Can. J. Math., Vol. XXXIII, No. 3, 1981, pp. 685-700

COMPACT PERTURBATIONS OF REFLEXIVE ALGEBRAS

KENNETH R. DAVIDSON

1. Introduction. In this paper we study lattice properties of operator algebras which are invariant under compact perturbations. It is easy to see that if $\mathscr{A}$ and $\mathscr{B}$ are two operator algebras with $\mathscr{A}$ contained in $\mathscr{B}$, then the reverse inclusion holds for their lattices of invariant subspaces. We will show that in certain cases, the assumption that $\mathscr{A}$ is contained in $\mathscr{B} + \mathscr{K}(\mathscr{H})$, where $\mathscr{K}(\mathscr{H})$ is the ideal of compact operators, implies that the lattice of $\mathscr{B}$ is "approximately" contained in the lattice of $\mathscr{A}$. In particular, suppose $\mathscr{A}$ and $\mathscr{B}$ are reflexive and have commutative subspace lattices containing "enough" finite dimensional elements. We show (Corollary 2.8) that if $\mathscr{A}$ is unitarily equivalent to a subalgebra of $\mathscr{B} + \mathscr{K}(\mathscr{H})$, then there is a unitary operator which carries all "sufficiently large" subspaces in lat $\mathscr{B}$ into lat $\mathscr{A}$.

Reflexive algebras with commutative subspace lattices were studied in [**1**]. Since then, there has been much interest in this family of non self-adjoint algebras. Related questions have been studied in [**8**] in the context of quasitriangular algebras. It is shown there that if two quasitriangular algebras are similar, then the corresponding lattices are unitarily equivalent for "sufficiently large" lattice elements. We show (Corollary 2.10) that if a reflexive algebra $\mathscr{A}$ is similar to a subalgebra of a quasitriangular algebra $\mathscr{Q}\mathscr{T}$ and has a commutative lattice, then lat $\mathscr{A}$ contains a chain unitarily equivalent to an implementing lattice of $\mathscr{Q}\mathscr{T}$. We also show that if $\mathscr{A}$ is a reflexive algebra with commutative lattice such that $\mathscr{A} + \mathscr{K}(\mathscr{H})$ contains a quasitriangular algebra, then $\mathscr{A} + \mathscr{K}(\mathscr{H})$ is quasitriangular itself (Theorem 5.3).

When a lattice $\mathscr{M}$ is not commutative, the problems are much more complicated, and our results are not as definitive. We show (Theorem 4.2) that if alg$\mathscr{M} + \mathscr{K}(\mathscr{H})$ contains alg$\mathscr{L}$ for some commutative lattice $\mathscr{L}$, then "sufficiently large" projections in $\mathscr{M}$ are lattice isomorphic to a sublattice of $\mathscr{L}$ and are asymptotically close to $\mathscr{L}$ in norm. However, we cannot determine whether these lattices are similar. In Section 5, we examine whether alg $\mathscr{M} + \mathscr{K}(\mathscr{H})$ need be quasitriangular if it contains a quasitriangular algebra. We introduce a large class of lattices for which the theorem is true. This result is of interest because the answer given is surprising and it shows that the general question is subtle.

Received January 2, 1980.

https://doi.org/10.4153/CJM-1981-054-0 Published online by Cambridge University Press

If $\mathscr{A}$ is any algebra of operators on a Hilbert space $\mathscr{H}$, lat $\mathscr{A}$ will denote its lattice of invariant subspaces. Such lattices are always strongly closed, and we shall reserve the term lattice for strongly closed subspace lattices. If $\mathscr{L}$ is a lattice, then alg $\mathscr{L}$ is the algebra of all operators leaving the subspaces of $\mathscr{L}$ invariant. The lattice operations of span and intersection will be denoted by $\vee$ and $\wedge$ respectively. We shall often identify $\mathscr{L}$ with the set of self adjoint projections onto the elements of $\mathscr{L}$, and in this setting

$$\text{alg}\,\mathscr{L} = \{A : P^\perp A P = 0 \text{ for all } P \text{ in } \mathscr{L}\},$$

where $P^\perp = I - P$. We also set $Q\,\text{alg}\,\mathscr{L} = \text{alg}\,\mathscr{L} + \mathscr{K}(\mathscr{H})$. All Hilbert spaces in this paper are separable.

