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A NOTE ON CERTAIN EQUIVALENT NORMS ON TSIRELSON'S SPACE

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Abstract. We prove that the norm $\|\cdot\|_n$ of the space $T[\mathcal{S}_n, \theta]$ and the norm $\|\cdot\|_n^M$ of its modified version $T_M[\mathcal{S}_n, \theta]$ are 3-equivalent. As a consequence, using the results of E. Odell and N. Tomczak-Jaegermann, we obtain that there exists a $K < \infty$ such that for all n, $\|\cdot\|_n^M$ does not $K-$ distort any subspace of Tsirelson's space T.

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1. Introduction. An important and still open question is whether or not there exists a distortable Banach space which is not arbitrarily distortable. The primary candidate for such a space is Tsirelson's space T. It is known that for every $1 < \lambda < 2$, there exists an equivalent norm on T which is a λ-distortion [**11**]. Tsirelson's space does not belong to any general class of Banach spaces known to be arbitrarily distortable and in fact recent results show that if there exists a distortable not arbitrarily distortable Banach space X, then X must contain a subspace with structure similar to Tsirelson's space. For these and relevant results we refer to the recently published handbooks of Banach spaces [**7**], [**8**]. Thus it is of interest to examine all known equivalent norms on T to see if they can arbitrarily distort T or any subspace of T. E. Odell and N. Tomczak-Jaegermann, [**10**], have begun such examination, by considering on T the equivalent norms $\|\cdot\|_n$ of the spaces $T_n = T[\mathcal{S}_n, \frac{1}{2^n}]$, $n \in \mathbb{N}$. They have shown that there exists a constant $K > 1$ so that for all $n \in \mathbb{N}$, $\|\cdot\|_n$ does not K-distort any subspace Y of T. Some conditions on the equivalent norms of T which imply that they do not distort T, were given earlier in [**11**]. Our work is a continuation of the work of E. Odell and N. Tomczak-Jaegermann. We consider the modified Tsirelson's space T_M and certain equivalent norms of this space. Modified Tsirelson's space T_M, was introduced by W. B. Johnson [**6**]. P. Casazza and E. Odell [**4**], have proved that T_M is isomorphic to the original one. Later S. Bellenot [**3**], gave an elegant argument that shows that T_M and T are 2-isomorphic. Our work is inspired by the work of S. Bellenot. We consider the norms $\|\cdot\|_n^M$ of the spaces $T_n^M = T_M[\mathcal{S}_n, \frac{1}{2^n}]$. These norms are equivalent to the norm of T and have much more complexity than the norms of the spaces $T[\mathcal{S}_n, \frac{1}{2^n}]$. It is known that for every $n \in \mathbb{N}$ the spaces $T[\mathcal{S}_n, \frac{1}{2^n}]$ and $T_M[\mathcal{S}_n, \frac{1}{2^n}]$ are 2^n-isomorphic. Our main result is that for every $0 < \theta < 1$ and every $n \in \mathbb{N}$, the spaces $T[\mathcal{S}_n, \theta]$ and $T_M[\mathcal{S}_n, \theta]$ are 3-isomorphic. Combining this result with the work of E. Odell and N. Tomczak-Jaegermann it follows that there exists $K > 1$ so that for every $n \in \mathbb{N}$ the norm $\|\cdot\|_n^M$ does not K-distort any subspace of T.

Section 2 contains the relevant terminology and background material. Otherwise our notation is standard as may be found in **[9]**. More detailed information about Tsirelson's space and some of its variations can be found in **[5]**.

2. Preliminaries. We denote by c_{00} the linear space of finitely supported sequences, and by (e_i) the unit vector basis of c_{00}. If $x = \sum_i a_i e_i \in c_{00}$, then $\operatorname{supp} x = \{i : a_i \neq 0\}$. If E is a subset of $\mathbb{N}$, then $Ex \in c_{00}$ is defined by $Ex = \sum_{i\in E} a_i e_i$. $X, Y, \ldots$ will denote separable infinite-dimensional Banach spaces.

A space $(X, \|\cdot\|)$ is arbitrarily distortable, if for all $\lambda > 1$, there exists an equivalent norm $|\cdot|$ on X such that

$$\sup\left\{\frac{|y|}{|z|} : y, z \in S_Y\right\} > \lambda \ \text{ for all } \ Y \subset X. \tag{2.1}$$

The norm $|\cdot|$ satisfying (2.1) is said to $\lambda-$distort X. X is $\lambda-$distortable if some norm $\lambda-$distorts X. X is distortable if it is $\lambda-$distortable for some $\lambda > 1$.

If M is a subset of $\mathbb{N}$, we denote the set of all finite subsets of M by $[M]^<$. For $E, F \subset \mathbb{N}$, we write $E < F$ (respectively $E \leq F$), if $\max E < \min F$ (resp. $\max E \leq \min F$) or if either one is empty. A subset $\mathcal{F} \subset [\mathbb{N}]^<$ is hereditary if $G \in \mathcal{F}$ whenever $G \subset F \in \mathcal{F}$. $\mathcal{F}$ is spreading if whenever $F = \{n_1, \ldots, n_k\} \in \mathcal{F}$ with $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ satisfies $n_i \leq m_i$ for $1 \leq i \leq k$, then $\{m_1, \ldots, m_k\} \in \mathcal{F}$. $\mathcal{F}$ is compact if it is compact in the product topology in $2^{\mathbb{N}}$. A set $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called regular if it has all three properties. Given $\mathcal{F} \subset [\mathbb{N}]^<$, a sequence of finite subsets $\{E_1, \ldots, E_n\}$ is said to be $\mathcal{F}$-admissible (resp. $\mathcal{F}$-allowable) if $\{\min E_1, \ldots, \min E_n\} \in \mathcal{F}$ and $E_1 < \cdots < E_n$ (respectively $\{\min E_1, \ldots, \min E_n\} \in \mathcal{F}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$). Similarly the vectors $x_i \in c_{00}$, $i = 1, \ldots, n$ are said to be $\mathcal{F}$-admissible (resp. allowable) if the sequence $\{\operatorname{supp} x_1, \ldots, \operatorname{supp} x_n\}$ is $\mathcal{F}$-admissible (resp. allowable).

