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ON A THEOREM OF LELONG

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Let $\Gamma_n = \{z = (z_1, z_2, \ldots, z_n): z_i \in \mathbb{C}$ and $\mathrm{Re}(z_i) > 0, \forall i\}$. For a multisequence M_j, $j = (j_1, j_2, \ldots, j_n)$ and $0 < M_{(j)} \leq \infty$, let $z^j = z_1^{j_1}, z_2^{j_2}, \ldots, z_n^{j_n}$, $|j| = \sum_{k=1}^n j_k$, $q(r) = \sup_{|j|}\{r^{|j|}/M_j\}$ and $\mathcal{T} = \int_0^\infty (\log q(r)/1 + r^2)\, dr$.

In [1], Lelong proved the following theorem.

THEOREM. *There exists a function* $f \not\equiv 0$, *holomorphic in* Γ_n *s.t.* $|f(z)| \leq M_j/|z^j|$, $\forall_j$ *if and only if* $\mathcal{T}$ *converges.*

For f defined in $\mathbb{C}^n$ and $\|j\| = \sum_{k=1}^n j_k^2$ we define $\lambda_f = \sup_j\{\sup_z(|z^j f(z)|^{1/\|j\|})\}$ and prove the following theorem.

THEOREM. *For any* $\beta > 1$, *there exists* $f \not\equiv 0$, *holomorphic in* Γ_n *s.t.* $1 < \lambda_f \leq \beta$.

Let $C\{M_j\}$ denote the class of infinitely differentiable functions f on R^n, $n \geq 1$, s.t.

$$\left|\frac{\partial^{|j|} f}{\partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \cdots \partial_{x_n}^{j_n}}\right| \leq \alpha_f B_f^{|j|} M_j$$

where α_f and B_f are positive constants depending only on f.

It is further proved in [1] that the convergence of $\mathcal{T}$ is equivalent to the existence in $C\{M_j\}$ of a function with compact support (in which case $C\{M_j\}$ is called non quasi-analytic). For $n = 1$, the later condition is equivalent to the existence of a function g vanishing with all its derivatives at a **point** x_0 (condition which is itself equivalent to the convergence of $\sum_{j=1}^\infty M_{j-1}/M_j$ by the Denjoy–Carleman Theorem [3]).

For $n > 1$, Lelong [1] has shown that the same is not true and Ronkin [2] showed that the necessary and sufficient condition for the existence of such a function g is that each of the classes $C\{M_{j,0,0,\ldots,0}\}$, $C\{M_{0,j,0,\ldots,0}\} \cdots$ and $C\{M_{0,0,\ldots,0,j}\}$ is non quasi-analytic (in one variable).

Proof of the Theorem. We consider a class $C\{N_k\}$ of functions in one variable and prove first that there exists $f \in \mathrm{Hol}(\Gamma_n)$, $f \not\equiv 0$, s.t.

$$|f(z)| \leq N_{j_1} N_{j_2} \cdots N_{j_n}/|z^j| \qquad \forall_j = (j_1, j_2, \ldots, j_n)$$

if and only if the class $C\{N_k\}$ is non quasi-analytic.

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We set $M_j = N_{j_1}, N_{j_2}, \ldots, N_{j_n}$.

Since the class $C\{N_k\}$ is non quasi-analytic, there exists $h \in C\{N_k\}$ with compact support. $\psi(x_1, x_2, \ldots, x_n) = h(x_1)\, h(x_2) \cdots h(x_n)$ is hence a function with compact support in $C\{M_j\}$ and implies that $\mathcal{T}$ converges. The existence of the above function f follows from Lelong's Theorem. Conversely if T converges, the class $C\{M_j\}$ is non quasi-analytic and implies that $C\{N_k\}$ is so too, by Ronkin's results.

Let $\beta \geq 1$. We set $N_k = \beta^{k^2}$ $k = 0, 1, 2, \ldots$.

By the Denjoy–Carleman Theorem, $C\{N_k\}$ is then non quasi-analytic for $\beta > 1$ and quasi-analytic for $\beta = 1$. Hence for $\beta > 1$, $\exists f \in \mathrm{Hol}(\Gamma_n)$, $f \not\equiv 0$

$$\text{s.t.} \quad \forall_j, |z^j f(z)| \leq N_{j_1} \cdots N_{j_n} = \beta^{\|j\|}.$$

So, $\lambda_f \leq \beta$.

For $\beta = 1$, the quasi-analyticity of $C\{N_k\}$ implies that for $f \in \mathrm{Hol}(\Gamma_n)$, $f \not\equiv 0$, there exist j_0 and z_0 s.t. $|z_0^{j_0} f(z_0)| > \beta^{\|j_0\|}$ and so $\lambda_f > 1$. This completes the proof of the Theorem.

References

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