L}_Y$. We let $p_i' \in \tilde{Y}$ be such that $\pi^{-1}(\pi(p_i)) = \{p_i, p_i'\}$ for $i \leqslant l_1$. Then $i_0(p_i') = k$ as well, but $k \notin \mathbb{I}_{p_i'}^{\mathrm{pri}}$ because of Propositions 3.9 and 3.11. This in particular implies that the interior linking nodes $\tilde{N}_Y \backslash \tilde{L}_Y$ contribute once in $A_{k,1}$ but twice in $A_{k,3}$; namely, only $\rho_{i_0(p_i)}$ appears in $A_{k,1}$, but both $\rho_{i_0(p_i)}$ and $\rho_{i_0(p_i')}$ appear in $A_{k,3}$. Therefore, $A_{k,1} = l$, $A_{k,2} = l - l_1$ and $A_{k,3} = 2l_1/2 = l_1$. Hence the left-hand side of (5.20) is 0. This proves (5.20) in this case; hence for all cases. This proves the lemma. □

We continue our proof of Proposition 5.6. We apply Lemmas 5.7 and 5.8 with $k = m_0$. Noticing $\rho_{i_0(p)} = 0$ for $i_0(p) > m_0$, we obtain

$$\begin{aligned}
\frac{E_Y(\lambda,\rho)}{2} - \frac{\ell_Y}{2} &= \sum_{\alpha=\bar{r}+1}^{r} \frac{E_\alpha^\epsilon(\lambda,\rho)}{2} - \frac{\ell_Y}{2} \\
&\leqslant \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right)\left(\dim W_Y \cap W_{m_0} - \ell_Y\right) - \frac{1}{2} \sum_{\pi(p) \in \mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y} \rho_{i_0(p)} \\
&\leqslant \left(1 + \frac{C^{-1}\epsilon}{\deg X}\right)\left(\dim W_Y \cap W_{m_0} - \ell_Y\right) - \frac{1}{2} \sum_{p \in \tilde{S}_{\mathrm{reg}}} \hat{\rho}_{i_0(p)}. \qquad (5.25)
\end{aligned}$$

Here we used that for all $p' \in \pi(\tilde{S}_{\mathrm{reg}}) - X \cap \pi(\tilde{\Lambda}) \cap Y$, $\rho_{i_0}(p') = 0$. And the last inequality holds since by the definition of $\tilde{S}_{\mathrm{reg}}$ and $\hat{\rho}_i$ (cf. (5.14)), we have $\sum_{q \in \tilde{S}_{\mathrm{reg}}} \hat{\rho}_{i_0(q)} \leqslant \sum_{\pi(p) \in \mathbf{x} \cap \pi(\tilde{\Lambda}) \cap Y} \rho_{i_0(p)}$.

Using $\deg X - g = m$ (hence $2/(m+1) \geqslant 1/\deg X$) and $E_{Y^\complement}(\lambda,\rho) = 2\deg Y^\complement$, we obtain

$$\begin{aligned}
\frac{E_X(\lambda,\rho)}{2} &= \left(\deg Y^\complement + \frac{\ell_Y}{2}\right) + \left(\frac{E_Y(\lambda,\rho)}{2} - \frac{\ell_Y}{2}\right) \\
&\leqslant \left(\deg Y^\complement + \frac{\ell_Y}{2}\right) + \left(1 + \frac{2C^{-1}\epsilon}{m+1}\right)\left(m_0 + 1 - \dim W_{Y^\complement}\right) - \frac{1}{2} \sum_{p \in \tilde{S}_{\mathrm{reg}}} \rho_{i_0(p)}.
\end{aligned}$$

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Here the last inequality follows from

$$\begin{aligned}\dim W_{m_0} &\geqslant \dim(W_{m_0}\cap W_Y + W_{m_0}\cap W_{Y^{\complement}})\\ &= \dim W_{m_0}\cap W_Y + \dim W_{m_0}\cap W_{Y^{\complement}} - \dim W_{m_0}\cap W_Y\cap W_{Y^{\complement}}\\ &= \dim W_{m_0}\cap W_Y + \dim W_{Y^{\complement}} - \ell_Y.\end{aligned}$$

Now we consider the right-hand side of (5.15) for ρ chosen as in (5.16), which gives $\sum_{i=0}^m \rho_i = m_0+1$. Since, by our assumption, the embedding $X\subset \mathbb{P}W$ is given by a non-special complete linear system of a very ample line bundle $\mathcal{O}_X(1)$ (cf. Corollary 5.3), using our choice of weights ρ_i (cf. (5.17)) and the proof of Lemma 5.5, we obtain

$$\sum_{\alpha=1}^r\left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1\right)\cdot \rho_{\hbar_\alpha} = \deg Y^{\complement} + \frac{\ell_Y}{2} - \dim W_{Y^{\complement}}.$$

We claim that $\sum_{i=0}^m \hat\rho_i = m_0+1-\dim W_{Y^{\complement}}$. Indeed, from our choice of ρ and the definition of $\hat\rho$ (cf. (5.14)), for any $0\leqslant i\leqslant m$, $\hat\rho_i = 1$ or 0, and it is 0 if and only if either $i>m_0$ or there is an X_α with $i\in\mathbb{I}_\alpha$ (cf. (3.3)) such that $\rho_{\hbar_\alpha}=1$, that is, $i\in\mathbb{I}_{Y^{\complement}} = \bigcup_{X_\alpha\subset Y^{\complement}}\mathbb{I}_\alpha$. This proves $\sum_{i=0}^m\hat\rho_i = m_0+1-|\mathbb{I}_{Y^{\complement}}|$.

Our claim will follow once we prove $|\mathbb{I}_{Y^{\complement}}| = \dim W_{Y^{\complement}}$, but this follows from the following criterion.

Criterion: $i\in\mathbb{I}_{Y^{\complement}}$ *if and only if* $\dim W_{i+1}\cap W_{X_\alpha} - \dim W_i\cap W_{X_\alpha} = 1$ *for some* $X_\alpha\subset Y^{\complement}$.

To justify this criterion, we notice that $\dim W_{i+1}\cap W_{X_\alpha} = \dim W_i\cap W_{X_\alpha}$ for all $X_\alpha\subset Y^{\complement}$ is equivalent to $Y^{\complement}\cap\{s_i=\cdots=s_m=0\} = Y^{\complement}\cap\{s_{i+1}=\cdots=s_m=0\}$ as subschemes of $Y^{\complement}$; that is, $\mathrm{inc}(s_i)\cap Y^{\complement}=\emptyset$. Since λ is a staircase,

$$i\notin\mathbb{I}_\alpha\quad\text{for all } X_\alpha\subset Y^{\complement} \text{ if and only if } \mathrm{inc}(s_i)\cap Y^{\complement}=\emptyset \text{ (cf. (3.3))}.$$

This proves the criterion.

With those in hand, we obtain

$$\begin{aligned}\frac{E_X(\lambda,\rho)}{2} &= \left(\deg Y^{\complement} + \frac{\ell_Y}{2}\right) + \left(\frac{E_Y(\lambda,\rho)}{2} - \frac{\ell_Y}{2}\right)\\ &\leqslant \left(\deg Y^{\complement} + \frac{\ell_Y}{2}\right) + \left(1+\frac{2C^{-1}\epsilon}{m+1}\right)\left(m_0+1-\dim W_{Y^{\complement}}\right) - \sum_{q\in\tilde S_{\mathrm{reg}}}\frac{\hat\rho_{i_0(q)}}{2}\\ &\leqslant m_0+1-\sum_{q\in\tilde S_{\mathrm{reg}}}\frac{\hat\rho_{i_0(q)}}{2} + \left(\deg Y^{\complement} + \frac{\ell_Y}{2} - \dim W_{Y^{\complement}}\right) + \frac{2C^{-1}\epsilon}{m+1}\left(m_0+1-\dim W_{Y^{\complement}}\right)\\ &= \sum_{i=0}^m\rho_i - \sum_{q\in\tilde S_{\mathrm{reg}}}\frac{\hat\rho_{i_0(q)}}{2} + \sum_{\alpha=1}^r\left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1\right)\cdot\rho_{\hbar_\alpha} + \frac{2C^{-1}\epsilon}{m+1}\sum_{i=0}^m\hat\rho_i.\end{aligned}$$

So the proof of proposition is completed. □

Let

$$\hat\omega(\lambda) = \hat\omega(\lambda,\rho) := \frac{2\deg X}{m+1}\sum_{i=0}^m\rho_i - E_X^{\epsilon}(\lambda,\rho). \tag{5.26}$$

We now state and prove the main result of this section.

Theorem 5.10. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ such that $\chi_{\mathbf{a},g} > 0$, we let C be the constant given in Lemma 5.2. Suppose $1 > \epsilon > 0$ such that $(2C^{-1}+1)\epsilon < \chi_{\mathbf{a},g}$. Then there exists a constant $M_5 = M_5(g,n,\mathbf{a},\epsilon)$ such that for any slope stable (respectively semistable) weighted pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ of genus g and $\deg X > M_5$, and for any staircase 1-PS λ, we have*

$$\omega_{\mathbf{a}}(\lambda) = \omega(\lambda) + \mu_{\mathbf{a}}(\lambda) \geqslant \hat{\omega}(\lambda) + \mu_{\mathbf{a}}(\lambda) > (\text{respectively } \geqslant) \frac{2\epsilon}{m+1}\sum_{i=0}^{m} \hat{\rho}_i. \tag{5.27}$$

Proof. We will give a proof of the stable case, from which the semistable case follows easily.

First, let us justify the first inequality. Given $(2C^{-1}+1)\epsilon < \chi_{\mathbf{a},g}$, we define $M_5 = M_5(g, n, \mathbf{a}, \epsilon) := \max\{M_4(g,n,\mathbf{a}), M_1(g, 6(g+n), n, \epsilon)/C\} > 0$, where C is the constant introduced in Lemma 5.2. Then by the slope stability assumption and Lemma 5.2, whenever $\deg X > M_5$, either X_α is exceptional or

$$\deg X_\alpha > CM_5 > M_1(g, 6(g+n), n, \epsilon) > M_1(g_\alpha, \ell_\alpha, n, \epsilon),$$

where the last inequality follows from $g \geqslant g_\alpha, 6(g+n) \geqslant \ell_\alpha$ (cf. Corollary 5.3) and the definition of M_1 in the proof of Theorem 4.1. Hence the assumption of Theorem 4.1 is satisfied. Applying Theorem 4.1 to $\hat{\omega}(\lambda)$ and using (2.15), we obtain $\omega(\lambda) > \hat{\omega}(\lambda)$. Thus the first inequality is proved.

By Proposition 5.6, it suffices to prove

$$\sum_{i=0}^{m} \rho_i - \sum_{q\in \tilde{S}_{\mathrm{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} + \sum_{\alpha=1}^{r}\left(\deg X_\alpha + \frac{\ell_\alpha}{2} - m_\alpha - 1\right)\cdot \rho_{\hbar_\alpha} + \frac{(2C^{-1}+1)\epsilon}{m+1}\sum_{i=0}^{m}\hat{\rho}_i$$
$$< \frac{\deg X}{m+1}\sum_{i=0}^{m}\rho_i + \frac{\mu_{\mathbf{a}}(\lambda,\rho)}{2}. \tag{5.28}$$

By linear programming, we only need to prove the above estimate for ρ of the form (5.16). We will break the verification into several inequalities. First, we have (defining $a_X = \sum_{j=1}^n a_j$)

$$\mu_{\mathbf{a}}(\lambda,\rho) = \frac{m_0+1}{m+1}a_X - \sum_{x_j\in Y\cap \mathbb{P}W_{m_0}} a_j - \sum_{x_j \in Y^{\complement}\cap \mathbb{P}W_{m_0}} a_j. \tag{5.29}$$

Here x_j runs through all marked points of the curve. We claim that

$$\sum_{q\in\tilde{S}_{\mathrm{reg}}} \frac{\hat{\rho}_{i_0(q)}}{2} = \frac{|\mathbf{x}\cap \pi(\tilde{\Lambda})\cap Y \cap \mathbb{P}W_{m_0}|}{2} \geqslant \sum_{x_j\in Y\cap\mathbb{P}W_{m_0}} \frac{a_j}{2}. \tag{5.30}$$

To this purpose, we first show that

$$\mathbf{x}\cap \pi(\tilde{\Lambda})\cap Y\cap \mathbb{P}W_{m_0} = \mathbf{x}\cap Y\cap\mathbb{P}W_{m_0}. \tag{5.31}$$

Indeed, for any x_i in $\mathbf{x}$ that lies in $Y\cap\mathbb{P}W_{m_0}$, $s_k(x_j) = 0$ for $k \geqslant m_0$. On the other hand, letting $x_j \in X_\alpha \subset Y$, since $Y^{\complement}$ is the largest subcurve of X contained in $\mathbb{P}W_{m_0}$, for some $k \geqslant m_0$, $s_k|_{X_\alpha} \neq 0$. Combined with $s_k(x_j) = 0$, we conclude $x_j \in \pi(\tilde{\Lambda})$ (cf. Definition 3.1). In particular, $\mathbf{x}\cap Y\cap \mathbb{P}W_{m_0} \subset \pi(\tilde{\Lambda})$. This proves (5.31).

Applying (5.31), and using that, for any colliding subset $\{x_{i_1},\dots,x_{i_s}\}$ (i.e. $x_{i_1} = \dots = x_{i_s}$), necessarily $a_{i_1} + \dots + a_{i_s} \leqslant 1$, we obtain

$$\sum_{x_j\in Y\cap\mathbb{P}W_{m_0}} \frac{a_j}{2} - \frac{|\mathbf{x}\cap\pi(\tilde{\Lambda})\cap Y\cap\mathbb{P}W_{m_0}|}{2} = \sum_{x_j\in Y\cap\mathbb{P}W_{m_0}}\frac{a_j}{2} - \frac{|\mathbf{x}\cap Y\cap\mathbb{P}W_{m_0}|}{2} \leqslant 0, \tag{5.32}$$

and hence (5.30).