If $\mathcal{M}$ and $\mathcal{N}$ are regular subsets of $[\mathbb{N}]^<$ we let

$$\mathcal{M}[\mathcal{N}] = \left\{\bigcup_{i=1}^{k} F_i : F_i \in \mathcal{N} \text{ for all } i \text{ and } \{F_1, \ldots, F_k\} \text{ is } \mathcal{M}\text{-admissible}\right\},$$

and

$$(\mathcal{M})^3 = \{M_1 \cup M_2 \cup M_3 : M_1 < M_2 < M_3 \text{ and } M_1, M_2, M_3 \in \mathcal{M}\}.$$

The Schreier families $\mathcal{S}_n$, $n \in \mathbb{N}$, were introduced in **[1]**, and are defined as follows: $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, $\mathcal{S}_1 \equiv \mathcal{S} = \{F \subset \mathbb{N} : \#F \leq \min F\}$, where $\#F$ denotes the cardinality of F. For $n \geq 1$, $\mathcal{S}_{n+1} = \mathcal{S}_1[\mathcal{S}_n]$. It is easy to see that the Schreier families are regular families of finite subsets of $\mathbb{N}$.

We shall consider also the modified Schreier families $\mathcal{S}_n^M$, $n \in \mathbb{N}$, and their relation to the original ones. Their definition and their relation to the original ones are given in the following lemma from **[2]** whose proof is provided since the relation of $\mathcal{S}_n$ and $\mathcal{S}_n^M$ and especially the relation (2.3) below is an essential tool in our proof.

LEMMA 2.1. **[2]** *For $n < \omega$ define the family $\mathcal{S}_n^M$ inductively as follows: $\mathcal{S}_0^M = \mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$, $\mathcal{S}_1^M = \mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leq \min F\}$ and for $n \geq 1$,*

$$\mathcal{S}_{n+1}^M = \left\{\bigcup_{i=1}^{k} F_i : \ F_i \in \mathcal{S}_n^M \textit{ for all } i \textit{ and } \{F_1, \ldots, F_k\} \textit{ is } \ \mathcal{S}\textit{-allowable}\right\} \cup \{\emptyset\}.$$

Then $S_n^M = \mathcal{S}_n$ for all $n \in \mathbb{N}$.

Proof. The proof of the lemma is an immediate consequence of the following.

CLAIM. *If* $A_1, \dots, A_k$, $k \in \mathbb{N}$, *are pairwise disjoint non-empty sets in* $\mathcal{S}_n$ *with* $\min A_1 < \cdots < \min A_k$, *then there exist non-empty sets* $A'_1 < \cdots < A'_k$ *in* $\mathcal{S}_n$ *such that* $\cup_{i=1}^k A_i = \cup_{i=1}^k A'_i$ *and* $\min A_i \le \min A'_i$ *for* $i = 1, \dots, k$.

Proof of the Claim. It is done by induction on $\mathbb{N}$. For $n = 0$ it is trivial. Suppose it is true for n.

Let $A_1, \dots, A_k$, $k \in \mathbb{N}$, be pairwise disjoint non-empty sets in $\mathcal{S}_{n+1}$ and $\min A_1 < \dots < \min A_k$. Each A_i is of the form $A_i = \cup_{j=1}^{m_i} B_j^i$ where $B_j^i \in \mathcal{S}_n$ and, for each i, $m_i \le B_1^i < B_2^i < \cdots < B_{m_i}^i$. Let $\{B_j\}_{j=1}^{m_1+\cdots+m_k}$ be a rearrangement of the family $\{B_j^i : i = 1, \dots, k, j = 1, \dots, m_i\}$, which satisfies $\min B_1 < \min B_2 < \cdots < \min B_{m_1+\cdots+m_k}$. It is easy to see that, for every i,

$$\min A_i = \min B_1^i \le \min B_{m_1+\ldots+m_{i-1}+1}. \tag{2.2}$$

By the inductive assumption, there exists sets B'_j, $j = 1, \dots, m_1 + \cdots + m_k$, with $B'_j \in \mathcal{S}_n$, $\cup_{j=1}^{m_1+\cdots+m_k} B'_j = \cup_{j=1}^{m_1+\cdots+m_k} B_j$ and such that $B'_1 < B'_2 < \cdots < B'_{m_1+\cdots+m_k}$ and $\min B_j \le \min B'_j$ for $j = 1, \dots, m_1 + \cdots + m_k$. For $i = 1, \dots, k$, we set

$$A'_i = \bigcup_{j=m_1+\cdots+m_{i-1}+1}^{m_1+\cdots+m_i} B'_j.$$

Then, $A'_1 < A'_2 < \cdots < A'_k$, $\cup_{i=1}^k A'_i = \cup_{i=1}^k A_i$, and for each $i = 1, \dots, k$ we have by (2.2),

$$m_i \le \min B_{m_1+\cdots+m_{i-1}+1} \le \min B'_{m_1+\cdots+m_{i-1}+1}$$

so $A'_i \in \mathcal{S}_{n+1}$. Moreover, using (2.2) again, we see that

$$\min A_i \le \min B_{m_1+\cdots+m_{i-1}+1} = \min A'_i.$$

This completes the proof of the Claim. The lemma follows. □

If $\mathcal{F}$ is a regular family of finite subsets of $\mathbb{N}$ and θ is number with $0 < \theta < 1$, the Tsirelson space $T[\mathcal{F}, \theta]$ is the completion of c_{00} under the implicit norm

$$\|x\| = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^k \|E_i x\| : (E_i)_{i=1}^k \text{ is } \mathcal{F}\text{-admissible}\right\}\right\}.$$

The modified Tsirelson space $T_M[\mathcal{F}, \theta]$ is the completion of c_{00} under the implicit norm

$$\|x\|_M = \max\left\{\|x\|_\infty, \theta \sup\left\{\sum_{i=1}^k \|E_i x\|_M : (E_i)_{i=1}^k \text{ is } \mathcal{F}\text{-allowable}\right\}\right\}.$$

The classical Tsirelson's space is $T \equiv T[\mathcal{S}_1, \frac{1}{2}]$, and we write $\|\cdot\|$ for the norm of T. We also consider the modified Tsirelson's space $T_M \equiv T_M[\mathcal{S}, \frac{1}{2}]$, whose norm is denoted by $\|\cdot\|_M$. Our main concern are the spaces $T_n \equiv T[\mathcal{S}_n, \frac{1}{2^n}]$ and their modified version $T_n^M \equiv T_M[\mathcal{S}_n, \frac{1}{2^n}]$. We shall denote the norms of these spaces by $\|\cdot\|_n$ and $\|\cdot\|_n^M$ respectively. It is clear from the definitions that $\|x\|_n \leq \|x\|_n^M$ for every $x \in c_{00}$.

We shall also make use of the following alternative definition of the norm of $X = T[\mathcal{F}, \theta]$ and $X_M = T_M[\mathcal{F}, \theta]$, where $\mathcal{F}$ is a regular family of finite subsets of $\mathbb{N}$, and θ is a number with $0 < \theta < 1$.

Inductively, we define a subset $K = \cup_{j=0}^{\infty} K_j$ of B_{X^*} as follows:

$$K_0 = \{\pm e_n : n \in \mathbb{N}\}.$$

Assume that K_j has been defined. We set

$$K_{j+1} = K_j \cup \big\{\theta(f_1 + \cdots + f_d) : d \in \mathbb{N}, f_i \in K_j,\ i = 1, \ldots, d, \\ \text{and } (f_i)_{i=1}^d \text{ is } \mathcal{F}\text{-admissible}\big\}.$$

Let $K = \cup_{j=0}^{\infty} K_j$. Then K is a norming set for X that is, for $x \in X$,

$$\|x\| = \sup_{f \in K} \langle x, f \rangle.$$

The norming set $K^M = \cup_{j=0}^{\infty} K_j^M$ of X_M is defined similarly to K, namely $K_0^M = \{\pm e_n : n \in \mathbb{N}\}$ and for $j \geq 0$,

$$K_{j+1}^M = K_j^M \cup \big\{\theta(f_1 + \cdots + f_d) : d \in \mathbb{N}, f_i \in K_j^M,\ i = 1, \ldots, d, \\ \text{and } (f_i)_{i=1}^d \text{ is } \mathcal{F}\text{-allowable}\big\}.$$

In what follows, by a tree $\mathcal{T}$ we shall mean a finite set of finite sequences of positive integers, partially ordered by the relation

$$\alpha \prec \beta \text{ iff } \alpha \text{ is an initial segment of } \beta,$$

and with the following property: For every $\alpha \in \mathcal{T}$, $\{\beta : \beta \prec \alpha\} \subseteq \mathcal{T}$. The elements of $\mathcal{T}$ are called nodes. $\mathcal{T}$ has a unique root, the empty sequence which we denote by 0. The length or level of a sequence $\alpha \in \mathcal{T}$ is denoted by $|\alpha|$. The height of $\mathcal{T}$, denoted by $\mathrm{h}(\mathcal{T})$, is the maximum length of the nodes of $\mathcal{T}$. If $\alpha \in \mathcal{T}$ we denote $S_\alpha = \{\beta \in \mathcal{T} : \alpha \prec \beta \text{ and } |\beta| = |\alpha| + 1\}$. The elements of S_α are called immediate successors of α. If $\alpha \in \mathcal{T}$ and $S_\alpha = \emptyset$ we say that α is terminal node. If $\alpha, \beta \in \mathcal{T}$ and $\beta \prec \alpha$ we say that β is a predecessor of α or α is a successor of β.

Let K be the norming set of the space $T[\mathcal{F}, \theta]$ and K^M be the norming set of its modified version $T_M[\mathcal{F}, \theta]$.

Definition 2.2. Let $\phi \in K$ (resp. $\phi \in K^M$). An analysis of ϕ is a subset $\{f_\alpha\}_{\alpha \in \mathcal{T}}$ of K (resp. K^M), indexed by a tree $\mathcal{T}$ of finite height, such that:

(1) $\phi = f_0$

(2) For every $0 \leq s \leq \mathrm{h}(\mathcal{T})$, the elements of the set $\{f_\alpha : \alpha \in \mathcal{T} \text{ and } |\alpha| = s\}$ are disjointly supported and $\cup_{|\alpha|=s} \operatorname{supp} f_\alpha \subset \operatorname{supp} \phi$.

(3) For every $\beta \in \mathcal{T}$, either $f_\beta = \pm e^*_{m_\beta}$ for some $m_\beta \in \mathbb{N}$, if β is a terminal node of $\mathcal{T}$, or $f_\beta = \theta \sum_{\alpha \in S_\beta} f_\alpha$, and the set $\{f_\alpha : \alpha \in S_\beta\}$ is $\mathcal{F}$-admissible (resp. $\mathcal{F}$-allowable).

It is easy to see that every $\phi \in K$ has an analysis. One of the main properties of the norming set is that it is closed under projections on subsets of $\mathbb{N}$, (i.e. if $f \in K$ and $E \subset \mathbb{N}$ then $Ef \in K$). This gives us also that (e_i) is an unconditional basis.

Given a functional f and an analysis $(f_\alpha)_{\alpha \in \mathcal{T}}$ of f, we shall adopt the terminology of the tree for the functionals, i.e. we shall call the functionals f_α nodes, if $\beta \prec \alpha$ we shall say that f_α is successor of f_β and so on.

DEFINITION 2.3. Let $f = (f_\alpha)_{\alpha\in\mathcal{T}}$ be a subsequence of c_{00} indexed by a tree $\mathcal{T}$. We say that f or $\mathcal{T}$ is admissible, if for every $\alpha \in \mathcal{T}$ and every $\beta, \gamma \in S_\alpha$ with minsupp $f_\beta <$ minsupp f_γ it holds that supp $f_\beta <$ supp f_γ.

It is easy to see that if $\mathcal{T}$ is admissible and $(\beta_i)_{i\le n}$ are the nodes in a level of $\mathcal{T}$ such that (minsupp $f_{\beta_i})_{i\le n}$ is an increasing sequence, we have that supp$f_{\beta_i} <$ supp$f_{\beta_{i+1}}$ for every $i < n$.

The norms $\|\cdot\|_n^M$ are equivalent to the norm $\|\cdot\|_M$ of T_M and it holds that

$$\|x\|_n^M \le \|x\|_M \le 2^{n-1}\|x\|_n^M. \tag{2.3}$$

From (2.3) and the 2-equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_M$, ([**3**, **4**]), we get that the norms $\|\cdot\|_n^M$ are equivalent norms on T. Let us explain (2.3).

Let f be a functional in the norming set of $T_M[\mathcal{S}_n, \frac{1}{2^n}]$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of f. Using the definition of the modified Schreier family $\mathcal{S}_n^M$, and in particular that $\mathcal{S}_n^M = \underbrace{\mathcal{S}_1^M[\cdots[\mathcal{S}_1^M]\cdots]}_{n-times}$ we define a functional h in the norming set of the space T_M and an analysis $(h_\beta)_{\beta\in\mathcal{T}_1}$ of h, so that for every node f_α, $\alpha \in \mathcal{T}$, with $|\alpha| = k$ there exists a node h_β, $\beta \in \mathcal{T}_1$ with $|\beta| = kn$ and $f_\alpha = h_\beta$. This gives us that the norming set of the space $T_M[\mathcal{S}_n, \frac{1}{2^n}]$ is contained in the norming set of the space $T_M[\mathcal{S}, \frac{1}{2}]$, and thus the left hand inequality.