By putting (5.29) and (5.30) together, we obtain

$$-\sum_{q\in\tilde{S}_{\mathrm{reg}}}\frac{\hat{\rho}_{i_0(q)}}{2}-\frac{\mu_{\mathbf{a}}(\lambda,\rho)}{2}\leqslant -\frac{m_0+1}{m+1}\frac{a_X}{2}+\sum_{x_j\in Y^{\complement}\cap\mathbb{P}W_{m_0}}\frac{a_j}{2}. \tag{5.33}$$

On the other hand, for ρ of the form in (5.16), we have

$$\begin{aligned}&\sum_{i=0}^{m}\rho_i+\sum_{\alpha=1}^{r}\Big(\deg X_\alpha+\frac{\ell_\alpha}{2}-m_\alpha-1\Big)\cdot\rho_{\hbar_\alpha}+\frac{(2C^{-1}+1)\epsilon}{m+1}\sum_{i=0}^{m}\hat{\rho}_i\\&\quad=m_0+1+\left(\deg Y^{\complement}+\frac{\ell_Y}{2}-\dim W_{Y^{\complement}}\right)+\frac{(2C^{-1}+1)\epsilon}{m+1}(m_0+1-\dim W_{Y^{\complement}}).\end{aligned} \tag{5.34}$$

Plugging (5.34) and (5.33) into (5.28), we obtain

$$\begin{aligned}&-\frac{\mu_{\mathbf{a}}(\lambda,\rho)}{2}+\frac{E_X^{\epsilon}(\lambda,\rho)}{2}+\frac{(2C^{-1}+1)\epsilon}{m+1}\sum_{i=0}^{m}\hat{\rho}_i\\&\leqslant m_0+1+\left(\deg Y^{\complement}+\frac{\ell_Y}{2}+\sum_{x_j\in Y^{\complement}\cap\mathbb{P}W_{m_0}}\frac{a_j}{2}-\dim W_{Y^{\complement}}\right)-\frac{m_0+1}{m+1}\frac{a_X}{2}\\&\quad+\frac{(2C^{-1}+1)\epsilon}{m+1}(m_0+1-\dim W_{Y^{\complement}})\\&=\frac{\deg Y^{\complement}+\ell_Y/2+\sum_{x_j\in Y^{\complement}\cap\mathbb{P}W_{m_0}}a_j/2}{\dim W_{Y^{\complement}}}\dim W_{Y^{\complement}}-\frac{m_0+1}{m+1}\frac{a_X}{2}\\&\quad+\left(1+\frac{(2C^{-1}+1)\epsilon}{m+1}\right)(m_0+1-\dim W_{Y^{\complement}}).\end{aligned} \tag{5.35}$$

Since $\dim W_{Y^{\complement}}\leqslant m_0<m$, we have

$$\frac{\deg Y^{\complement}+\ell_Y/2+\sum_{x_j\in Y^{\complement}\cap\mathbb{P}W_{m_0}}a_j/2}{\dim W_{Y^{\complement}}}<\frac{\deg X+a_X/2}{m+1}$$

by our stability assumption and Lemma 5.5. Hence we have

$$\begin{aligned}\text{LHS of (5.35)}&<\frac{\deg X+a_X/2}{m+1}\dim W_{Y^{\complement}}+\left(1+\frac{(2C^{-1}+1)\epsilon}{m+1}\right)(m_0+1-\dim W_{Y^{\complement}})-\frac{m_0+1}{m+1}\frac{a_X}{2}\\&\leqslant\frac{\deg X+a_X/2}{m+1}(\dim W_{Y^{\complement}}+m_0+1-\dim W_{Y^{\complement}})-\frac{m_0+1}{m+1}\frac{a_X}{2}\\&=\frac{\deg X}{m+1}\cdot(m_0+1)=\frac{\deg X}{m+1}\sum_{i=0}^{m}\rho_i,\end{aligned}$$

where we have used the assumption $(2C^{-1}+1)\epsilon<\chi_{\mathbf{a},g}$ to conclude

$$\frac{\deg X+a_X/2}{m+1}>1+\frac{(2C^{-1}+1)\epsilon}{m+1}$$

in the second inequality. This completes the proof. □

Proof of Theorem 1.5. We first prove that slope stable implies Chow stable. By our assumption $\chi_{\mathbf{a},g} > 0$, we may choose an $0 < \epsilon < 1$ such that $(2C^{-1}+1)\epsilon < \chi_{\mathbf{a},g}$. Fixing such ϵ, we let $M_5 = M_5(g,n,\mathbf{a},\epsilon)$ be the constant given in Theorem 5.10. Then by Theorem 5.10, whenever $\deg X > M_5$ and $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is slope stable we have $\omega_{\mathbf{a}}(\lambda) > (2\epsilon/(m+1))\sum_{i=0}^m \hat{\rho}_i \geqslant 0$; hence $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is stable by Definition 1.3. The proof for the semistable case is similar. This proves the sufficient part.

We now prove the other direction: Chow stable implies slope stable. Let $Y \subset X$ be any proper subcurve, let $W_Y \subset W$ be the linear subspace spanned by Y, and let $m_0+1 = \dim W_Y$. We choose a 1-PS $\lambda = \mathrm{diag}[t^{\rho_0}, \dots, t^{\rho_m}] \cdot t^{-\rho_{\mathrm{ave}}}$ such that the corresponding filtration $\{W_i\}_{i=0}^m$ satisfies $W_{m_0+1} = W_Y$; we choose the weights $\{\rho_i\}$ as in (5.16). Then

$$\mu_{\mathbf{a}} = a_X\left(\frac{m_0+1}{m+1}\right) - \sum_{x_j \in \mathbb{P}W_Y}^{n} a_j.$$

Thus, by the proof of [Mum77, Proposition 5.5, p. 60], $e(\tilde{\mathcal{I}}(\lambda))/2 \geqslant \deg Y + \ell_Y/2$; hence

$$\begin{aligned} 0 \leqslant \frac{\omega(\lambda)+\mu_{\mathbf{a}}(\lambda)}{2} &= \frac{\sum_{i=0}^m \rho_i}{m+1}\cdot \deg X - \frac{e(\tilde{\mathcal{I}}(\lambda))}{2} + \frac{\mu_{\mathbf{a}}(\lambda)}{2} \\ &\leqslant \frac{m_0+1}{m+1}\cdot \deg X - \left(\deg Y + \frac{\ell_Y}{2}\right) + \frac{m_0+1}{m+1}\frac{a_X}{2} - \sum_{x_j\in\mathbb{P}W_Y}^{n} \frac{a_j}{2} \\ &= (m_0+1)\left(\frac{\deg X + a_X/2}{m+1} - \frac{\deg Y + \ell_Y/2 + a_Y/2}{m_0+1}\right), \end{aligned} \tag{5.36}$$

which is equivalent to the slope semistability (cf. Definition 1.4) provided that $(X, \mathcal{O}_X(1))$ is *non-special*, which will be proved in Proposition 6.2.

Finally, if we assume further that $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is stable, then (5.36) becomes equality only if $m_0 = m$. Since $M_5 \geqslant M_4$ by our choice, by Lemma 5.5, $m_0 = m$ only when $Y^{\complement}$ is a disjoint union of exceptional components provided $\deg X > M_5$. By choosing $M(g,n,\mathbf{a}) := \max\{M_5, M_6\}$ with M_6 being determined in Proposition 6.2, we complete the proof of the theorem. □

Proof of Theorem 1.6. By our choice that $M \geqslant M_5 \geqslant M_4$, our claim follows from Proposition 5.4. □

6. Re-construction of the moduli of weighted pointed curves

In this section, we use the GIT quotient of the Hilbert scheme to construct the moduli of weighted pointed stable curves, first introduced and constructed by Hassett [Has03] using a different method. First, following Caporaso [Cap94, § 3.3], we introduce the following definition.

Definition 6.1. A *weighted pointed quasistable* curve is a weighted pointed nodal curve $(X, \mathbf{x}, \mathbf{a})$ such that:

(1) $\omega_X(\mathbf{a}\cdot\mathbf{x})$ is numerically non-negative;
(2) the total degree $2\chi_{\mathbf{a},g}(X) = \deg \omega_X(\mathbf{a}\cdot\mathbf{x})$ is positive;
(3) any connected subcurve $E \subset X$ satisfying $\deg \omega_X(\mathbf{a}\cdot\mathbf{x})|_E = 0$ must have $E \cap \mathbf{x} = \emptyset$ and $E \cong \mathbb{P}^1$ and is called an *exceptional component.*

We say $(X, \mathbf{x}, \mathbf{a})$ is weighted pointed stable if it is weighted pointed quasistable and does not contain exceptional components.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

6.1 As a GIT quotient

We fix integers n, g and weights $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g}(X) > 0$; for a large integer k such that $k \cdot a_i \in \mathbb{Z}$ for all i, we let $d = (|\mathbf{a}| + 2g - 2) \cdot k$, and form

$$P(t) = d \cdot t + 1 - g \in \mathbb{Z}[t] \quad \text{and} \quad m + 1 = P(1). \tag{6.1}$$

We denote by $\mathrm{Hilb}^P_{\mathbb{P}^m}$ the Hilbert scheme of subschemes of $\mathbb{P}^m$ of Hilbert polynomial P; we define $\mathcal{H}$ to be the fine moduli scheme of flat families of $(X, \iota, \mathbf{x})$, where

$$[\iota : X \to \mathbb{P}^m] \in \mathrm{Hilb}^P_{\mathbb{P}^m} \quad \text{and} \quad \mathbf{x} = (x_1, \ldots, x_n) \in X^n.$$

Using that Hilbert schemes are projective, we see that $\mathcal{H}$ exists and is projective. We denote by

$$(\pi_{\mathcal{H}}, \varphi) : \mathcal{X} \longrightarrow \mathcal{H} \times \mathbb{P}^m, \quad \mathfrak{x}_i : \mathcal{H} \to \mathcal{X}, \quad i = 1, \ldots, n, \tag{6.2}$$

the universal family of $\mathcal{H}$.

We introduce a parallel space for the Chow variety. We let $\mathrm{Chow}^d_{\mathbb{P}^m}$ be the Chow variety of degree d dimension one effective cycles in $\mathbb{P}^m$. For any such cycle Z, we denote by $\mathrm{Chow}(Z) \in \mathrm{Div}^{d,d}[(\mathbb{P}^{m\vee})^2]$ its associated Chow point (cf. § 1). We define

$$\mathcal{C} := \{(Z, \mathbf{x}) \in \mathrm{Chow}^d_{\mathbb{P}^m} \times (\mathbb{P}^m)^n \mid \mathbf{x} = (x_1, \ldots, x_n) \in (\mathrm{Supp}\, Z)^n\}.$$

By the Chow theorem, $\mathcal{C}$ is projective. Using the Chow coordinate, we obtain an injective morphism

$$\mathcal{C} \xrightarrow{\subset} \mathrm{Div}^{d,d}[(\mathbb{P}^{m\vee})^2] \times (\mathbb{P}^m)^n. \tag{6.3}$$

As before (cf. § 1), we endow it with the ample $\mathbb{Q}$-line bundle $\mathcal{O}_{\mathcal{C}}(1, \mathbf{a})$, which is canonically linearized by the diagonal action of $G := \mathrm{SL}(m+1)$ on $\mathcal{C}$. We let $\mathcal{C}^{\mathrm{ss}} \subset \mathcal{C}$ be the (open) set of *semistable* points with respect to the G linearization on $\mathcal{O}_{\mathcal{C}}(1, \mathbf{a})$.

For any one-dimensional subscheme $X \subset \mathbb{P}^m$, we denote by $[X]$ its associated one-dimensional cycle. By sending $(X, \iota, \mathbf{x}) \in \mathcal{H}$ to $([X], \mathbf{x}) \in \mathcal{C}$, we obtain the G-equivariant *Hilbert–Chow* morphism (cf. [MFK94, § 5.4])

$$\Phi : \mathcal{H} \longrightarrow \mathcal{C}.$$

To characterize the members in $\Phi^{-1}(\mathcal{C}^{\mathrm{ss}})$, we need the following proposition.

Proposition 6.2. *For g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g} > 0$, there is an integer $M_6 = M_6(g, n, \mathbf{a})$ so that for $d > M_6$, a connected one-dimensional closed subscheme $X \subset \mathbb{P}^m$ satisfies $(X, \iota, \mathbf{x}) \in \Phi^{-1}(\mathcal{C}^{\mathrm{ss}})$ if and only if the associated data $(X, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \mathbf{x}, \mathbf{a})$ is a slope semistable polarized weighted pointed nodal curve.*

The proof is a slight modification of the one given in [Mum77, Proposition 3.1] by incorporating the weighted points. To do that, we need the following.

Lemma 6.3. *Let $\lambda = \mathrm{diag}[t, 1, \ldots, 1]$, $\lambda' = \mathrm{diag}[t^4, t^2, t, 1, \ldots, 1]$ and $x_0 := [1, 0, \ldots, 0] \in \mathbb{P}^m$. Then the $\mathbf{a}$-λ-weight (respectively $\mathbf{a}$-λ'-weight) of $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{P}^m)^n$ is*

$$\mu_{\mathbf{a}}(\lambda) = -\frac{m \cdot \sum_{x_j = x_0} a_j}{m+1} \left(\text{respectively } \mu_{\mathbf{a}}(\lambda') = -\frac{(4m-3) \cdot \sum_{x_j = x_0} a_j}{m+1} \right).$$

In particular, $\mu_{\mathbf{a}}(\lambda), \mu_{\mathbf{a}}(\lambda') \leqslant 0$ as long as $m \geqslant 1$.

Proof. It directly follows from (5.10). □

Proof. Let $(X,\iota,\mathbf{x})\in\Phi^{-1}(\mathcal{C}^{\mathrm{ss}})$. We claim that when $\deg X/(m+1)<8/7$, then every irreducible component of the cycle $[X]$ has multiplicity one; X_{red} is a nodal curve, and X differs from X_{red} by embedded points. First, if $x\in X$ has multiplicity at least 3, then we choose coordinates so that $x=[1,0,\dots,0]$ and $\lambda=\mathrm{diag}[t,1,\dots,1]$; in this case the weight

$$\begin{aligned}\omega_{\mathbf{a}}(\lambda)=\omega(\lambda)+\mu_{\mathbf{a}}(\lambda)&=\frac{2\deg X\sum_{i=0}^m}{m+1}-e(\mathcal{I}(\lambda)-\frac{m\cdot\sum_{x_j=x_0}a_j}{m+1}\\&\leqslant\frac{2\deg X\cdot\sum_{i=0}^m\rho_i}{m+1}-e(\mathcal{I}(\lambda)<\frac{16}{7}\sum_{i=0}^m\rho_i-3<0.\end{aligned}$$

Next, if $x\in X$ is a non-ordinary double point then, by choosing the coordinates in the proof of [Mum77, Proposition 3.1] and $\lambda=\mathrm{diag}[t^4,t^2,t,1,\dots,1]$ accordingly, we obtain

$$\begin{aligned}\omega_{\mathbf{a}}(\lambda)=\omega(\lambda)+\mu_{\mathbf{a}}(\lambda)&=\frac{2\deg X\sum_{i=0}^m}{m+1}-e(\mathcal{I}(\lambda)-\frac{(4m-3)\cdot\sum_{x_j=x_0}a_j}{m+1}\\&\leqslant\frac{2\deg X\cdot\sum_{i=0}^m\rho_i}{m+1}-e(\mathcal{I}(\lambda)<\frac{16}{7}\sum_{i=0}^m\rho_i-16=0.\end{aligned}$$

Both cases contradict our assumption that $(X,\iota,\mathbf{x})\in\Phi^{-1}(\mathcal{C}^{\mathrm{ss}})$. This proves the claim.