In order to get the right hand inequality, for every f in the norming set of the space $T_M[\mathcal{S}, \frac{1}{2}]$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ an analysis of f, we transform its analysis so that the terminal nodes have levels $n, 2n, 3n, \ldots$.

More precisely, let f be a functional and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of f. For $x = \sum a_i e_i \in c_{00}$ we set $A = \operatorname{supp} x \cap \{i : e_i^* \text{ is a terminal node of } f\}$. Then we have that

$$f(x) = \sum_{i\in A} \frac{a_i}{2^{r_i}} \tag{2.4}$$

where r_i is the level of the terminal node e_i^*, $i \in A$ (we may assume that $f(e_i) \ge 0$ and $a_i \ge 0$ since the basis is unconditional).

If $r_i \notin \{n, 2n, \ldots\}$ we replace e_i^* by $\frac{1}{2^{l_i}} e_i^*$ where $0 < l_i < n$ and $r_i + l_i \in \{n, 2n, \ldots\}$. This gives us a new tree $\mathcal{T}_1$ in which all terminal nodes have levels $n, 2n, \ldots$ Next, for every node f_α with $|\alpha| = 0, n, 2n, \ldots$ we consider as its immediate successors the $\mathcal{S}_n$-allowable set, $\{f_\beta : \alpha \prec \beta,\ \beta \in \mathcal{T}_1 \text{ and } |\beta| = |\alpha| + n\}$. This gives us a functional h in the norming set of $T_M[\mathcal{S}_n, \frac{1}{2^n}]$, which has the same support as f. This follows from the replacement of the terminal nodes of f. Since every terminal node e_i^* of f have been moved down l_i levels and $0 \le l_i \le n-1$, we get that

$$f(x) = \sum_{i\in A} \frac{a_i}{2^{r_i}} \le \sum_{i\in A} 2^{l_i} \frac{a_i}{2^{r_i+l_i}} \le 2^{n-1} h(x) \le 2^{n-1}\|x\|_n^M.$$

From (2.4) it is clear that if we change the position of a terminal node in the tree without changing its level, the estimation remains the same. This will be the main ingredient in the proof for the equivalence of the norms $\|\cdot\|_n$ and $\|\cdot\|_n^M$.

3. The spaces T_n and T_n^M are 3-isomorphic. We consider the regular families $\mathcal{A}_n = \{F \subset \mathbb{N} : \#F \le n\}$, $n \in \mathbb{N}$, of finite subsets of $\mathbb{N}$. Let $\mathcal{F}_n = \mathcal{S}_n[\mathcal{A}_2]$. First we shall consider the auxiliary space $Y = T[\mathcal{S}_n[\mathcal{A}_2], \frac{1}{2^n}]$. We prove in Proposition 3.2, that the spaces T_n and Y are 3-isomorphic. Next we show, Proposition 3.4, that given a functional f in the norming set of the space T_n^M, we can produce a functional g in the norming set of the space Y, that dominates the estimation of f for every $x \in c_{00}$. This will give us that the norm of the space T_n^M is dominated by the norm of the space Y. From the 3-equivalence of the norms of the spaces Y and T_n we get that the spaces T_n and T_n^M are 3-isomorphic.

We begin with an auxiliary lemma.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then it holds that $\mathcal{F}_n[\mathcal{A}_3] = [\mathcal{S}_n[\mathcal{A}_2]][\mathcal{A}_3] \subset (\mathcal{S}_n)^3$.*

Proof. We shall prove the result by induction on $n \in \mathbb{N}$. For $n = 1$,

$$[\mathcal{S}_1[\mathcal{A}_2]][\mathcal{A}_3] = \mathcal{S}_1[\mathcal{A}_2[\mathcal{A}_3]] = \mathcal{S}_1[\mathcal{A}_6]\,.$$

Let $k \leq m_1 < \cdots < m_k$ and

$$m_i \leq \lambda_{(i-1)6+1} < \lambda_{(i-1)6+2} < \lambda_{(i-1)6+3} < \lambda_{(i-1)6+4} < \lambda_{(i-1)6+5} < \lambda_{6i} < m_{i+1}$$

for every $i = 1, \ldots, k$. We set $G = \{\lambda_1, \ldots, \lambda_{6k}\}$. From the assumptions we have that $k \leq G$. We consider the sets

$$F_1 = \{k, k+1, \ldots, 2k-1\}\,, \quad F_2 = \{2k, 2k+1, \ldots, 4k-1\}\,,$$
$$F_3 = \{4k, 4k+1, \ldots, 7k-1\}\,.$$

Each of the sets F_i, $i = 1, 2, 3$ belongs to the family $\mathcal{S}$ since $\#F_i \leq \min F_i$. We may write $G = \cup_{i=1}^{3} G_i$ where $G_i = \{\lambda_{j+1-k} : j \in F_i\}$ for $i = 1, 2, 3$. Then G_i is a spreading of F_i and hence belongs to the Schreier family $\mathcal{S}$. Thus $G \in (\mathcal{S}_1)^3$.

Assume that the result holds for n, i.e. $\mathcal{F}_n[\mathcal{A}_3] = [\mathcal{S}_n[\mathcal{A}_2]][\mathcal{A}_3] \subset (\mathcal{S}_n)^3$. Then we have that

$$\mathcal{F}_{n+1}[\mathcal{A}_3] = [\mathcal{S}_{n+1}[\mathcal{A}_2]][\mathcal{A}_3] = \mathcal{S}_1[\mathcal{S}_n[\mathcal{A}_2]][\mathcal{A}_3] = \mathcal{S}_1\,[\mathcal{F}_n[\mathcal{A}_3]] \subset \mathcal{S}_1[(\mathcal{S}_n)^3] \subset (\mathcal{S}_{n+1})^3\ .$$

The last inclusion relation, i.e. $\mathcal{S}_1[(\mathcal{S}_n)^3] \subset (\mathcal{S}_{n+1})^3$, follows immediately from the associativity of the operation $\mathcal{M}[\mathcal{N}]$, and the fact that $(\mathcal{M})^3 = \mathcal{A}_3[\mathcal{M}]$. $\square$

Proposition 3.2. *For every $x \in c_{00}$ we have the inequality*

$$\|x\|_n \leq \|x\|_Y \leq 3\|x\|_n\,.$$

Proof. An easy inductive argument, using that $S_n \subset \mathcal{S}_n[\mathcal{A}_2]$, gives us that the norming set of $T_n = T[\mathcal{S}_n, \frac{1}{2^n}]$ is contained in the norming set of $Y = T[\mathcal{S}_n[\mathcal{A}_2], \frac{1}{2^n}]$, and hence the left hand inequality.

For the right hand inequality we shall prove that for every functional f in the norming set $K = \cup_{j=0}^{\infty} K_j^Y$ of the space Y there exist three functionals g_1, g_2, g_3 in the norming set of the space $T_n = T[\mathcal{S}_n, \frac{1}{2^n}]$ so that

$$f(x) \leq g_1(x) + g_2(x) + g_3(x), \ \bigcup_{i=1}^{3} \operatorname{supp} g_i \subset \operatorname{supp} f \ \text{ and } \ g_1 < g_2 < g_3. \tag{3.1}$$

For $f = e_k^* \in K_0^Y$ (3.1) trivially holds. Assume that the result holds for every $f \in K_j^Y$, and let $f = \frac{1}{2^n} \sum_{i \in F} f_i \in K_{j+1}^Y$. Then we have that $\{\text{minsupp } f_i : i \in F\} \in \mathcal{S}_n[\mathcal{A}_2]$ and the functionals $(f_i)_{i \in F} \subset K_j^Y$ have successive support.

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By the inductive hypothesis for every $i \in F$ there exist three functionals g_i^1, g_i^2, g_i^3 in the norming set of the space $T[\mathcal{S}_n, \frac{1}{2^n}]$ so that (3.1) holds for f_i and g_i^1, g_i^2, g_i^3 i.e.

$$f_i(x) \leq g_i^1(x) + g_i^2(x) + g_i^3(x), \quad \bigcup_{j=1}^{3} \operatorname{supp} g_i^j \subset \operatorname{supp} f_i \ \text{ and } \ g_i^1 < g_i^2 < g_i^3 . \tag{3.2}$$

We have that $\{\text{minsupp } f_i : i \in F\} \in \mathcal{S}_n[\mathcal{A}_2]$ and therefore

$$\left\{\text{minsupp } g_i^1, \text{minsupp } g_i^2, \text{minsupp } g_i^3 : i \in F\right\} \in [\mathcal{S}_n[\mathcal{A}_2]][\mathcal{A}_3] .$$

From Lemma 3.1 we have that $\left\{\text{minsupp } g_i^1, \text{minsupp } g_i^2, \text{minsupp } g_i^3 : i \in F\right\} \in (\mathcal{S}_n)^3$. Therefore the functional $\frac{1}{2^n}\sum_{i\in F}(g_i^1 + g_i^2 + g_i^3)$ can be written as the sum of the three successive functionals $g_k = \frac{1}{2^n}\sum_{A_k} g_i^j$, $k = 1, 2, 3$, where $A_1 < A_2 < A_3$ are such that $A_1 \cup A_2 \cup A_3 = \left\{\text{minsupp } g_i^1, \text{minsupp } g_i^2, \text{minsupp } g_i^3 : i \in F\right\} \in (\mathcal{S}_n)^3$. It follows from (3.2) that

$$f(x) = \frac{1}{2^n}\sum_{i\in F} f_i(x) \leq \frac{1}{2^n}\sum_{i\in F}\left(g_i^1 + g_i^2 + g_i^3\right)(x) = (g_1 + g_2 + g_3)(x),$$

$$\bigcup_{i=1}^{3} \operatorname{supp} g_i \subset \operatorname{supp} f \ \text{ and } \ g_1 < g_2 < g_3 .$$

Therefore (3.1) holds for f and g_1, g_2, g_3. This completes the proof. □

REMARK 3.3. Let us observe that Proposition 3.2 is independent of the number $\frac{1}{2^n}$, that is, it holds also for the norms of the spaces $T[\mathcal{S}_n, \theta]$ and $T[\mathcal{S}_n[\mathcal{A}_2], \theta]$, for every $0 < \theta < 1$.

Let f be a functional in the norming set of the space $T_M[\mathcal{S}_n, \frac{1}{2^n}]$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of f. In the next proposition, following the analysis of f, we shall construct a functional g in the norming set of the space $Y = T[\mathcal{S}_n[\mathcal{A}_2], \frac{1}{2^n}]$ which gives the same estimation as f. As a consequence we get that $\|x\|_n^M \leq \|x\|_Y$, for every $x \in c_{00}$. Our proof is based on the properties of the tree, and the basic tool in the proof is the notion of the good terminal node, Definition 3.5. The definition of a good terminal and Lemma 3.6, with slight modifications, have been taken from [**3**], Theorem 2.1, where the terminology good leaf is used instead. Our proof is a descendant of the proof in [**3**].

PROPOSITION 3.4. *Let f be a functional in the norming set of the space $T_M[\mathcal{S}_n, \frac{1}{2^n}]$ and $(f_\alpha)_{\alpha\in\mathcal{T}}$ be an analysis of f. Then there exists a functional g in the norming set of the space $Y = T[\mathcal{S}_n[\mathcal{A}_2], \frac{1}{2^n}]$ and an analysis $(g_a)_{\alpha\in\mathcal{T}_1}$ of g, so that*

$$\begin{gathered}\textit{the trees } \mathcal{T} \textit{ and } \mathcal{T}_1 \textit{ have the same terminal nodes,}\\ \textit{and the level of each terminal node is the same in both trees } \mathcal{T} \textit{ and } \mathcal{T}_1.\end{gathered} \tag{3.3}$$

It follows that $\|x\|_n^M \leq \|x\|_Y$ for every $x \in c_{00}$.

Proof. It is clear from (3.3) that the functional g has the same support as f and gives the same estimation. This gives us also that $\|x\|_n^M \leq \|x\|_Y$ for every $x \in c_{00}$..

We shall transform $\mathcal{T}$ in several steps in order to get the functional g. Actually in every step we construct a part of its analysis, starting from the level $\mathrm{h}(\mathcal{T})$.

CLAIM 1. *We may transform the analysis* $(f_\alpha)_{\alpha\in\mathcal{T}}$ *to satisfy the following:*

$$\text{If } \alpha, \beta \text{ are nodes on the same level with } \operatorname{minsupp} f_\alpha < \operatorname{minsupp} f_\beta \text{ and } \gamma \in S_\alpha,\ \delta \in S_\beta \text{ then } \operatorname{minsupp} f_\gamma < \operatorname{minsupp} f_\delta. \tag{3.4}$$

In particular the nodes in every level of $\mathcal{T}$ *are ordered following the natural order of the minimum of their support.*

Proof of Claim 1. To achieve this property first note that the nodes in level j form an $\mathcal{S}_{nj}$-allowable set, and therefore $\{\operatorname{minsupp} f_\alpha : |\alpha| = j\} \in \mathcal{S}_{nj}^M$. From Lemma 2.1 we have $S_k^M = S_k$ for every $k \in \mathbb{N}$, and especially the Claim holds. Working from the level $\mathrm{h}(\mathcal{T})$ and moving toward the root we rearrange in every step the (new) nodes of every level j, so that $\{\operatorname{minsupp} f_\alpha : |\alpha| = j\} \in \mathcal{S}_{nj}$. This gives us also that the nodes in every level of $\mathcal{T}$ are ordered following the natural order of the minimum of their support.