We now show that $X=X_{\mathrm{red}}\subset\mathbb{P}^m$ and is embedded by a complete non-special linear system. Let $X_\alpha\subset X_{\mathrm{red}}$ be an irreducible component, and write $X_{\mathrm{red}}=X_\alpha\cup X_\alpha^\complement$. For $W_{X_\alpha}\subset W$ the linear subspace spanned by X_α, we choose a basis $\{s_i\}$ so that $W_{X_\alpha}=\{s_{m_\alpha+1}=\cdots=s_m=0\}$, and define a 1-PS λ by the rule

$$\lambda=\mathrm{diag}[\overbrace{t,\dots,t}^{m_\alpha+1},1,\dots,1].$$

Since $\mathrm{Chow}(X,\iota,\mathbf{x})\in\mathcal{C}^{\mathrm{ss}}$, by the same calculation as in the proof of Theorem 1.5 (cf. (5.36)) we obtain

$$\begin{aligned}0\leqslant\frac{\omega_{\mathbf{a}}(\lambda)}{2}&=\frac{\omega(\lambda)+\mu_{\mathbf{a}}(\lambda)}{2}\\&\leqslant\frac{m_\alpha+1}{m+1}\cdot\deg X-\left(\deg Y+\frac{\ell_Y}{2}\right)+\frac{m_\alpha+1}{m+1}\frac{a_X}{2}-\sum_{x_j\in\mathbb{P}W_{X_\alpha}}\frac{a_j}{2}\\&=(m_\alpha+1)\left(\frac{\deg X+a_X/2}{m+1}-\frac{\deg X_\alpha+\ell_{X_\alpha}/2+a_{X_\alpha}/2}{m_\alpha+1}\right).\end{aligned}$$

Now we choose $M_6\geqslant 8(g-1)+n/2$, and assume $d=\deg X>M_6$; then

$$\frac{\deg X_\alpha+\ell_{X_\alpha}/2+a_{X_\alpha}/2}{m_\alpha+1}\leqslant\frac{\deg X+a_X/2}{m+1}\leqslant\frac{8}{7}. \tag{6.4}$$

We claim that $h^1(\mathcal{O}_{X_\alpha}(1))=0$. Suppose not, then by Saint-Donat's extension of Clifford's theorem [GM84, Lemma 9.1] we have $h^0(\mathcal{O}_{X_\alpha}(1))\leqslant\frac{1}{2}\deg X_\alpha+1$ which, combined with (6.4), implies

$$\deg X_\alpha\leqslant\tfrac{8}{7}h^0(\mathcal{O}_{X_\alpha}(1))\leqslant\tfrac{8}{14}\deg X_\alpha+\tfrac{8}{7}.$$

Note that this is possible only if $\deg X_\alpha\leqslant 2$, and then $X_\alpha\cong\mathbb{P}^1$ and $h^1(X_\alpha,\mathcal{O}_{X_\alpha}(1))=0$, a contradiction. This proves the claim.

We next claim that $h^1(\mathcal{O}_{X_\alpha}(1)(-L_\alpha))=0$, where $L_\alpha=X_\alpha\cap X_\alpha^\complement$ (cf. (3.9)). Indeed, by (6.4) and using $h^1(\mathcal{O}_{X_\alpha}(1))=0$ just proved, we deduce

$$\deg X_\alpha+\frac{a_{X_\alpha}}{2}+\frac{\ell_{X_\alpha}}{2}\leqslant\frac{8}{7}(\deg X_\alpha+1-g(X_\alpha)).$$

Hence

$$\deg(\mathcal{O}_{X_\alpha}(1)(-L_\alpha)) = \deg X_\alpha - \ell_{X_\alpha} \geqslant 8(g(X_\alpha)-1) + \tfrac{5}{2}\ell_{X_\alpha} + \tfrac{7}{2}a_{X_\alpha}$$

which is greater than $2g(X_\alpha)-2$ unless either when $g(X_\alpha)=1$, $\ell_{X_\alpha}=0$ and $a_{X_\alpha}/2=0$, or when $g(X_\alpha)=0$, $\ell_{X_\alpha}=1$ or 2. The first case cannot happen, since $L_\alpha \neq \emptyset$ by our assumption on the connectedness of X; in the second case, we have $\mathcal{O}_{X_\alpha}(1)(-L_\alpha) = \mathcal{O}_{\mathbb{P}^1}(e)$ with $e \geqslant -1$. Thus for both cases we have $h^1(\mathcal{O}_{X_\alpha}(1)(-L_\alpha)) = 0$. This settles the claim.

Finally, we show that $h^1(X, \mathcal{O}_X(1)) = 0$ and $X = X_{\text{red}}$. First, $h^1(\mathcal{O}_{X_\alpha}(1)(-L_\alpha)) = 0$ for all X_α and that X_{red} is a nodal curve implies that $h^1(\mathcal{O}_{X_{\text{red}}}(1)) = 0$. Since X differs from X_{red} by embedded points, we have $h^1(\mathcal{O}_X(1)) = 0$. This proves that $(X, \mathcal{O}_X(1))$ is non-special.

It remains to show that $X = X_{\text{red}}$. As $(X, \iota, \mathbf{x}) \in \mathcal{H}$, by the vanishing already proved, we have $m+1 = h^0(X, \mathcal{O}_X(1))$. Suppose $X_{\text{red}} \neq X$, then X_{red} lies in a hyperplane, say $\{s_m = 0\}$ for a basis $\{s_i\}$. Let $\lambda = \operatorname{diag}[t, \ldots, t, 1]$, then the λ-weight for $\operatorname{Chow}(X, \iota, \mathbf{x})$ (by letting $Y = X_{\text{red}}$ in (5.36)) is

$$\frac{\omega(\lambda) + \mu_{\mathbf{a}}(\lambda)}{2} = m\left(\frac{\deg X + a_X/2}{m+1} - \frac{\deg X + a_X/2}{m}\right) < 0,$$

contradicting the fact that $(X, \iota, \mathbf{x}) \in \Phi^{-1}(\mathcal{C}^{\text{ss}})$. So $X = X_{\text{red}}$ is a nodal curve. This implies that $X \subset \mathbb{P}W$ is non-degenerate and is embedded by a complete linear system.

Our next step is to show that $(X, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \mathbf{x}, \mathbf{a})$ is a weighted pointed nodal curve. For this, we need to verify that the weighted points are away from the nodes of X, and the total weight at any point is no more than one. Let $p \in X$ be any point. We choose a 1-PS λ as in Example 2.5; the associated λ-weight for $\operatorname{Chow}(X, \iota, \mathbf{x})$ is

$$\omega(\lambda) + \mu_{\mathbf{a}}(\lambda) = \frac{2\deg X}{m+1} - \epsilon_p + \frac{1}{m+1}a_X - \sum_{x_j = p} a_j = 2 - \epsilon_p + \frac{2\chi_{\mathbf{a},g}}{m+1} - \sum_{x_j=p} a_j,$$

where $\epsilon_p = 2$ if p is a node and 1 otherwise. Since $\operatorname{Chow}(X, \iota, \mathbf{x})$ is semistable, we must have $0 \leqslant \omega(\lambda) + \mu_{\mathbf{a}}(\lambda)$. Now we choose M so that $M \geqslant g + 2\chi_{\mathbf{a},g}/\min\{a_i\}$; then $0 \leqslant \omega(\lambda) + \mu_{\mathbf{a}}(\lambda)$ implies that the weighted points must be away from the nodes, and the total weight of marked points at p does not exceed one.

In the end, Theorem 1.5 implies that such an $(X, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \mathbf{x}, \mathbf{a})$ is slope semistable. This proves that for the choice $M_6(g, n, \mathbf{a}) := \max\{g + 2\chi_{\mathbf{a},g}/\min\{a_i\}, 8(g-1) + n/2\}$, the lemma holds. ☐

We define

$$\mathcal{H}^{\text{ss}} = \Phi^{-1}(\mathcal{C}^{\text{ss}}) \subset \mathcal{H}.$$

Corollary 6.4. *For $d \geqslant M$ specified in Proposition 6.2, the restriction*

$$\Phi^{\text{ss}} := \Phi|_{\mathcal{H}^{\text{ss}}} : \mathcal{H}^{\text{ss}} \to \mathcal{C}^{\text{ss}}$$

is injective and hence an isomorphism.

Proof. We only need to prove that Φ^{ss} is injective. Suppose not, and say there are $(X, \iota, \mathbf{x}) \neq (X', \iota', \mathbf{x}') \in \mathcal{H}^{\text{ss}}$ such that $\Phi(X, \iota, \mathbf{x}) = \Phi(X', \iota', \mathbf{x}') \in \mathcal{C}^{\text{ss}}$; then by Lemma 6.2, both X and X' are nodal subcurves of $\mathbb{P}^m$. Since $\Phi(X, \iota, \mathbf{x}) = \Phi(X', \iota', \mathbf{x}') \in \mathcal{C}^{\text{ss}}$, the cycles $[X] = [X']$ and $\mathbf{x} = \mathbf{x}' \subset \mathbb{P}^m$. Since both X and X' are nodal, we must have $X = X'$; thus $(X, \iota, \mathbf{x}) = (X', \iota', \mathbf{x}')$, a contradiction. This proves that Φ^{ss} is injective. Finally, since $\mathcal{C}^{\text{ss}}$ is normal, we conclude that Φ^{ss} is an isomorphism by Zariski's main theorem. ☐

To construct the moduli of weighted pointed curves, taking the k specified before (6.1), we form

$$\mathcal{K} = \{(X, \iota, \mathbf{x}) \in \mathcal{H} \mid X \text{ smooth weighted pointed curves}, \iota^* \mathcal{O}_{\mathbb{P}^m}(1) \cong \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}\}.$$

It is locally closed and is a smooth subscheme of $\mathcal{H}$. (Note that $\mathcal{H}$ is smooth near $\mathcal{K}$.) Since X in $(X, \iota, \mathbf{x}) \in \mathcal{K}$ are smooth, applying Theorem 1.5, we conclude that $\Phi(\mathcal{K}) \subset \mathcal{C}^{\mathrm{ss}}$, and thus $\mathcal{K} \subset \mathcal{H}^{\mathrm{ss}}$. Let $\overline{\mathcal{K}} \subset \mathcal{H}^{\mathrm{ss}}$ be the closure of $\mathcal{K}$ in $\mathcal{H}^{\mathrm{ss}}$. Because Φ^{ss} is finite, and $\mathcal{C}$ is projective, the GIT quotients $\mathcal{H}^{\mathrm{ss}} /\!\!/ G \to \mathcal{C}^{\mathrm{ss}} /\!\!/ G$ exist and the arrow is finite [Gie77, Lemma 4.6]; thus $\mathcal{H}^{\mathrm{ss}} /\!\!/ G$ is projective. Because $\overline{\mathcal{K}}$ is closed in $\mathcal{H}^{\mathrm{ss}}$, the GIT quotient

$$\mathbf{q} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{K}} /\!\!/ G \tag{6.5}$$

exists and is projective.

There is a natural transformation from the category of flat families of pointed curves in $\overline{\mathcal{K}}$ to the category of stable genus g, $\mathbf{a}$-weighted pointed nodal curves. For any $(X, \iota, \mathbf{x}) \in \overline{\mathcal{K}}$, since the associated weighted pointed nodal curve $(X, \mathbf{x}, \mathbf{a})$ is semistable, we can form a new weighted pointed curve by contracting all of its exceptional components (cf. Definition 6.1). We denote the resulting curve by

$$(X^{\mathrm{st}}, \mathbf{x}^{\mathrm{st}}, \mathbf{a}), \tag{6.6}$$

and call it the *stabilization of* $(X, \mathbf{x}, \mathbf{a})$. Since $(X, \iota, \mathbf{x}) \in \mathcal{H}^{\mathrm{ss}}$ and the marked points never lie on the contracted components, by Lemma 5.2 the stabilization produces a weighted pointed *stable* curve of the same genus. Furthermore, the stabilization applies to families of quasistable weighted pointed curves. The mentioned transformation is obtained by applying this contraction to the restriction to $\overline{\mathcal{K}}$ of the universal family of $\mathcal{H}$, resulting in a family of weighted pointed stable curves on $\overline{\mathcal{K}}$.

Let $\overline{\mathcal{M}}_{g,\mathbf{a}}$ be the coarse moduli space of stable genus g, $\mathbf{a}$-weighted nodal curves constructed by Hassett [Has03]. This transformation induces a morphism

$$\overline{\Psi} : \overline{\mathcal{K}} \longrightarrow \overline{\mathcal{M}}_{g,\mathbf{a}}. \tag{6.7}$$

As this morphism is G-equivariant with G acting trivially on $\overline{\mathcal{M}}_{g,\mathbf{a}}$, it descends to a morphism

$$\psi : \overline{\mathcal{K}} /\!\!/ G \longrightarrow \overline{\mathcal{M}}_{g,\mathbf{a}}. \tag{6.8}$$

Theorem 6.5. *The morphism ψ is an isomorphism.*

It is worth mentioning that the two coarse moduli schemes $\overline{\mathcal{K}} /\!\!/ G$ and $\overline{\mathcal{M}}_{g,\mathbf{a}}$ parameterize different moduli objects. For g, $\mathbf{a}$ and sufficiently divisible k, we define $\widetilde{\mathcal{P}} \subset \mathcal{H}$ via

$$\widetilde{\mathcal{P}} = \{(X, \iota, \mathbf{x}) \in \mathcal{H} \mid (X, \mathbf{x}, \mathbf{a}) \text{ weighted pointed stable curves}, \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k} \cong \iota^* \mathcal{O}_{\mathbb{P}^m}(1)\}.$$

A direct check shows that $\widetilde{\mathcal{P}}$ with reduced structure is a smooth, locally closed and G-invariant subscheme of $\mathcal{H}$. We let $\mathcal{P} \subset \widetilde{\mathcal{P}}$ be the open subset of $(X, \iota, \mathbf{x})$ such that the X are smooth. By definition, $\mathcal{P} = \mathcal{K}$. However, the following example shows that $\widetilde{\mathcal{P}} \not\subseteq \mathcal{H}^{\mathrm{ss}}$. The theorem states that this change of moduli objects does not alter the resulting coarse moduli schemes.