For any such transformation we obtain new nodes which are also denoted by f_α. Lemma 2.1 and the Claim ensure that the new nodes remain functionals in the norming set of the space $T_M[\mathcal{S}_n, \frac{1}{2^n}]$.

Note also that if we have arranged the nodes in the $j-$level to be an $\mathcal{S}_{nj}$-admissible set, any such transformation in the $(j-1)$-level does not have any affect in the support of nodes in the $j-$level.

Indeed, in the level $\mathrm{h}(\mathcal{T})$ we just rearrange the terminal nodes in successive order with respect to their support, while in any level $< \mathrm{h}(\mathcal{T})$ any such transformation takes place either between terminal nodes or between terminal nodes and nodes which have successors.

No such operation changes the estimation of the functional, since the terminal nodes do not change level and hence the new functional gives the same estimation. $\square$

After we complete these transformations, it is easy to see that the nodes which prevent $(f_\alpha)_{\alpha\in\mathcal{T}}$ to be an admissible functional are the terminal nodes. To proceed we give the following definitions.

DEFINITION 3.5. A node f_β is to the *lower left* of node f_α, if $|\beta| \geq |\alpha|$, and if γ is $\beta's$ unique predecessor in the level of α, then $\operatorname{minsupp} f_\gamma < \operatorname{minsupp} f_\alpha$. A terminal node $f_\alpha = \pm e^*_{m_\alpha}$ is said to be a *good terminal node* if for all nodes f_β to the lower left of f_α, we have that $\operatorname{minsupp} f_\beta < \operatorname{minsupp} f_\alpha$. If the terminal node is not good, we will say f_α is bad.

In the sequel when we consider terminal nodes $f_\alpha = \pm e^*_{m_\alpha}$ and examine the property of being good or bad terminal node, in many cases we shall omit the sign of $e^*_{m_\alpha}$ since this property is independent of the sign.

LEMMA 3.6. *If* $\mathcal{T}$ *has only good terminal nodes, then* $\mathcal{T}$ *is admissible.*

Proof of Lemma. Let α be a node in $\mathcal{T}$ and $\beta, \gamma \in S_\alpha$ so that $\operatorname{minsupp} f_\beta < \operatorname{minsupp} f_\gamma$. Let $\lambda_\beta = \operatorname{maxsupp} f_\beta$ and $m_\gamma = \operatorname{minsupp} f_\gamma$. It follows that $e^*_{\lambda_\beta}$ and $e^*_{m_\gamma}$ are terminal nodes. We need to show that $e^*_{\lambda_\beta} < e^*_{m_\gamma}$ which is equivalent to $\operatorname{maxsupp} f_\beta < \operatorname{minsupp} f_\gamma$. If $|e^*_{\lambda_\beta}| \geq |e^*_{m_\gamma}|$ it follows from (3.4) that $e^*_{\lambda_\beta}$ is to the lower left of $e^*_{m_\gamma}$.

Indeed, let $\{f_\beta = f_{\beta_1} \prec f_{\beta_2} \prec \cdots \prec f_{\beta_m} = e^*_{\lambda_\beta}\}$ be the predecessors of $e^*_{\lambda_\beta}$ until the level of f_β, and $\{f_\gamma = f_{\gamma_1} \prec f_{\gamma_2} \prec \cdots \prec f_{\gamma_k} = e^*_{m_\gamma}\}$ be the predecessors of $e^*_{m_\gamma}$ until the level of f_γ. Since $|e^*_{\lambda_\beta}| \geq |\, e^*_{m_\gamma}|$, it follows that $k \leq m$. Inductively, from (3.4), we get that

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minsupp $f_{\beta_i} <$ minsupp f_{γ_i} for every $i \leq k$. Since $e^*_{\lambda_\beta}$ is a successor of f_{β_k} we get that $e^*_{\lambda_\beta}$ is to the lower left of $e^*_{m_\gamma}$ and therefore $e^*_{\lambda_\beta} < e^*_{m_\gamma}$, since $e^*_{m_\gamma}$ is a good terminal.

If $|e^*_{\lambda_\beta}| < |e^*_{m_\gamma}|$, let f_δ be the predecessor node of $e^*_{m_\gamma}$ in the level of $e^*_{\lambda_\beta}$. Let $m_\delta =$ minsupp f_δ. From the definition of the analysis it holds that $e^*_{m_\delta} \leq e^*_{m_\gamma}$, and from property (3.4), as in the previous case, it follows easily that $e^*_{\lambda_\beta} < e^*_{m_\delta}$ and hence $e^*_{\lambda_\beta} < e^*_{m_\gamma}$. □

Our aim is to transform $\mathcal{T}$ in such a way that it contains only good terminal nodes, but losing the admissibility in the way that every node to have at most $\mathcal{S}_n[\mathcal{A}_2]$-admissible immediate successors, and obtaining the same estimation.

Inductive hypothesis. Assume that we have transformed $\mathcal{T}$ in such way so that for the levels $\geq \mathrm{h}(\mathcal{T}) - i$, $i \geq 0$, the following holds:

(1) The terminal nodes in the levels $\geq \mathrm{h}(\mathcal{T}) - i$ are good terminal nodes.

(2) Every node f_α in the levels $\geq \mathrm{h}(\mathcal{T}) - i$ belongs to the norming set of the space Y.

(3) For every node f_α in the levels $\geq \mathrm{h}(\mathcal{T}) - i - 1$, the set $\{f_\beta : \beta \in S_\alpha\}$ is $\mathcal{S}_n[\mathcal{A}_2]$–admissible if $i > 0$, and $\mathcal{S}_n$-admissible if $i = 0$.

(4) For every node f_α in the levels $< \mathrm{h}(\mathcal{T}) - i - 1$, the set $\{f_\beta : \beta \in S_\alpha\}$ is $\mathcal{S}_n$ allowable.

Properties (2) and (3) give us also that every node in level $\mathrm{h}(\mathcal{T}) - i - 1$ is also in the norming set of the space Y.

For $i = 0$ properties (1) and (3) are consequences of (3.4), while property (2) follows from the definitions of the norming sets.