Example 6.6. Let X be a nodal curve with one node and two smooth irreducible components X_1 and X_2, of genus $g(X_1) = g(X_2) = 2$. We assume that the marked points $\mathbf{x}$ are contained in X_2 with total weight $a_X = 6$. For $Y = X_1$, the left-hand side of the inequality (1.6) is

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

$$\begin{aligned}
&\left|\left(\deg X_1+\frac{a_{X_1}}{2}\right)-\frac{\deg_{X_1}\omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg\omega_X(\mathbf{a}\cdot\mathbf{x})}\left(\deg X+\frac{a_X}{2}\right)\right|\\
&\quad=\left|\left(\deg X_1+\frac{\deg_{X_1}\omega_X}{2}+\frac{a_{X_1}}{2}-\frac{\deg_{X_1}\omega_X}{2}\right)\right.\\
&\qquad\left.-\frac{\deg_{X_1}\omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg\omega_X(\mathbf{a}\cdot\mathbf{x})}\left(\deg X+\frac{\deg\omega_X}{2}+\frac{a_X}{2}-\frac{\deg\omega_X}{2}\right)\right|\\
&\quad=\left|-\frac{\deg_{X_1}\omega_X}{2}-\frac{\deg_Y\omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg\omega_X(\mathbf{a}\cdot\mathbf{x})}\left(-\frac{\deg\omega_X}{2}\right)\right|\\
&\quad=\left|g_1-\frac{1}{2}-\frac{g(X_1)-1/2}{g(X_1)+g(X_2)-1+a_X/2}(g(X_1)+g(X_2)-1)\right|=\frac{3}{4}>\frac{1}{2}=\frac{\ell_{X_1}}{2},
\end{aligned}$$

violating (1.6). Hence $(X,\omega_X(\mathbf{a}\cdot\mathbf{x})^{\otimes k},\mathbf{x},\mathbf{a})\in\tilde{\mathcal{P}}$ but not in $\mathcal{H}^{\mathrm{ss}}$. In particular, one notices that this example is contrary to what was claimed in [Swi12, after Theorem 7.2]. The readers may consult [WX14] for more general discussion of this.

We break the proof of the theorem into several steps.

6.2 Surjectivity

Let $(X,\mathbf{x},\mathbf{a})$ be a weighted pointed stable curve. We endow it with the polarization $\mathcal{O}_X(1)=\omega_X(\mathbf{a}\cdot\mathbf{x})^{\otimes k}$ together with the embedding $\iota:X\to\mathbb{P}H^0(\mathcal{O}_X(1))^\vee$. When X is smooth, $(X,\iota,\mathbf{x},\mathbf{a})$ lies in $\mathcal{K}$; when X is singular, this may not necessarily hold. Our solution is to replace $\omega_X(\mathbf{a}\cdot\mathbf{x})^{\otimes k}$ by its twist, to be defined momentarily.

Given $(X,\mathbf{x})$, we choose a smoothing $\pi:\mathcal{X}\to T$ over a pointed curve $0\in T$ such that $\mathcal{X}$ is smooth and $\mathcal{X}_0=\mathcal{X}\times_T 0\cong X$. By an étale base change of T, we can extend the n-marked points of X to sections $\mathfrak{x}_i:T\to\mathcal{X}$ so that, defining $\mathfrak{x}=(\mathfrak{x}_1,\dots,\mathfrak{x}_n)$, the $(\mathcal{X},\mathfrak{x},\mathbf{a})$ form a flat family of weighted pointed stable curves. Let $X_1,\dots,X_r$ be the irreducible components of X. The following proposition gives the surjectivity of ψ.

Proposition 6.7. *Given g, n and $\mathbf{a}\in\mathbb{Q}^n_+$ satisfying $\chi_{\mathbf{a},g}>0$, there is a constant $K=K(g,n,\mathbf{a})$ such that for a weighted pointed stable curve $(X,\mathbf{x},\mathbf{a})$ and sufficiently divisible $k\geqslant K$, and for $(\mathcal{X},\mathfrak{x},\mathbf{a})$ the constructed T-family, there are integers $\{b_\alpha\}_{\alpha=1}^r$ independent of k such that after letting*

$$\mathcal{O}_\mathcal{X}(1)=\omega_{\mathcal{X}/T}(\mathbf{a}\cdot\mathbf{s})^{\otimes k}\otimes_{\mathcal{O}_\mathcal{X}}\mathcal{O}_\mathcal{X}\left(\sum b_\alpha X_\alpha\right),\tag{6.9}$$

$(\mathcal{X},\mathcal{O}_\mathcal{X}(1),\mathbf{s},\mathbf{a})$ is a family of slope semistable weighted pointed nodal curves.

The proposition was essentially proved by Caporaso in [Cap94]. Since we need to use the same technique to prove the injectivity, we outline its proof, following [Cap94].

For any line bundle $\mathcal{L}$ on X, we define $\delta_\alpha(\mathcal{L})=\deg\mathcal{L}|_{X_\alpha}$ and the numerical class of $\mathcal{L}$ to be

$$\vec{\delta}(\mathcal{L}):=(\delta_1(\mathcal{L}),\dots,\delta_r(\mathcal{L}))\in\mathbb{Z}^{\oplus r}.$$

We next let

$$\ell_{\alpha,\beta}=\ell_{\alpha,\beta}(X)=|X_\alpha\cap X_\beta|\quad\text{if }\alpha\neq\beta\quad\text{and}\quad\ell_{\alpha,\alpha}=\ell_{\alpha,\alpha}(X)=-|X_\alpha\cap\ X_\alpha^{\complement}|.\tag{6.10}$$

We define $\vec{\ell}_\alpha=\vec{\ell}_\alpha(X)=(\ell_{\alpha,1}(X),\ell_{\alpha,2}(X),\dots,\ell_{\alpha,r}(X))$. Letting

$$\mathbb{Z}_0^{\oplus r}=\left\{\vec{v}\in\mathbb{Z}^{\oplus r}\;\middle|\;\sum_{i=1}^r v_i=0\right\},$$

then $\vec{\ell}_\alpha\in\mathbb{Z}_0^{\oplus r}$, for every α. We define $\Gamma_X\subset\mathbb{Z}_0^{\oplus r}$ to be the subgroup generated by $\vec{\ell}_1,\dots,\vec{\ell}_r$.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Remark 6.8. Let $\mathcal{L} = \omega_X(\mathbf{a}\cdot\mathbf{x})^{\otimes k}$. Since $\mathcal{X}$ is smooth, for the invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ defined in (6.9) depending on the integers $b_1, \dots, b_r$, we have

$$\vec{\delta}(\mathcal{O}_{\mathcal{X}}(1)|_X) = \vec{\delta}(\mathcal{L}) + \sum_{\alpha=1}^{r} b_\alpha \vec{\ell}_\alpha.$$

This says that any two choices of $\mathcal{O}_{\mathcal{X}}(1)$ restricted to the central fiber have equivalent numerical classes modulo Γ_X.

We introduce one more piece of notation. For any vector $\vec{v} = (v_1, \dots, v_r) \in \mathbb{Z}^{\oplus r}$ and any subcurve $Y \subset X$, mimicking the notion of degree, we define

$$\deg_Y \vec{v} = \sum_{X_\alpha \subset Y} v_\alpha.$$

Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed nodal curve, and let d be a positive integer. For any subcurve $Y \subset X$, we introduce the *d-extremes* of Y as

$$\mathbf{M}_Y^{d,\pm} := \frac{\deg_Y \omega_X(\mathbf{a}\cdot\mathbf{x})}{\deg \omega_X(\mathbf{a}\cdot\mathbf{x})}\left(d + \frac{a_X}{2}\right) - \frac{a_Y}{2} \pm \frac{\ell_Y}{2}. \tag{6.11}$$

Then Proposition 5.4 can be reformulated as follows.

PROPOSITION 6.9. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ satisfying $\chi_{\mathbf{a},g} > 0$, let $M_3 = M_3(g, n, \mathbf{a})$ be the constant defined in Corollary 5.3. Then any polarized weighted pointed nodal curve $(X, \mathcal{L}, \mathbf{x}, \mathbf{a})$ of $\deg \mathcal{L} = d \geqslant M_3$ is slope semistable if and only if*

$$\deg_Y \mathcal{L} \in [\mathbf{M}_Y^{d,-}, \mathbf{M}_Y^{d,+}] \quad \textit{for any subcurve } Y \subset X. \tag{6.12}$$

Proof. This follows from Proposition 5.4 and the fact that (6.12) is equivalent to (1.6). □

Let $\mathbb{Z}_{\geqslant 0}^{\oplus r}$ be those $\vec{v} = (v_i) \in \mathbb{Z}^{\oplus r}$ such that $v_i \geqslant 0$. We define

$$\mathfrak{B}_{X,\mathbf{x},\mathbf{a}}^d = \{\vec{v} \in \mathbb{Z}_{\geqslant 0}^{\oplus r} \mid \deg_X \vec{v} = d, \vec{v} \text{ satisfies (6.12) with } \deg_Y \mathcal{L} \text{ replaced by } \deg_Y \vec{v}\}.$$

Then we have the following proposition.

PROPOSITION 6.10. *Let $(X, \mathbf{x}, \mathbf{a})$ be a weighted pointed quasistable (cf. Definition 6.1) curve. Then for any $\vec{v} \in \mathbb{Z}^{\oplus r}$ we have*

$$(\vec{v} + \Gamma_X) \cap \mathfrak{B}_{X,\mathbf{x},\mathbf{a}}^d \neq \emptyset.$$

Proof. Since the proof is completely parallel to [Cap94, Proposition 4.1], we will omit it. □

Proof of Proposition 6.7. By applying Proposition 6.10 to $\vec{v} = \vec{\delta}(\omega_X(\mathbf{a}\cdot\mathbf{x})^{\otimes k})$, one easily sees that there are $\{b_\alpha\}$ independent of k such that for the $\mathcal{O}_{\mathcal{X}}(1)$ given in (6.9) and $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)|_X$, $\vec{\delta}(\mathcal{L}) \in \mathbb{Z}_{\geqslant 0}^{\oplus r}$ and satisfies (6.12). Since $\mathcal{X}$ has smooth fibers other than the central fiber, we only need to show that the central fiber $(X, \mathcal{L}, \mathbf{x}, \mathbf{a})$ is slope semistable. To achieve that, notice that $\vec{\delta}(\mathcal{L})$ satisfies (6.12) already, and by Proposition 6.9, to show that $(X, \mathcal{L}, \mathbf{x}, \mathbf{a})$ is a slope semistable polarized weighted pointed nodal curve all we need to show is that $\mathcal{L}$ is ample and $\deg_X \mathcal{L} = k\chi_{\mathbf{a},g} \geqslant M_3$. First, by our assumption, $(X, \mathbf{x}, \mathbf{a})$ is a *weighted pointed stable curve* (cf. Definition 6.1), and hence $\omega_X(\mathbf{a}\cdot\mathbf{x})$ is ample. It follows from the proof of Lemma 5.2 that $\deg_Y \mathcal{L} \geqslant C \deg X > 0$ for any $Y \subset X$ as long as $\deg X > M_3 \geqslant M_2$; in particular, $\mathcal{L}$ is very ample, by Corollary 5.3. Now we define $K(g, n, \mathbf{a}) := M_3(g, n, \mathbf{a})/\chi_{\mathbf{a},g}$; then in the case $k > K = K(g, n, \mathbf{a})$, by Proposition 6.9, $(X, \mathcal{L}, \mathbf{x}, \mathbf{a})$ is a polarized slope semistable weighted pointed nodal curve with $\mathcal{L}$ being very ample. □

6.3 Injectivity

We use the separatedness of $\overline{\mathcal{K}}/\!\!/G$ to prove that ψ in (6.8) is injective.

Definition 6.11. For $(\bar{X}, \bar{\mathbf{x}})$ and $(X, \mathbf{x})$ two pointed nodal curves, we say the former is a *blow-up* of the latter if there is a morphism $\pi : \bar{X} \to X$ that is derived by contracting some exceptional components of $(\bar{X}, \bar{\mathbf{x}})$.

Since the restriction of ψ to $\mathcal{K}/\!\!/G$ is an isomorphism and $\mathcal{K}/\!\!/G$ is irreducible, ψ is a birational morphism. By the deformation theory of pointed nodal curves, we see that $\overline{\mathcal{M}}_{g,\mathbf{a}}$ has only finite quotient singularities, and thus is normal. By Zariski's main theorem and the properness of $\overline{\mathcal{K}}/\!\!/G$, the injectivity of ψ follows from the following lemma.

Lemma 6.12. *$\psi^{-1}(\psi(\xi))$ is zero-dimensional for each $\xi \in \overline{\mathcal{K}}/\!\!/G$.*

Proof. Let $\xi \in \overline{\mathcal{K}}/\!\!/G \backslash (\mathcal{K}/\!\!/G)$, and let $\psi(\xi) = (X, \mathbf{x}, \mathbf{a}) \in \overline{\mathcal{M}}_{g,\mathbf{a}}$ be the associated weighted pointed stable curve. We describe the set $\Theta_\xi = \mathbf{q}^{-1}(\psi^{-1}(\psi(\xi))) \subset \overline{\mathcal{K}}$, where $\mathbf{q} : \overline{\mathcal{K}} \to \overline{\mathcal{K}}/\!\!/G$ is the projection.

For any $\eta = (\bar{X}, \iota, \bar{\mathbf{x}}) \in \Theta_\xi \subset \overline{\mathcal{K}}$, there is a smooth affine curve $\phi : 0 \in T \to \overline{\mathcal{K}}$ such that the pullback of the universal family of $\overline{\mathcal{K}}$, say $\pi : (\mathcal{X}, \mathcal{L}, \mathbf{s}) \to T$, contains $(\bar{X}, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}})$ as its central fiber, $\phi(T \backslash \{0\}) \subset \mathcal{K}$, and the total space $\mathcal{X}$ is smooth.

By Proposition 6.2, the central fiber $(\bar{X}, \bar{\mathbf{x}}, \mathbf{a})$ is weighted pointed quasistable (cf. Definition 6.1) and is a blow-up of $(X, \mathbf{x}, \mathbf{a})$. Since $\mathcal{X}$ is smooth, there are integers $\{b_\alpha\}$ indexed by the irreducible components $\bar{X}_\alpha$ of $\bar{X}$ such that if we view $\bar{X}_\alpha$ as divisor in $\mathcal{X}$ then

$$\iota^*\mathcal{O}_{\mathbb{P}^m}(1) = \omega_{\bar{X}/T}(\mathbf{a} \cdot \mathbf{x})^{\otimes k}\left(\sum_{\alpha=1}^{\bar{r}} b_\alpha \bar{X}_\alpha\right).$$

Since the collection of blow-ups of X coupled with integers $\{b_\alpha\}_{\alpha=1}^{\bar{r}}$ is a discrete set, the choices of $(\bar{X}, \mathcal{L}, \bar{\mathbf{x}})$ are discrete. Thus $\{(\bar{X}, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}}) \mid (\bar{X}, \iota, \bar{\mathbf{x}}) \in \Theta_\xi\}$ is discrete. Finally, any two $(\bar{X}, \iota, \bar{\mathbf{x}})$ with isomorphic $(\bar{X}, \iota^*\mathcal{O}_{\mathbb{P}^m}(1), \bar{\mathbf{x}})$ lie in the same G-orbit. Thus Θ_ξ consists of a discrete collection of G-orbits. Hence $\psi^{-1}(\psi(\xi))$ is discrete. □

6.4 The coarse moduli space

We prove that $\overline{\mathcal{K}}/\!\!/G$ is a coarse moduli space of weighted pointed stable curves, thus proving that ψ is an isomorphism.