Assume that the inductive hypothesis holds for some $i \geq 0$. We shall transform the tree so that all the terminal nodes in the level $\mathrm{h}(\mathcal{T}) - i - 1$ become good terminal nodes. Let $f_\alpha = \pm e^*_{m_\alpha}$ be a bad terminal node in the level $\mathrm{h}(\mathcal{T}) - i - 1$, such that minsupp $f_\alpha \leq$ minsupp f_γ for all the bad terminals nodes f_γ in the level $\mathrm{h}(\mathcal{T}) - i - 1$.

Then we have that all the terminal nodes f_β with $|\beta| = |\alpha|$ and minsupp $f_\beta <$ minsupp f_α are good. Also from the inductive hypothesis we have that for all the terminal nodes f_β, with $|\beta| > |\alpha|$ are good.

Let f_β be the node on the level of α with largest minsupp $f_\beta <$ minsupp f_α. Such a node exists otherwise there is nothing to the lower left of f_α and f_α would be a good terminal node. Also $S_\beta \neq \emptyset$, otherwise f_β would be a good terminal node and anything to the lower left of f_α would also be to the lower left of f_β, which would imply that f_α is also a good terminal node.

Let $m_\beta =$ minsupp f_β. Since $\pm e^*_{m_\beta}$ is a good terminal node it follows that for every node f_γ to the lower left of f_β it holds that max supp$f_\gamma <$ minsupp f_β.

Indeed let f_γ be a node to the lower left of f_β. If $|\gamma| = |\beta|$ and f_γ is terminal node, it follows readily that maxsupp $f_\gamma =$ minsupp $f_\gamma <$ minsupp f_β. If $|\gamma| \geq |\beta|$ and f_γ has successors or $|\gamma| > |\beta|$ and f_γ is a terminal node, then from the inductive hypothesis we have that $\pm e^*_{\lambda_\gamma}$, where $\lambda_\gamma =$ maxsupp f_γ, is a good terminal node. Arguing as in Lemma 3.6, we get that maxsupp $f_\gamma <$ minsupp f_β.

From this observation and (3.4) we get that any node that causes $f_\alpha = \pm e^*_{m_\alpha}$ to be a bad terminal node is successor of f_β.

Let E^1_β be the interval [minsupp f_β, m_α) of $\mathbb{N}$ and E^2_β be the interval $(m_\alpha$, maxsupp $f_\beta]$. We set $f^1_\beta = E^1_\beta f_\beta$ and $f^2_\beta = E^2_\beta f_\beta$. By the inductive hypothesis f_β belongs to the norming set of the space Y, and therefore f^1_β, f^2_β belong also to the norming set of Y (see comment after Definition 2.2). Since the intervals E^1_β, E^2_β is a

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partition of the support of f_β, we have that $f_\beta = f_\beta^1 + f_\beta^2$ and moreover by the choice of these intervals it holds that $f_\beta^1 < f_\alpha = e^*_{m_\alpha} < f_\beta^2$. We replace the node f_β and its successors by f_β^1 and the successors of f_β^1, i.e. we take the intersection of the successors of f_β with the interval E_β^1.

Note that this replacement of f_β by the node f_β^1 changes automatically the immediate predecessor f_δ of f_β to f_δ^1. It follows readily that minsupp $f_\delta =$ minsupp f_δ^1 and hence this replacement changes neither the number nor the admissibility of the nodes of the previous levels.

We have now to add f_β^2 and its successors, which are the intersection of the successors of f_β with the interval E_β^2, as nodes of the tree, and this in combination with the replacement of f_β by f_β^1 will give us the new tree. Let

$$\mathcal{A} = \{\gamma \in \mathcal{T} : |\gamma| = |\alpha| \text{ and } \operatorname{minsupp} f_\alpha \leq \operatorname{minsupp} f_\gamma\}$$

and enumerate the elements of $\mathcal{A}$ as $\alpha_1 = \alpha, \alpha_2, \dots$ so that the sequence $(\operatorname{minsupp} f_{\alpha_i})_{i\geq 1}$ is increasing.

For the nodes in $\mathcal{A}$ which have successors the following key property holds:

$$\text{If } \alpha_i \in \mathcal{A} \text{ and } f_{\alpha_i} \text{ has successors then } \operatorname{maxsupp} f_\beta^2 = \operatorname{maxsupp} f_\beta < \operatorname{minsupp} f_{\alpha_i} \quad (3.5)$$

Indeed, let $\alpha_i \in \mathcal{A}$ be such that f_{α_i} has successors. Then if $m_{\alpha_i} =$ minsupp f_{α_i} it holds that $e^*_{m_{\alpha_i}}$ is a good terminal node, since $|f_\alpha| < |e^*_{m_{\alpha_i}}|$. Also $e^*_{\lambda_\beta}$, where $\lambda_\beta =$ maxsupp f_β, is a terminal node. Arguing as in the proof of Lemma 3.6 we get that either $e^*_{\lambda_\beta}$ is to the lower left of $e^*_{m_{\alpha_i}}$ and hence $e^*_{\lambda_\beta} < e^*_{m_{\alpha_i}}$ or $e^*_{\lambda_\beta} <$ minsupp f_δ, where f_δ is the predecessor of $e^*_{m_{\alpha_i}}$ in the level of $e^*_{\lambda_\beta}$. Thus in any case we have that $e^*_{\lambda_\beta} < e^*_{m_{\alpha_i}}$ or equivalently maxsupp $f_\beta^2 =$ maxsupp $f_\beta <$ minsupp f_{α_i}.

Let

$$j = \max\{i : \operatorname{minsupp} f_{\alpha_i} < \operatorname{minsupp} f_\beta^2\}. \quad (3.6)$$

From (3.5) it follows that f_{α_i}, $i \leq j$, are terminal nodes. In order to preserve (3.4), f_β^2 must be added between f_{α_j} and $f_{\alpha_{j+1}}$.

We add f_β^2 to the immediate successors of the predecessor node f_δ of f_{α_j} and the successors of f_β^2 are the intersection of the successors of f_β with the interval E_β^2. (3.7)

Note that since we add f_β^2 after the terminal node f_{α_j}, and as an immediate successor of the predecessor node f_δ of the terminal node, any such operation changes neither the number nor the admissibility of the nodes of the previous level. This gives us that property (4) of the inductive hypothesis holds. This completes the inductive step.

Let $\mathcal{T}_1$ be the resulting tree. Every terminal node in $\mathcal{T}$ is on the same level in $\mathcal{T}_1$. The condition (3.4) remains true in $\mathcal{T}_1$ by construction. We have to show now that any such transformation does not change a good terminal node in the levels $\geq \mathrm{h}(\mathcal{T}) - i$ into a bad terminal node.