Proposition 6.13. *Let T be any scheme and $(\mathcal{X}, \mathfrak{x}, \mathbf{a})$ be a T-family of weighted pointed stable curves. Then there is a unique morphism $f : T \to \overline{\mathcal{K}}/\!\!/G$, canonical under base changes, such that for any closed point $c \in T$, the image $\psi(f(c)) \in \overline{\mathcal{M}}_{g,\mathbf{a}}$ is the closed point associated to the weighted pointed stable curve $(\mathcal{X}, \mathfrak{x}, \mathbf{a})|_c$.*

We define a subscheme $\widetilde{\mathcal{P}} \subset \mathcal{H}$:

$$\widetilde{\mathcal{P}} = \{(X, \iota, \mathbf{x}) \in \mathcal{H} \mid (X, \mathbf{x}, \mathbf{a}) \text{ weighted pointed stable curves, } \omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k} \cong \iota^*\mathcal{O}_{\mathbb{P}^m}(1)\}.$$

A direct check shows that $\widetilde{\mathcal{P}}$ is a smooth, locally closed and G-invariant subscheme of $\mathcal{H}$. We let $\mathcal{P} \subset \widetilde{\mathcal{P}}$ be the open subset of $(X, \iota, \mathbf{x})$ such that the X are smooth. By definition, we have $\mathcal{P} = \mathcal{K}$.

Lemma 6.14. *The composition $F : \mathcal{P} \to \overline{\mathcal{K}} \to \overline{\mathcal{K}}/\!\!/G$ extends to a unique morphism $\widetilde{F} : \widetilde{\mathcal{P}} \to \overline{\mathcal{K}}/\!\!/G$.*

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Proof. Applying the deformation theory of nodal curves, we know that $\mathcal{P}$ is dense in $\widetilde{\mathcal{P}}$. Let $\Gamma \subset \mathcal{P} \times \overline{\mathcal{K}}/\!\!/G$ be the graph of the morphism F stated in the lemma; we let

$$\overline{\Gamma} \subset \widetilde{\mathcal{P}} \times \overline{\mathcal{K}}/\!\!/G$$

be the closure of Γ. Let $p : \overline{\Gamma} \to \widetilde{\mathcal{P}}$ be the projection. We claim that p is bijective. Indeed, given $\xi = (X, \iota, \mathbf{x}) \in \widetilde{\mathcal{P}}$, we let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1), \mathfrak{x})$ be the family given by Proposition 6.7, which shows that $\xi \in p(\overline{\Gamma})$. This proves that p is surjective. On the other hand, repeating the proof of Lemma 6.12, we see that p is one-to-one. This proves that p is bijective.

Next, we claim that p is an isomorphism. Since $\widetilde{\mathcal{P}}$ is smooth, $\mathcal{P} \subset \widetilde{\mathcal{P}}$ is dense and Γ is isomorphic to $\mathcal{P}$, we conclude that $\overline{\Gamma}$ is reduced. Then since $p : \overline{\Gamma} \to \widetilde{\mathcal{P}}$ is birational and a homeomorphism and $\widetilde{\mathcal{P}}$ is smooth, p must be étale. Thus p is an isomorphism. Finally, by composing the isomorphism p^{-1} with the projection to the second factor of $\widetilde{\mathcal{P}} \to \overline{\mathcal{K}}/\!\!/G$, we obtain the desired extension $\widetilde{F}$ of F. □

Proof of Proposition 6.13. We cover T with a collection of affine open $\{T_a\}_{a\in A}$. Let $\pi_a : \mathcal{X}_a \to T_a$ with sections $\mathfrak{x}_{a,i} : T_a \to \mathcal{X}_a$ be the restriction of $\mathfrak{x}_i$ to T_a of the family on T. By fixing a trivialization $(\pi_a)_*\omega_{\mathcal{X}_a/T_a}(\mathbf{a}\cdot\mathfrak{x}_a)^{\otimes k} \cong \mathcal{O}_{T_a}^{\oplus(m+1)}$, we obtain morphisms $f_a : T_a \to \widetilde{\mathcal{P}}$. Composed with the morphism $\widetilde{F}$ constructed in the previous lemma, we obtain $\widetilde{F} \circ f_a : T_a \to \overline{\mathcal{K}}/\!\!/G$.

Since the choice of the trivializations does not alter the morphism $\widetilde{F} \circ f_a$, this collection $\{\widetilde{F} \circ f_a\}_{a\in A}$ patches to a morphism $T \to \overline{\mathcal{K}}/\!\!/G$. This proves the first part of Proposition 6.13.

Finally, that $\psi(f(c))$ is the point associated to the weighted pointed curve $(\mathcal{X}, \mathfrak{x}, \mathbf{a})|_c$ follows from the construction. □

Proof of Theorem 6.5. It follows from Propositions 6.7 and 6.13 and Lemma 6.12. □

We remark that, for convenience, in the proof given above we use the existence of the coarse moduli space $\overline{\mathcal{M}}_{g,\mathbf{a}}$ constructed by Hassett. A modification of the argument should give an independent GIT construction of it.

For completeness, we give a complete description of polystable points in $\mathcal{C}^{\mathrm{ss}}$, generalizing the case $\mathbf{x} = \emptyset$ proved in [Cap94]. Let the *exceptional set* $E(X) \subset X$ be the union of exceptional components of $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$.

DEFINITION 6.15 ([Cap94] when $\mathbf{x} = \emptyset$). We say $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is *extremal* if each proper subcurve $Y \subsetneq X$ satisfying $\vec{\delta}_Y(\mathcal{O}_X(1)) = \mathbf{M}_Y^{d,-}$ (cf. (6.12)) has $L_Y = Y \cap Y^{\complement} \subset E(X)$.

Recall that $\mathrm{Chow}(X, \mathbf{x}) \in \mathcal{C}^{\mathrm{ss}}$ is *polystable* if the G-orbit $G \cdot \mathrm{Chow}(X, \mathbf{x})$ is closed in $\mathcal{C}^{\mathrm{ss}}$. Here is an equivalent characterization of polystable points.

LEMMA 6.16. *Let G be a reductive group and $(Z, \mathcal{O}_Z(1))$ be a G-polarized projective scheme. Then a* semistable *point $z \in Z^{\mathrm{ss}}$ is* polystable *if and only if for any 1-PS λ either the λ-weight of z is > 0 or $\lim_{t\to 0} \lambda(t) \cdot z \in G \cdot z$.*

Proof. Without loss of generality, we may assume that $Z = \mathbb{P}W$ for a $\Bbbk$-vector space W. Let $\lambda : \mathbb{G}_m \to G$ be any 1-PS, with $W = \bigoplus_{i\in\mathbb{Z}} W_i$ its weight decomposition such that λ acts on W_i by multiplying by t^i. Let $z \in \mathbb{P}W$, and let $0 \neq \hat{z} \in W$ be a lift of z, with associated decomposition $\hat{z} = \bigoplus_i \hat{z}_i$, $\hat{z}_i \in W_i^\vee$. Then the λ-weight of $\hat{z}$ is $\omega_\lambda(\hat{z}) = \max\{-i \mid \hat{z}_i \neq 0\}$.

Suppose z is polystable, then $G \cdot \hat{z}$ is closed in $W^\vee$ (cf. [Gie98]). Suppose $\omega_\lambda(\hat{z}) = 0$; we have $0 \neq \hat{z}_0 = \lim_{t\to 0} \lambda(t) \cdot \hat{z} \in \overline{G \cdot \hat{z}} = G \cdot \hat{z}$. Thus $z_0 \in G \cdot z$, and this verifies one direction of the lemma.

Conversely, suppose z is semistable but not polystable. Then there are a 1-PS λ and $g \in G$ such that $\lim_{t\to 0} \lambda(t)\cdot(g\cdot z) = z_0$, z_0 is polystable and $z_0 \notin G\cdot z$. In particular, since $\lambda^g = g^{-1}\cdot\lambda\cdot g$ fixes z, the λ^g-weight of z is 0 while $\lim_{t\to 0} \lambda^g(t)\cdot z \notin G\cdot z$. This proves the lemma. □

We characterize curves having positive dimensional stabilizers.

Lemma 6.17. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ such that $\chi_{\mathbf{a},g} > 0$, let $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ be a genus g semistable polarized weighted pointed nodal curve such that $(X, \mathbf{x}) \subset \mathbb{P}W$ is invariant under a 1-PS λ and $\deg X \geqslant M$, the constant given in Theorem 1.5. Suppose under its diagonalizing basis, λ has k weights. Then there are k mutually disjoint subcurves $Y_1, \ldots, Y_k \subsetneq X$ such that:*

(1) *the complement $(\bigcup_{i=1}^k Y_i)^{\complement}$ is a union of exceptional components of X; and*
(2) *each Y_i has $\vec{\delta}_{Y_i}(\mathcal{O}_X(1)) = M_{Y_i}^{d,-}$, where $d = \deg X$.*

Proof. We let $Y \subset X$ be the union of irreducible components of X that are fixed by λ; we let $E = Y^{\complement}$. As $E \subset \mathbb{P}W$ is λ-invariant but not fixed, it is a union of $\mathbb{P}^1$ components. Since $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is semistable, by Theorem 1.5, Lemma 5.2 and the assumption $\deg X \geqslant M$, E is a union of exceptional components.

By the assumption that λ has k weights, we have the weight decomposition $W = \bigoplus_{i=1}^k W_i$. We let $Y_i = Y \cap \mathbb{P}W_i^\vee$. Because Y is fixed by λ, we have that $Y = \bigcup_{i=1}^k Y_i$ is a disjoint union. Since $X = Y \cup E$ is connected and since E is a union of exceptional components, the first part of the lemma follows.

For the second part, for each Y_i let λ_i be the 1-PS that acts on W_i via multiplying t and fixes $\bigcup_{j\neq i} W_j$. Under such λ_i, $(X, \mathbf{x}) \subset \mathbb{P}W$ is invariant, thus by semistability the λ_i-weight of $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ is 0, which is equivalent to $\vec{\delta}_{Y_i}(\mathcal{O}_X(1)) = M_{Y_i}^{d,-}$. This proves the lemma. □

Proposition 6.18. *Given g, n and $\mathbf{a} \in \mathbb{Q}_+^n$ such that $\chi_{\mathbf{a},g} > 0$, a stable polarized weighted pointed nodal curve $(X, \mathcal{O}_X(1), \mathbf{x}, \mathbf{a})$ of $d \geqslant M$ (the constant given in Theorem 1.5) is polystable if and only if it is extremal.*

Proof. Suppose $\mathrm{Chow}(X, \mathbf{x})$ is polystable and $Y \subsetneq X$ such that $\vec{\delta}_Y(\mathcal{O}_X(1)) = \mathbf{M}_Y^{d,-}$. We pick a decomposition $W_0 \oplus W_1 = W = H^0(X, \mathcal{O}_X(1))$ such that $Y = X \cap \mathbb{P}W_0$, which is possible for $d \geqslant M$. We pick a 1-PS λ such that it acts on W_0 (respectively W_1) via multiplication by t (respectively by 1). Since $\vec{\delta}_Y(\mathcal{O}_X(1)) = \mathbf{M}_Y^{d,-}$, the λ-weight of $\mathrm{Chow}(X, \mathbf{x})$ is 0. By Lemma 6.16, $(X', \mathbf{x}') = \lim_{t\to 0} \lambda(t)\cdot(X, \mathbf{x})$ lies in the G-orbit of $(X, \mathbf{x})$. Further, since λ leaves $(X', \mathbf{x}')$ invariant and $\mathrm{Chow}(X', \mathbf{x}')$ is semistable, by Lemma 6.17 we have $L_{Y'} \subset E(X')$, which is equivalent to $L_Y \subset E(X)$. This proves the sufficient part of the proposition.

Conversely, suppose ξ is semistable but not polystable. Then there is a 1-PS λ such that $\lim_{t\to 0} \lambda(t)\cdot\xi = \xi'$ is polystable. Let $\mathcal{X} \to \mathbb{A}^1$ be the total space of this family of curves. Since $\mathcal{X} \times_{\mathbb{A}^1} (\mathbb{A}^1 - 0)$ is a constant family, the special fiber will be a 'blow-up' of the general fibers. Because $\xi' = (X', \mathcal{O}_{X'}(1), \mathbf{x}', \mathbf{a})$ is polystable and is invariant under λ, we have the decomposition $X' = \bigcup_{i=1}^k Y_i' \cup \bigcup_{j,j'} E_{jj'}$, where $E_{jj'}$ is the union of exceptional components in $(\bigcup_i Y_i')^{\complement}$ that intersects with both Y_j' and $Y_{j'}'$, given by Lemma 6.17. Since X is a 'blow-down' of X', and λ does not fix $(X, \mathbf{x}) \subset \mathbb{P}W$, the blow-down map $X' \to X$ must contract at least one exceptional component, say, in $E_{jj'}$. Suppose $j < j'$. We let $Y \subset X$ be the image of $\bigcup_{i=1}^j Y_i'$ under $X' \to X$. Then it can be checked directly that $Y \subsetneq X$, $\vec{\delta}_Y(\mathcal{O}_X(1)) = \mathbf{M}_Y^{d,-}$ and $L_Y \not\subset E(X)$. This proves the proposition. □

7. K-stability of nodal curves

In this section, we apply Theorem 1.5 to study the K-stability of polarized nodal curves.

Theorem 7.1. *For a polarized connected nodal curve $(X, \mathcal{O}_X(1))$ the following statements are equivalent:*

(1) $(X, \mathcal{O}_X(1))$ *is K-stable;*
(2) $(X, \mathcal{O}_X(1))$ *is K-semistable;*
(3) $\mathcal{O}_X(1)$ *is numerically proportional to ω_X.*

One direction of the theorem is proved by Odaka who in [Oda13b] proved that a nodal curve X polarized by $\omega_X^{\otimes k}$ is K-stable for a $k \in \mathbb{N}$. He used birational geometry and a weight formula proved by himself and by the second named author independently [Wan12]. He also informed us that he can generalize his method to prove the stated theorem.

7.1 K-stability of curves

We recall the notion of K-stability of polarized curves. (See [RT07, § 3] and [Sto11] for the general case.)

Definition 7.2. A *test configuration* for a polarized curve $(X, \mathcal{O}_X(1))$ consists of a $\mathbb{G}_m$-equivariant flat projective morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ and a $\mathbb{G}_m$-linearized π-relative very ample line bundle $\mathcal{L}$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ via multiplication, such that for any $t \neq 0 \in \mathbb{A}^1$, $(\mathcal{X}, \mathcal{L}) \times_{\mathbb{A}^1} \{t\} \cong (X, \mathcal{O}_X(1))$.

For a closed subset $\Sigma \subset \mathcal{X}_0$, we call it a trivial configuration away from Σ if there is a closed subset $\Sigma_0 \subset X$ such that there is a $\mathbb{G}_m$-equivariant isomorphism $\mathcal{X} - \Sigma \cong X \times \mathbb{A}^1 - \Sigma_0 \times \{0\}$, such that the line bundle $\mathcal{L}|_{\mathcal{X}-\Sigma}$ is the pullback of a line bundle on X, and the $\mathbb{G}_m$-action on $X \times \mathbb{A}^1$ is the product action that acts trivially on X. When $\Sigma \subset \mathcal{X}$ has codimension at least 2, we say $(\mathcal{X}, \mathcal{L})$ is *trivial in codimension* 2.

Given a test configuration $(\mathcal{X}, \mathcal{L})$ for a polarized curve $(X, \mathcal{O}_X(1))$ as above, we let $w(l)$ be the weight of the induced $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}}(\pi_* \mathcal{L}^{\otimes l}|_0)$. By Riemann–Roch, $w(l) = a_2 l^2 + a_1 l + a_0$ is quadratic in l (for $l \gg 1$). We expand the following quotient in l^{-1}:

$$\frac{w(l)}{l \cdot \chi(\mathcal{O}_X(l))} = e_0 + e_{-1} l^{-1} + \cdots .$$

Using $\chi(\mathcal{O}_X(l)) = b_1 l + b_0$, the Donaldson–Futaki invariant of the test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X(1))$ is defined to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = e_{-1} = -\frac{a_2 b_0 - a_1 \cdot b_1}{b_1^2}.$$

Remark 7.3. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for $(X, \mathcal{O}_X(1))$. Then the $\mathbb{G}_m$-linearization of $\mathcal{L}$ induces a $\mathbb{G}_m$-linearization of $\mathcal{L}^{\otimes l}$, which makes $(\mathcal{X}, \mathcal{L}^{\otimes l})$ a test configuration for $(X, \mathcal{O}_X(l))$, with $\mathrm{DF}(\mathcal{X}, \mathcal{L}^{\otimes l}) = \mathrm{DF}(\mathcal{X}, \mathcal{L})$.

Definition 7.4. A polarized nodal curve $(X, \mathcal{O}_X(1))$ is K-stable (respectively K-semistable) if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$ (respectively $\leqslant 0$) for any test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, \mathcal{O}_X(1))$ that is non-trivial in codimension 2.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

7.2 Proof of the main result

For $(X, \mathcal{O}_X(1))$ and integer k, we let $W_{(k)}^\vee = H^0(\mathcal{O}_X(k))$ with $X \subset \mathbb{P}W_{(k)}$ the tautological embedding. Then, given any 1-PS subgroup λ of $\operatorname{Aut}\mathbb{P}W_{(k)}$, the $\mathbb{G}_m$-orbit of X in $\mathbb{P}W_{(k)} \times \mathbb{A}^1$ via the diagonal action produces a test configuration of $(X, \mathcal{O}_X(k))$; we denote such a test configuration by $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$. Conversely, any test configuration of $(X, \mathcal{O}_X(k))$ can be constructed from a 1-PS of $\operatorname{Aut}\mathbb{P}W_{(k)}$ (cf. [RT07, Proposition 3.7]). Thus, to prove the K-stability of $(X, \mathcal{O}_X(1))$, it suffices to show that when $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ is non-trivial in codimension 2, the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda) < 0$ for sufficiently large k and all 1-PS λ of $\operatorname{Aut}\mathbb{P}W_{(k)}$.

In the following, for notational simplicity, we replace $(X, \mathcal{O}_X(1))$ by $(X, \mathcal{O}_X(k))$, and say that $\mathcal{O}_X(1)$ is sufficiently ample instead of saying k is sufficiently large. This way, we only need to study test configuration $(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ for any 1-PS λ of $\operatorname{Aut}\mathbb{P}W$, assuming $\mathcal{O}_X(1)$ is sufficiently ample. To proceed, we first relate $\mathrm{DF}(\mathcal{X}_\lambda, \mathcal{L}_\lambda)$ to the Chow weights of $(X, \mathcal{O}_X(l))$. We pick a λ-diagonalizing basis $\mathbf{s} = \{s_0, \ldots, s_m\}$ of $W^\vee$ and represent λ as a 1-PS of $\mathrm{GL}(W^\vee)$ of the form

$$\lambda(t) := \operatorname{diag}[t^{\rho_0}, \ldots, t^{\rho_m}], \quad \rho_0 \geqslant \rho_1 \geqslant \cdots \geqslant \rho_m = 0, \quad \rho_i \in \mathbb{Z}. \tag{7.1}$$

By Remark 7.3, replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes l}$, the test configuration $(\mathcal{X}_\lambda, \mathcal{L})$ introduces a test configuration $(\mathcal{X}_\lambda, \mathcal{L}^{\otimes l})$, which by [RT07, Proposition 3.7] is induced by a 1-PS λ_l of $\operatorname{Aut}\mathbb{P}W_{(l)}$. We now construct explicitly such λ_l. Since $\mathcal{O}_X(1)$ is a sufficiently high multiple of an ample line bundle, the tautological

$$\phi_l : S^l W^\vee \longrightarrow W_{(l)}^\vee = H^0(\mathcal{O}_X(l)) \tag{7.2}$$

is surjective. We fix our convention. For $I = (i_0, \ldots, i_m)$, we define $s^I = s_0^{i_0} \cdots s_m^{i_m}$, which has weight $\rho(I) = \sum_j \rho_j \cdot i_j$ under the induced λ action on $S^l W^\vee$.

We let $\mathfrak{S}_l$ be the set of monomials in $S^l W^\vee$. We order $\mathfrak{S}_l$ as follows: We define $s^I \succ s^{I'}$ *when either* $\rho(I) < \rho(I')$, *or* $\rho(I) = \rho(I')$ *and there is a* $0 \leqslant j_0 \leqslant m$ *such that* $i_j = i'_j$ *for all* $j > j_0$ *and* $i_{j_0} > i'_{j_0}$. Thus, the set $\{t^{\rho_i} s_i\}$ is ordered increasingly as $t^{\rho_0} s_0, \ldots, t^{\rho_m} s_m$. Following the definition, we see that $s^I \succ s^{I'}$ if and only if $s^J \cdot s^I \succ s^J \cdot s^{I'}$ for any non-trivial monomial s^J.

We pick a basis of $W_{(l)}^\vee$, which will be a diagonalizing basis for λ_l. Let $m_l + 1 = \dim W_{(l)}^\vee$ and set $s_{l,m_l} = s_m^l$, with weight $\varrho_{l,m_l} = l \cdot \rho_m$. Suppose for an integer $0 \leqslant k < m_l$ we have picked $s_{l,k+1}, \ldots, s_{l,m_l}$ and their weights $\varrho_{l,j}$; let $\Theta_{l,k+1}$ be the linear span of $\{s_{l,k+1}, \ldots, s_{l,m_l}\}$ and let s^{I_k} be the largest element in

$$\{s^I \in \mathfrak{S}_l \mid \phi_l(s^I) \notin \phi_l(\Theta_{l,k+1})\}.$$

We set $s_{l,k} = \phi_l(s^{I_k})$ and define $\varrho_{l,k} = \rho(I_k)$ to be the weight of $s_{I,k}$. Then $s_{l,0}, \ldots, s_{l,m_l}$ form a basis of $W_{(l)}^\vee$. We let λ_l be the 1-PS of $\operatorname{Aut}\mathbb{P}W_{(l)}$ with diagonalizing basis $\{s_{l,0}, \ldots, s_{l,m_l}\}$ and weights

$$\lambda_l(t) \cdot s_{l,k} = t^{\varrho_{l,k}} s_{l,k}. \tag{7.3}$$

Note that for $l = 1$, $s_{1,k} = t^{\rho_k} s_k$ and $\lambda_1 = \lambda$.

Lemma 7.5. *Let* $(\mathcal{X}_{\lambda_l}, \mathcal{L}_{\lambda_l})$ *be the test configuration of* λ_l. *Then we have an isomorphism of test configurations* $(\mathcal{X}_{\lambda_l}, \mathcal{L}_{\lambda_l}) \cong (\mathcal{X}_\lambda, \mathcal{L}_\lambda^{\otimes l})$.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Proof. We let $R_{l,k}$ be the $\mathbb{k}[t]$-submodule of $H^0(\mathcal{O}_X(kl))\otimes\mathbb{k}[t]$ generated by monomials of degree k in elements in $\{t^{\varrho_{l,i}}s_{l,i}\}$ and let $R_l=\bigoplus_{k\geqslant 0}R_{l,k}$, where $R_{l,0}=\mathbb{k}[t]$. Clearly, R_l is a graded $\mathbb{k}[t]$-algebra and is generated by $R_{l,1}$. Following [Mum77, p. 28], we have

$$\mathcal{X}_{\lambda_l}=\operatorname{Proj}_{\mathbb{k}[t]}R_l\subset\mathbb{P}W_{(l)}^{\vee}\times\mathbb{A}^1. \tag{7.4}$$

For $l=1$, we have $\mathcal{X}_\lambda=\operatorname{Proj}_{\mathbb{k}[t]}R_1\subset\mathbb{P}W\times\mathbb{A}^1$.

We claim that for any $k\geqslant 1$, $R_{l,k}=R_{1,lk}\subset H^0(\mathcal{O}_X(kl))\otimes\mathbb{k}[t]$. Indeed, by definition, we have $R_{l,1}=R_{1,l}$. Since $R_{l,k}$ is generated by $R_{l,1}$ and $R_{1,kl}$ is generated by $R_{1,l}$, we conclude that $R_{l,k}=R_{1,lk}$ as $\mathbb{k}[t]$-submodules of $H^0(\mathcal{O}_X(kl))\otimes\mathbb{k}[t]$. Consequently, they induce a homomorphism of graded $\mathbb{k}[t]$-algebra $R_l\to R_1$, which induces a $\mathbb{G}_m$-equivariant isomorphism $(\mathcal{X}_\lambda,\mathcal{L}_\lambda^{\otimes l})\cong(\mathcal{X}_{\lambda_l},\mathcal{L}_{\lambda_l})$. This proves the lemma. □

LEMMA 7.6. *Let the notation be as before. Then*

$$\lim_{l\to\infty}l^{-1}\cdot\omega(\lambda_l)=-b_1^{-1}\cdot\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)<\infty.$$

Proof. It follows from Lemma 7.5 that $\omega(\lambda_l)$ is the Chow weight for the test configuration $(\mathcal{X}_\lambda,\mathcal{L}_\lambda^{\otimes l})$. By [RT07, Theorem 3.9], we know that this Chow weight $\omega(\lambda_l)$ is a linear function of the form $-b_1^{-1}\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)\cdot l+\text{constant}$. Dividing by l and taking the limit, we complete the proof of the lemma. □

Thus, to prove $\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)<0$, it suffices to show that

$$\lim_{l\to\infty}l^{-1}\cdot\omega(\lambda_l)>0. \tag{7.5}$$

Proof of Theorem 7.1. We will prove the theorem in the following order:

$$(3)\Rightarrow(1)\Rightarrow(2)\Rightarrow(3).$$

Since the middle arrow trivially follows from Definition 7.4, we only need to establish the first and third arrows.

Suppose X is a stable (nodal) curve and $\mathcal{O}_X(1)$ is numerically proportional to ω_X; then $(X,\mathcal{O}_X(1))$ is slope stable. We will show in this case that for any 1-PS $\lambda\subset\mathrm{SL}(W)$ we have $\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)<0$ unless $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ is trivial in codimension 2.

We divide the study into two cases. The first case is when $e(\mathcal{I}(\lambda))=0$. In this case, we claim that there is some $0<i_0<m$ such that $\varrho_{i_0}=0$ and $\bigcap_{k\geqslant i_0}\{s_k=0\}=\emptyset$. Indeed, let i_0 be the smallest index such that $\rho_{i_0}=0$. Suppose $q\in\bigcap_{k\geqslant i_0}\{s_k=0\}\neq\emptyset$; then we have $\rho_{i_0(q)}>0$ and $\Delta_q\neq\emptyset$, and hence $e(\mathcal{I}(\lambda))_q>0$. By Corollary 2.8 and Lemma 2.4 we obtain $e(\mathcal{I}(\lambda))>0$, a contradiction. Therefore we have $\bigcap_{k\geqslant i_0}\{s_k=0\}\neq\emptyset$, and then $i_0<m$. This proves the claim. To continue, we quote a result of Stoppa ([Sto09, pp. 1405–1406], [Sto11]) that in this case either the test configuration $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ of λ is trivial in codimension 2 or $\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)<0$. This proves the theorem in this case.

The other case is when $e(\mathcal{I}(\lambda))>0$. We let λ_l' be the staircase constructed from λ_l using Proposition 3.5 with weights $\varrho_{l,i}'=\varrho_{l,i}$. We let $\hat{\varrho}_{l,i}$ be the shifted weights according to the rule (5.14) applied to λ_l'; namely, $\hat{\varrho}_{l,i}=\min_\beta\{\varrho_{l,i}-\varrho_{\hbar_\beta(\lambda_l')}\mid i\in\mathbb{I}_\beta(\lambda_l')\}$. Since $(X,\mathcal{O}_X(1))$ is slope stable, applying Theorem 5.10 and Proposition 3.5, we can find an $\epsilon>0$ such that for l sufficiently large,

$$l^{-1}\cdot\omega(\lambda_l)\geqslant l^{-1}\cdot\omega(\lambda_l')\geqslant\frac{1}{\deg X+l^{-1}(1-g_X)}\cdot\frac{\epsilon}{l^2}\cdot\sum_{i=0}^{m_l}\hat{\varrho}_{l,i}. \tag{7.6}$$

We state a sublemma which we will prove shortly.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Sublemma 7.7. *Suppose* $e(\mathcal{I}(\lambda))>0$. *Then* $\liminf_{l\to\infty}(1/l^2)\cdot\sum_{i=0}^{m_l}\hat{\varrho}_{l,i}>0$.

Applying Sublemma 7.7, we obtain $l^{-1}\cdot\omega(\lambda_l)>0$, which by Lemma 7.6 is equivalent to $\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)<0$. Since λ is arbitrary, we conclude that $(X,\mathcal{O}_X(1))$ is K-stable. This proves one direction of the theorem.