Claim 2. *If f_γ is a good terminal node in $\mathcal{T}$ with $|\gamma| > |\alpha|$ or $|\gamma| = |\alpha|$ and* minsupp $f_\gamma <$ minsupp f_α, *then f_γ remains good terminal node in $\mathcal{T}_1$.*

Proof of Claim 2. Let f_γ be a good terminal node in $\mathcal{T}$ with $|\gamma| > |\alpha|$ or $|\gamma| = |\alpha|$ and $\operatorname{minsupp} f_\gamma < \operatorname{minsupp} f_\alpha$. Let $f_\beta \in \mathcal{T}_1$ be a node which is to the lower left of f_γ. Then there is a node f_δ in $\mathcal{T}$ to the lower left of f_γ such that either $f_\beta = f_\delta$ or f_β is the projection of f_δ onto E^1_β or E^2_β. If $|\gamma| = |\alpha$ then it follows from the choice of α that $\operatorname{maxsupp} f_\delta < \operatorname{minsupp} f_\gamma$, and if $|\gamma| > |\alpha|$ then it follows from the inductive hypothesis. Hence $\operatorname{maxsupp} f_\beta < \operatorname{minsupp} f_\gamma$, and f_γ is a good terminal node in $\mathcal{T}_1$. □

We repeat the above steps, changing a bad terminal node into a good terminal one, for all the bad nodes in the level $\mathrm{h}(\mathcal{T}) - i - 1$. Since the nodes in the level $\mathrm{h}(\mathcal{T}) - i - 1$ is an $\mathcal{S}_{n(\mathrm{h}(\mathcal{T})-i-1)}$ allowable set and hence finite, a finite number of such transformations will turn every bad terminal node into a good.

In the final step we check that the tree $\mathcal{T}_1$ satisfies properties (1)–(4) of the inductive hypothesis.

Claim 3. *The tree $\mathcal{T}_1$ satisfies properties* (1)–(3) *of the inductive hypothesis, in every level* $\geq h(\mathcal{T}) - i - 1$ *and property* (4) *in every level*$< h(\mathcal{T}) - i - 2$

Proof of Claim 3. Property (1) follows from Claim 2. Property (2) is a consequence of the inductive hypothesis and the properties of the norming set (see remark after Definition 2.2).

From the construction it follows that in the level $\mathrm{h}(\mathcal{T}) - i - 1$ of $\mathcal{T}$ we add at most one node after each of its terminal nodes. Therefore after we complete the transformations in the level $\mathrm{h}(\mathcal{T}) - i - 1$, and f_δ is a node in the tree $\mathcal{T}$ in the level $\mathrm{h}(\mathcal{T}) - i - 2$, with $\mathcal{S}_n$-allowable immediate successors in the level $\mathrm{h}(\mathcal{T}) - i - 1$, then the immediate successors of (the new) f_δ in the new tree is at most $\mathcal{S}_n[\mathcal{A}_2]$-admissible set. The admissibility of the successors, that is they have successive support, follows from proof of Lemma 3.6. Therefore property (3) also holds. Also as we have already observe in the proof of the inductive step, any such transformation changes neither the number nor the admissibility of the nodes in the level $\mathrm{h}(\mathcal{T}) - i - 2$ and hence property (4) also holds. □

A finite number of such transformations will give us a new tree $\mathcal{T}_1$ and a functional $(g_\alpha)_{\alpha \in \mathcal{T}_1}$ in the norming set of Y, which is admissible and satisfies (3.3). This completes the proof of Proposition 3.4. □

Let us observe that the proof of Proposition 3.4 is independent of the number $\frac{1}{2^n}$, that is, it holds also for the spaces $T_M[\mathcal{S}_n, \theta]$ and $T[\mathcal{S}_n[\mathcal{A}_2], \theta]$ for every $0 < \theta < 1$. Since Proposition 3.2 is also independent of the number $\frac{1}{2^n}$, Remark 3.3, combining Propositions 3.2 and 3.4 we have the following theorem.

Theorem 3.7. *Let $n \in \mathbb{N}$ and $0 < \theta < 1$. Then the spaces $T_M[\mathcal{S}_n, \theta]$ and $T[\mathcal{S}_n, \theta]$ are* 3-*isomorphic.*

As we have mentioned, the special case of Theorem 3.7 when $n = 1$ has been proved in **[3]** and **[4]**. The isomorphic constant they provide is a function of θ.

Let us now turn to the distortion problem. E. Odell and N. Tomczak-Jaegermann **[10]**, have introduced for every $n \in \mathbb{N}$, the following equivalent norms on Tsirelson's space:

$$\|x\|^n_j = \max\left\{\|x\|_\infty, \frac{1}{2^j} \sup\left\{\sum_{i\in F} \|E_i x\|_n : \{E_i : i \in F\} \text{ is } \mathcal{S}_j\text{-admissible}\right\}\right\}$$

for $j = 0, 1, \ldots, n-1$. The main result of E. Odell and N. Tomczak-Jaegermann in **[10]**, is that there exists a constant K so that the norms $\|\cdot\|_j^n$ do not $K-$ distort any subspace of T. More precisely they have proved the following result.

THEOREM. (*E. Odell and N. Tomczak-Jaegermann*) *There exists a constant K, so that for all $Y \prec T$ and $n \in \mathbb{N}$ there exist $Z \prec Y$ and $d > 0$ such that for all $0 \leq j < n$ and $z \in S_Z$,*

$$d \leq \|z\|_j^n \leq Kd\,.$$

Let us define the modified version, $\|\cdot\|_{j,n}^M$ of the previous norms:

$$\|x\|_{j,M}^n = \max\left\{\|x\|_\infty, \frac{1}{2^j}\sup\left\{\sum_{i\in F}\|E_i x\|_n^M : \{E_i : i \in F\} \text{ is } \mathcal{S}_j\text{-allowable}\right\}\right\}$$

for $j = 0, 1, \ldots, n-1$. Combining the theorem of E. Odell and N. Tomczak-Jaegermann and the 3-equivalence of the norms $\|\cdot\|_n$ and $\|\cdot\|_n^M$, $n \in \mathbb{N}$ we have the following theorem.

THEOREM 3.8. *There exist a constant $K > 1$ so that for all $Y \prec T$ and $n \in \mathbb{N}$ there exists $Z \prec Y$ and $d > 0$ so that for every $z \in S_Z$,*

$$d \leq \|z\|_{j,M}^n \leq Kd.$$

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