Conversely, suppose $(X,\mathcal{O}_X(1))$ is K-semistable. Since $(X,\mathcal{O}_X(1))$ being K-semistable is equivalent to $(X,\mathcal{O}_X(k))$ being K-stable for all large k (cf. Remark 7.3), without loss of generality we assume that $\mathcal{O}_X(1)$ is *sufficiently ample*. In particular, this implies that $X\subset\mathbb{P}W$ contains no line. We claim that $(X,\mathcal{O}_X(1))$ satisfies (1.6) with $\mathbf{a}=0$. Suppose not; then there is a subcurve $Y\subset X$ destabilizing $(X,\mathcal{O}_X(1))$, that is,

$$\frac{\deg_Y\omega_X}{\deg\omega_X}\cdot\deg\mathcal{O}_X(1)-\deg\mathcal{O}_Y(1)-\frac{\ell_Y}{2}>0. \tag{7.7}$$

Let $W_Y=H^0(\mathcal{O}_Y(1))^\vee\subset W=H^0(\mathcal{O}_X(1))^\vee$, which is the linear subspace spanned by Y; let $m_0+1=\dim H^0(\mathcal{O}_Y(1))$. We choose a two-weight 1-PS λ as in the proof of Theorem 1.5 (at the end of § 5) so that λ acts with weight 1 on $W_Y\subset W$ and acts with weight 0 on a complement $W_Y^\perp$ of $W_Y\subset W$. Let $(\mathcal{X}_\lambda,\mathcal{L}_\lambda)$ be the test configuration associated to λ. We now evaluate

$$\frac{\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)}{b_1}=\lim_{l\to\infty}-\frac{\omega(\lambda_l)}{l}=-\lim_{l\to\infty}\frac{1}{l}\cdot\left(\frac{2l\deg X\sum_{i=0}^{m_l}\varrho_{l,i}}{l\deg X+1-g}-e(\mathcal{I}(\lambda_l))\right).$$

To evaluate this term, we identify the central fiber $(\mathcal{X}_\lambda)_0$ of $\mathcal{X}_\lambda$. As $\mathcal{O}_X(1)$ is sufficiently ample, $X\subset\mathbb{P}W$ contains no line. Further, as λ is a two-weight 1-PS, and the weight one eigenspace is W_Y, we see that $(\mathcal{X}_\lambda)_0=Y\cup E\cup Y'$, derived by inserting ℓ_Y-lines (whose union is E) into X at the nodes $Y\cap Y^\complement$, and $Y'\subset\mathbb{P}W_Y^\perp$ is isomorphic to $Y^\complement\subset X$, because $\mathcal{O}_X(1)$ is sufficiently ample. Consequently,

$$H^0(\mathcal{L}_\lambda^{\otimes l}|_{(\mathcal{X}_\lambda)_0})=H^0(\mathcal{O}_Y(l))\oplus H^0(\mathcal{O}_E(l)(-(Y\cup Y')\cap E))\oplus H^0(\mathcal{O}_{Y'}(l)),$$

and elements in $H^0(\mathcal{O}_X(l)|_Y)$ (respectively $H^0(\mathcal{O}_E(l)(-(Y\cup Y')\cap E))$, and respectively $H^0(\mathcal{O}_{Y'}(l))$) have weights l (respectively $l-1,\ldots,0$, and respectively 0). Therefore

$$\sum\varrho_{l,i}=h^0(\mathcal{O}_Y(l))\cdot l+\ell_Y\cdot\frac{l(l-1)}{2}=\left(\deg\mathcal{O}_Y(1)+\frac{\ell_Y}{2}\right)\cdot l^2+\left(1-g(Y)-\frac{\ell_Y}{2}\right)\cdot l.$$

By Definition 2.4 and Lemma 7.5, $\sum_{i=0}^{m_l}\varrho_{l,i}=e(\mathcal{I}(\lambda))\cdot(l^2/2)+O(l)$ is the weight of the $\mathbb{G}_m$-action on $\wedge^{\mathrm{top}}(H^0(\mathcal{L}_\lambda^{\otimes l})/tH^0(\mathcal{L}_\lambda^{\otimes l}))$. Thus,

$$e(\mathcal{I}(\lambda_l))=l^2e(\mathcal{I}(\lambda))=2l^2(\deg Y+\ell_Y/2).$$

Combining and simplifying by using $\deg\omega_X=g-1$, etc., we obtain

$$\begin{aligned}\frac{\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)}{b_1}&=-\lim_{l\to\infty}\frac{1}{l}\cdot\left(\frac{2l\deg X\sum_{i=0}^{m_l}\varrho_{l,i}}{l\deg X+1-g}-e(\mathcal{I}(\lambda_l))\right)\\&=-\lim_{l\to\infty}\frac{1}{l}\cdot\frac{2l^2(g-1)(\deg Y+\ell_Y/2)-l^2\deg_Y\omega_X}{l\deg X+1-g}\\&=\frac{g-1}{\deg X}\left(\frac{\deg_Y\omega_X}{\deg\omega_X}\cdot\deg X-\deg Y-\frac{\ell_Y}{2}\right).\end{aligned}\tag{7.8}$$

Since $Y\subset X$ is destabilizing, by (7.7) we have $\mathrm{DF}(\mathcal{X}_\lambda,\mathcal{L}_\lambda)>0$, contradicting $(X,\mathcal{O}_X(1))$ being K-semistable. This proves that $(X,\mathcal{O}_X(1))$ satisfies (1.6) with $\mathbf{a}=0$. In particular, we obtain that $(X,\mathcal{O}_X(l))$ satisfies (1.6) for all large l since $(X,\mathcal{O}_X(l))$ is also K-semistable for any $l>0$. This forces $\mathcal{O}_X(1)$ to be numerically proportional to ω_X. This proves the other direction of the theorem. □

It remains to prove Sublemma 7.7. We introduce a few more pieces of notation. Following the discussion in § 2, we define (for $q \in \tilde{X}$ and $\tilde{s}_{l,k}$ the lift of $s_{l,k}$ to the normalization $\tilde{X}$ of X)

$$\hbar_\alpha(\lambda_l) = \min\{i \mid \tilde{s}_{l,i+1}|_{\tilde{X}_\alpha} = 0\}, \quad \hbar(\lambda_l, q) = \max\{i \mid v(\tilde{s}_{l,i}, q) \neq \infty\}, \tag{7.9}$$

and

$$\tilde{\Lambda}_\alpha(\lambda_l) = \{p \in \tilde{X}_\alpha \mid \tilde{s}_{l,\hbar_\alpha(\lambda_l)}(p) = 0\}, \quad \tilde{\Lambda}(\lambda_l) = \bigcup_{\alpha=1}^{r} \tilde{\Lambda}_\alpha(\lambda_l). \tag{7.10}$$

(Here r is the number of irreducible components of X.) We claim that

$$\tilde{\Lambda}(\lambda_l) = \tilde{\Lambda}(\lambda) \quad \text{and} \quad e(\mathcal{I}(\lambda_l)) = \text{n.l.c.}\, \chi(\mathcal{O}_{X\times\mathbb{A}^1}(k)/\mathcal{I}(\lambda)^{kl}). \tag{7.11}$$

Indeed, by our choice of the basis $\{s_{l,0}, \ldots, s_{l,m_l}\}$, we have $\tilde{s}_{l,\hbar_\alpha(\lambda_l)} = \tilde{s}^l_{\hbar_\alpha}$ and $\varrho_{l,\hbar_\alpha(\lambda_l)} = l\rho_{\hbar_\alpha}$ for all $X_\alpha \subset X$, from which we deduce

$$\tilde{\Lambda}_\alpha(\lambda_l) = (\tilde{s}_{l,\hbar_\alpha(\lambda_l)})^{-1}(0) = (\tilde{s}^l_{\hbar_\alpha})^{-1}(0) = \tilde{\Lambda}_\alpha(\lambda) \subset \tilde{X}_\alpha$$

and hence $\tilde{\Lambda}(\lambda_l) = \tilde{\Lambda}(\lambda)$ by Definition 3.1. Also, by the construction of λ_l, we have the middle identity

$$(t^{\varrho_{l,0}} s_{l,0}, \ldots, t^{\varrho_{l,m_l}} s_{l,m_l}) = \mathcal{I}(\lambda_l) = \mathcal{I}(\lambda)^l = (t^{\rho_0} s_0, \ldots, t^{\rho_m} s_m)^l \subset \mathcal{O}_{X\times\mathbb{A}^1}(l), \tag{7.12}$$

where the first and the third are by the definition. This implies the second part of (7.11) and hence our claim. Furthermore, (7.12), together with (2.11), (2.12) and Lemma 2.6, actually implies $\Delta_q(\lambda_l) = l \cdot \Delta_q(\lambda)$ for each $q \in \tilde{\Lambda}(\lambda)$.

With those in hand, we conclude that for λ'_l, the staircase 1-PS obtained from λ_l by applying Proposition 3.5, we have (1) $\tilde{\Lambda}(\lambda_l) = \tilde{\Lambda}(\lambda'_l)$ and for each $q \in \tilde{\Lambda}(\lambda_l)$, $w(\tilde{\mathcal{I}}(\lambda_l), q) = w(\tilde{\mathcal{I}}(\lambda'_l), q)$; and (2) for each $q \in \tilde{\Lambda}(\lambda_l)$, $\Delta_q(\lambda_l) = l \cdot \Delta_q(\lambda) \subset \Delta_q(\lambda'_l)$.

Proof of Sublemma 7.7. We first prove that the sublemma holds when $\rho_{\hbar_\alpha} = 0$ for all irreducible components X_α. Indeed, applying [Mum77, Proposition 2.11], we have

$$\sum_{i=0}^{m_l} \varrho_{l,i} = e(\mathcal{I}(\lambda)) \cdot \frac{l^2}{2} + a_1 \cdot l + a_2, \quad a_i \text{ depending only on } \lambda. \tag{7.13}$$

Since all $\rho_{\hbar_\alpha} = 0$, we have $\varrho_{l,i} = \hat{\varrho}_{l,i}$. Therefore,

$$\liminf_{l\to\infty} \frac{1}{l^2} \cdot \sum_{i=0}^{m_l} \hat{\varrho}_{l,i} = \liminf_{l\to\infty} \frac{1}{l^2} \cdot \sum_{i=0}^{m_l} \varrho_{l,i} = \frac{e(\mathcal{I}(\lambda))}{2} > 0.$$

We now prove the general case. We claim that there is an irreducible component X_β and a $q \in \tilde{X}_\beta$ such that

$$|\Delta_q(\lambda)| - \rho_{\hbar_\beta(\lambda)} \cdot w(\tilde{\mathcal{I}}(\lambda), q) > 0. \tag{7.14}$$

Suppose not. Since the $\geqslant$ for (7.14) always holds, we have $\varrho_i = \varrho_{\hbar_\alpha}$ for every $i \in \mathbb{I}_\alpha$. Because of the prior discussion, we must have an X_α such that $\varrho_{\hbar_\alpha} > 0$. Since X is connected, we can find a pair $X_\alpha \neq X_\beta$ such that $X_\alpha \cap X_\beta \neq \emptyset$ and $\rho_{\hbar_\alpha(\lambda)} > \rho_{\hbar_\beta(\lambda)} = 0$.

Let $\pi : \tilde{X} \to X$ be the projection and let $q \in \tilde{X}_\beta \cap \pi^{-1}(X_\alpha \cap X_\beta)$. We claim that the pair (β, q) satisfies the inequality (7.14). Since $\pi(q) \in X_\alpha$, we have $\tilde{s}_j(q) = 0$ for all $j > \hbar_\alpha(\lambda)$, and

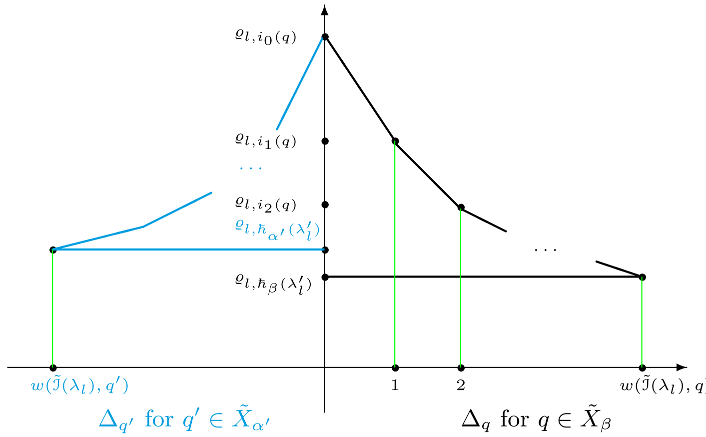

Figure 6. (Color online) The Newton polygon at $q, q' \in \tilde{X}$ when $\pi(q) = \pi(q') \in X_{\alpha'} \cap X_\beta$ is a linking node.

hence $i_0(q) \leqslant \hbar_\alpha(\lambda)$. Since $\rho_{\hbar_\alpha(\lambda)} > \rho_{\hbar_\beta(\lambda)} = 0$, we have $\rho_{i_0(q)} \geqslant \rho_{\hbar_\alpha(\lambda)} > 0$, and thus $|\Delta_q(\lambda)| > 0$, contradicting the assumption that (7.14) never holds and $\rho_{\hbar_\beta(\lambda)} = 0$. So a pair $q \in \tilde{X}_\beta$ satisfying (7.14) exists.

Let (β, q) be such a pair. We will show that

$$\sum_i \hat{\varrho}_{l,i} \geqslant \frac{1}{2}(|\Delta_q(\lambda'_l)| - \varrho_{l,\hbar_\beta(\lambda'_l)} \cdot w(\tilde{\mathfrak{I}}(\lambda'_l), q) - \rho_0 \cdot l) \tag{7.15}$$

and

$$\liminf_{l\to\infty} \frac{1}{l^2} \cdot (|\Delta_q(\lambda'_l)| - \varrho_{l,\hbar_\beta(\lambda'_l)} \cdot w(\tilde{\mathfrak{I}}(\lambda'_l), q)) \geqslant |\Delta_q(\lambda)| - \varrho_{l,\hbar_\beta(\lambda)} \cdot w(\tilde{\mathfrak{I}}(\lambda), q). \tag{7.16}$$

The sublemma follows after these two inequalities are established.

We prove (7.15). Following the notation introduced in § 4, we have $\sum_{i=0}^{m_l} \hat{\varrho}_{l,i} \geqslant \sum_{i\in\mathbb{I}_\alpha(\lambda'_l)} \hat{\varrho}_{l,i} \geqslant \sum_{i\in\mathbb{I}_q^{\text{pri}}(\lambda'_l)} \hat{\varrho}_{l,i}$, where $\mathbb{I}_\beta(\lambda'_l)$ is the set of indices for $\tilde{X}_\beta$, and $\mathbb{I}_q^{\text{pri}}(\lambda'_l)$ is the set of primary indices for $q \in \tilde{X}_\beta$, both with respect to the staircase λ'_l.

By Propositions 3.9 and 3.11, we know that for $i_0(q) \neq i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)$ we have $\hat{\varrho}_{l,i} = \varrho_{l,i} - \varrho_{l,\hbar_\beta(\lambda'_l)}$. (Note that it is possible that $\hat{\varrho}_{l,i_0(q)} = \varrho_{l,i_0(q)} - \varrho_{l,\hbar_{\alpha'}(\lambda'_l)} < \varrho_{l,i_0(q)} - \varrho_{l,\hbar_\beta(\lambda'_l)}$ for some $\alpha' \neq \beta$ (cf. (5.14)) and Figure 6.)

By the proof of Lemma 4.2, we have

$$\sum_{i\in\mathbb{I}_q^{\text{pri}}(\lambda'_l)\setminus\{i_0(q)\}} \hat{\varrho}_{l,i} \geqslant |\Delta_p^{\text{pri}}(\lambda'_l) \cap ([1, w^{\text{pri}}(q, \lambda'_l)] \times \mathbb{R})| - \varrho_{l,\hbar_\beta(\lambda'_l)} \cdot (w^{\text{pri}}(q, \lambda'_l) - 1).$$

Following (4.9), we continue to write $\bar{\jmath}_q(\lambda'_l) = \max\{i \in \mathbb{I}_q^{\text{pri}}(\lambda'_l)\}$ and $w^{\text{pri}}(q, \lambda'_l) = w(\tilde{\mathcal{E}}_{\bar{\jmath}_q(\lambda'_l)+1}(\lambda'_l), q)$. By the boundedness result from Corollary 3.12, for sufficiently large l, since the number of secondary indices is bounded by a uniform constant depend only on g and n (cf. Definition 3.10 and Corollary 5.3), the effects on the shape of $\Delta_q(\lambda'_l)$ from the *secondary indices* $\mathbb{I}_q(\lambda'_l)\setminus\mathbb{I}_q^{\text{pri}}(\lambda'_l)$

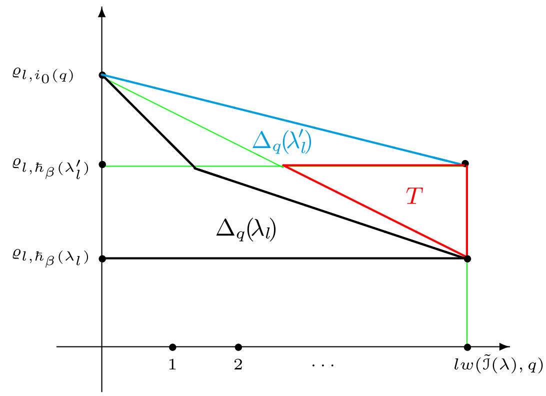

FIGURE 7. (Color online) Where T is triangle bounded by $y = \varrho_{l,\hbar_\beta(\lambda_l')}$, $x = l \cdot w(\tilde{\mathfrak{I}}(\lambda), q)$ and the line joining $(0, \varrho_{i_0(q)})$ and $(l \cdot w(\tilde{\mathfrak{I}}(\lambda), q), \varrho_{l,\hbar_\beta(\lambda_l)})$.

are marginal; thus for large l we have

$$\begin{aligned}
&|\Delta_q^{\text{pri}}(\lambda_l') \cap ([1, w^{\text{pri}}(q, \lambda_l')] \times \mathbb{R})| - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot (w^{\text{pri}}(q, \lambda_l') - 1) \\
&\quad \geqslant \tfrac{1}{2}(|\Delta_q(\lambda_l') \cap ([1, w(\tilde{\mathfrak{I}}(\lambda_l'), q) - 1)] \times \mathbb{R})| - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot (w(\tilde{\mathfrak{I}}(\lambda_l'), q) - 1)) \\
&\quad \geqslant \tfrac{1}{2}(|\Delta_q(\lambda_l')| - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot w(\tilde{\mathfrak{I}}(\lambda_l'), q) - \varrho_{l,i_0(q)}).
\end{aligned}$$

On the other hand, by our construction, $\varrho_{l,i_0(q)} \leqslant \varrho_{l,0} \leqslant \rho_0 \cdot l$. Combining, and adding $\hat{\varrho}_{i_0(p)} > 0$, we obtain

$$\sum_{i \in \mathbb{I}_q^{\text{pri}}(\lambda_l')} \hat{\varrho}_{l,i} \geqslant \tfrac{1}{2}(|\Delta_q(\lambda_l')| - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot w(\tilde{\mathfrak{I}}(\lambda_l'), q) - \rho_0 \cdot l).$$

This proves (7.15).

Before we move to (7.16), we claim that

$$A_\beta := \limsup_{l \to \infty} \frac{\varrho_{l,\hbar_\beta(\lambda_l')} - \varrho_{l,\hbar_\beta(\lambda_l)}}{l} = 0. \tag{7.17}$$

Suppose not. Say $A_\beta > 0$ (it is non-negative, by our construction of staircase in Proposition 3.5); then for l large, $\varrho_{l,\hbar_\beta(\lambda_l')} - \varrho_{l,\hbar_\beta(\lambda_l)} \geqslant \frac{1}{2} \cdot l \cdot A_\beta$.

By examining the geometry of $\Delta_q(\lambda_l) \subset \Delta_q(\lambda_l')$ (cf. Figure 7), we obtain

$$\begin{aligned}
|\Delta_q(\lambda_l')| - |\Delta_q(\lambda_l)| \geqslant |T| &= \frac{1}{2} \cdot \frac{(\varrho_{l,\hbar_\beta(\lambda_l')} - \varrho_{l,\hbar_\beta(\lambda_l)})^2 \cdot l \cdot w(\tilde{\mathfrak{I}}(\lambda), q)}{\varrho_{l,i_0(q)} - \varrho_{l,\hbar_\beta(\lambda_l)}} \\
&\geqslant \frac{1}{2} \cdot \left(\frac{A_\beta \cdot l}{2}\right)^2 \cdot \frac{l \cdot w(\tilde{\mathfrak{I}}(\lambda), q)}{l(\rho_{i_0(q)} - \rho_{\hbar_\beta(\lambda)})} := C \cdot A_\beta^2 \cdot l^2 > 0,
\end{aligned}$$

where we have used $l(\rho_{i_0(q)} - \rho_{\hbar_\beta(\lambda)}) = \varrho_{l,i_0(q)} - \varrho_{l,\hbar_\beta(\lambda_l)}$ because of Lemma 7.11. Therefore,

$$l^{-1} \cdot \omega(\lambda_l) = l^{-1} \cdot \omega(\lambda_l') + l^{-1} \cdot (e(\mathfrak{I}(\lambda_l')) - e(\mathfrak{I}(\lambda_l))) \geqslant l^{-1} \cdot (e(\mathfrak{I}(\lambda_l') - e(\mathfrak{I}(\lambda_l)), \tag{7.18}$$

where we have used Theorem 1.5 to deduce $\omega(\lambda_l') \geqslant 0$.

By Corollary 2.8 and our construction of staircase using Proposition 3.5, we deduce

$$l^{-1} \cdot (e(\mathfrak{I}(\lambda_l')) - e(\mathfrak{I}(\lambda_l))) \geqslant l^{-1} \cdot (|\Delta_q(\lambda_l')| - |\Delta_q(\lambda_l)|) \geqslant C \cdot A_\beta^2 \cdot l.$$

This is impossible since Lemma 7.6 implies that the left-hand side of (7.18) remains bounded for large l. This proves $A_\beta = 0$.

We now prove (7.16). Because $A_\beta = 0$, $|\Delta_q(\lambda_l')| \geqslant |\Delta_q(\lambda_l)|$, and by (2) after (7.12), we obtain

$$\begin{aligned}
&|\Delta_q(\lambda_l')| - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot w(\tilde{\mathfrak{I}}(\lambda_l'), q) \\
&\quad = |\Delta_q(\lambda_l')| - \varrho_{l,\hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathfrak{I}}(\lambda_l), q) + \varrho_{l,\hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathfrak{I}}(\lambda_l), q) - \varrho_{l,\hbar_\beta(\lambda_l')} \cdot w(\tilde{\mathfrak{I}}(\lambda_l'), q) \\
&\quad \geqslant |\Delta_q(\lambda_l)| - \varrho_{l,\hbar_\beta(\lambda_l)} \cdot w(\tilde{\mathfrak{I}}(\lambda_l), q) + (\varrho_{l,\hbar_\beta(\lambda_l)} - \varrho_{l,\hbar_\beta(\lambda_l')}) \cdot w(\tilde{\mathfrak{I}}(\lambda_l), q) \\
&\quad = l^2 \cdot (|\Delta_q(\lambda)| - \varrho_{l,\hbar_\beta(\lambda)} \cdot w(\tilde{\mathfrak{I}}(\lambda), q)) + l^2 \cdot \frac{\varrho_{l,\hbar_\beta(\lambda_l)} - \varrho_{l,\hbar_\beta(\lambda_l')}}{l} \cdot w(\tilde{\mathfrak{I}}(\lambda), q).
\end{aligned}$$

Taking $\liminf$ as $l \to \infty$, and using $A_\beta = 0$, we obtain (7.16).

Finally, by (7.13) and that $0 \leqslant \hat{\varrho}_{l,i} \leqslant \varrho_{l,i}$, we conclude that the $\liminf$ in the statement of the lemma is finite; thus the $\liminf$ is finite and positive by (7.14). This proves the lemma. □

Remark 7.8. It follows from the proof of Lemma 7.7 that $\omega(\lambda_l) \geqslant c \cdot \rho_0 \cdot l$ for l large (cf. [Sto09]). This can be viewed as a version of *uniform* Chow stability, an advantage of the GIT approach compared with that of [Oda13a].

Remark 7.9. Theorem 7.1 implies that the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ (for $g \geqslant 2$) is a K-stable compactification of the moduli of smooth curves. As K-stability is an analytic version of GIT stability via a CM-line bundle $\lambda^{\otimes 12} \otimes \delta^{-1}$ on the moduli of curves defined by Paul and Tian [PT06], it is interesting to see this generalized to moduli of high-dimensional polarized varieties. For recent progress, see Odaka [Oda12].

Remark 7.10. Theorem 7.1 can be easily generalized to the weighted pointed stable curve, that is, although a weighted pointed stable curve in general is not asymptotic Chow stable with respect to the polarization $\omega_X(\mathbf{a} \cdot \mathbf{x})^{\otimes k}$, it is *log K-stable* (cf. [OS11] for the definition). In other words, the asymptotic of *Chow instability* behaves in a controlled manner.

Acknowledgements

We thank the anonymous referee for pointing out quite a few inaccuracies and gaps in the first draft and for making several valuable suggestions. We want to thank Baosen Wu for assisting us to generate all the figures included in this paper. The first named author is partially supported by an NSF grant NSF0601002. The second named author is partially supported by RGC grants CUHK 403106 and 403709 from the Hong Kong government. The majority of the work was completed when he was a faculty member of the Department of Mathematics in the Chinese University of Hong Kong; he wants to express his gratitude to his colleagues there for the stimulating environment they provided. Also, part of the writing was completed when the second author was visiting the Mathematics Department of Stanford University in summer 2009 and the Mathematics Department of Harvard University in fall 2009. He thanks the faculty members there for their hospitality and the fine research environment they provided.

References

BS08 E. Baldwin and D. Swinarski, *A geometric invariant theory construction of moduli spaces of stable maps*, Int. Math. Res. Pap. IMRP (2008), Art. ID rpn 004.

BFMV14 G. Bini, F. Felici, M. Melo and F. Viviani, *Geometric invariant theory for polarized curves*, Lecture Notes in Mathematics, vol. 2122 (Springer, Cham, 2014).

Cap94 L. Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), 589–660.

Don02 S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.

Eis95 D. Eisenbud, *Commutative algebra*, in *With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150 (Springer, New York, 1995).

Gie77 D. Gieseker, *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. (2) **106** (1977), 45–60.

Gie82 D. Gieseker, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69 (Springer, New York, 1982).

Gie98 D. Gieseker, *Geometric invariant theory and the moduli of bundles*, in *Gauge theory and the topology of four-manifolds (Park City, UT, 1994)*, IAS/Park City Mathematics Series, vol. 4 (American Mathematical Society, Providence, RI, 1998).

GM84 D. Gieseker and I. Morrison, *Hilbert stability of rank-two bundles on curves*, J. Differential Geom. **19** (1984), 1–29.

Has03 B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. **173** (2003), 316–352.

HM98 J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187 (Springer, New York, 1998).

KK93 J. Kantor and A. Khovanskii, *Une application du théorème de Riemann–Roch combinatoire au polynôme d'Ehrhart des polytopes entiers de* $\mathbb{R}^d$, C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), 501–507.

Laz04 R. Lazarsfeld, *Positivity in algebraic geometry. II.*, in *Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3 (Springer, Berlin, 2004).

Mor09 I. Morrison, *GIT constructions of moduli spaces of stable curves and maps*, in *Surveys in differential geometry, Vol. XIV, Geometry of Riemann surfaces and their moduli spaces*, Surveys in Differential Geometry, vol. 14 (International Press, Somerville, MA, 2009), 315–369; arXiv:0810.2340.

Mum77 D. Mumford, *Stability of projective varieties*, Enseign. Math. (2) **23** (1977), 39–110.

MFK94 D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, third edition (Springer, Berlin, 1994).

Oda12 Y. Odaka, *On the moduli of Kähler–Einstein Fano manifolds*, Preprint (2012), arXiv:1211.4833.

Oda13a Y. Odaka, *The GIT-stability of polarized varieties via discrepancy*, Ann. of Math. (2) **177** (2013), 645–661; arXiv:0807.1716.

Oda13b Y. Odaka, *A generalization of Ross–Thomas' slope theory*, Osaka J. Math. **50** (2013), 171–185; arXiv:0910.1794.

OS11 Y. Odaka and S. Sun, *Testing log K-stability by blowing up formalism*, Ann. Fac. Sci. Toulouse Math., to appear, arXiv:1112.1353.

PT06 S. Paul and G. Tian, *CM stability and the generalized Futaki invariant I*, Preprint (2006), arXiv:math/0605278 [math.AG].

RT07 J. Ross and R. Thomas, *A study of the Hilbert–Mumford criterion for the stability of projective varieties*, J. Algebraic Geom. **16** (2007), 201–255.

Sch91 D. Schubert, *A new compactification of the moduli space of curves*, Compositio Math. **78** (1991), 297–313.

Sto09 J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221** (2009), 1397–1408.

Sto11 J. Stoppa, *A note on the definition of K-stability*, Preprint (2011), arXiv:1111.5826.

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press

Swi12 D. Swinarski, *GIT stability of weighted pointed curves*, Trans. Amer. Math. Soc. **364** (2012), 1737–1770.

Wan12 X. Wang, *Height and GIT weight*, Math. Res. Lett. **19** (2012), 906–926.

WX14 X. Wang and C. Xu, *Nonexistence of asymptotic GIT compactification*, Duke Math. J. **163** (2014), 2217–2241.

Jun Li jli@stanford.edu
Stanford University, Stanford, CA 94305, USA

Xiaowei Wang xiaowwan@rutgers.edu
The Chinese University of Hong Kong, Hong Kong

and

Rutgers University, Newark, NJ 07102, USA

https://doi.org/10.1112/S0010437X1500737X Published online by Cambridge University Press