If $\mathscr{P} = \{P_n\}$ is an increasing sequence of finite rank projections which span $\mathscr{H}$, then $\mathscr{T}(\mathscr{P}) = \text{alg}\,\mathscr{P}$ and $\mathscr{Q}\mathscr{T}(\mathscr{P}) = Q\,\text{alg}\,\mathscr{P}$ are the triangular and quasitriangular algebras associated with $\mathscr{P}$. In [**5**], Halmos defines an operator to be quasitriangular with respect to $\mathscr{P}$ if

$$\lim_{n\to\infty} \|P_n^\perp T P_n\| = 0.$$

In [**1**], Arveson shows that this is equivalent to being in $\mathscr{Q}\mathscr{T}(\mathscr{P})$. It is clear from this characterization that if $\mathscr{R} = \{R_n\}$ is another such sequence satisfying

$$\lim_{n\to\infty} \|P_n - R_n\| = 0,$$

then $\mathscr{Q}\mathscr{T}(\mathscr{R}) = \mathscr{Q}\mathscr{T}(\mathscr{P})$. It also follows that $\mathscr{Q}\mathscr{T}(\mathscr{P})$ is closed. It is not known whether $Q\,\text{alg}\,\mathscr{L}$ is closed in general, but it is closed if $\mathscr{L}$ is generated by its finite rank elements. The algebra $\mathscr{A} + \mathscr{K}(\mathscr{H})$ may fail to be closed for general operator algebras [**3**].

An operator algebra $\mathscr{A}$ is said to be *reflexive* if it is equal to alg lat $\mathscr{A}$. We will restrict our attention to these algebras because there can be no good results for algebras which are too small. For example, if U is the bilateral shift with respect to a basis $\{e_n\}$ and $V = UP$ where P is the projection orthogonal to $\{e_0\}$, then V is a compact perturbation of U. Also if $\mathscr{U}$ and $\mathscr{V}$ are the norm closed algebras generated by U and V, it is easy to check that $\mathscr{U} + \mathscr{K}(\mathscr{H}) = \mathscr{V} + \mathscr{K}(\mathscr{H})$. However, V is unitarily equivalent to $S^* \oplus S$ where S is the unilateral shift. The invariant subspaces of these two operators are well known, and it is clear that they are quite dissimilar. However, the weakly closed algebras generated by U and V are reflexive and are no longer compact perturbations of one another.

2. Almost finite lattices.

Definition 2.1. We will say that a lattice $\mathscr{L}$ is AF if every element of $\mathscr{L}$ is the union of finite rank projections in $\mathscr{L}$.

We note that a commutative lattice is AF if and only if the finite dimensional subspaces of $\mathscr{L}$ span $\mathscr{H}$. In this case, the minimal projections in the abelian von Neumann algebra $\mathscr{L}''$ are finite rank and span $\mathscr{H}$. There is a natural partial order on these minimal projections induced by $\mathscr{L}$. Namely, $M_1 \prec_{\mathscr{L}} M_2$ if and only if $M_2 \leqq L$ implies $M_1 \leqq L$ for L in $\mathscr{L}$, or equivalently, if and only if

$$M_1\mathscr{B}(\mathscr{H})M_2 \subseteq \operatorname{alg}\mathscr{L}.$$

We will write $M_1 \prec M_2$ if $\mathscr{L}$ is unambiguous.

LEMMA 2.2 *Let $\mathscr{L}$ be a commutative* AF *lattice and suppose that Q* alg $\mathscr{L}$ *is contained in a quasitriangular algebra $\mathscr{Q}\mathscr{T}(\mathscr{P})$. Then*

$$\lim_{n\to\infty} d(P_n, \mathscr{L}) = 0.$$

Proof. Suppose that for some $\epsilon > 0$ and a subset $\mathscr{Q} = \{Q_n\}$ of $\mathscr{P}$, we have $d(Q, \mathscr{L}) > \epsilon$ for all Q in $\mathscr{Q}$. Note that

$$\operatorname{alg}\mathscr{L} \subseteq \mathscr{Q}\mathscr{T}(\mathscr{P}) \subseteq \mathscr{Q}\mathscr{T}(\mathscr{Q}).$$

Set $\delta = \epsilon/10$ and $L_0 = 0$, and let $\mathscr{L}_1$ be any chain in $\mathscr{L}$ of finite rank projections with the identity operator as its supremum. Inductively we will choose increasing sequences Q_n in $\mathscr{Q}$ and L_n in $\mathscr{L}_1$ such that

(1) $\quad \|Q_n^{\perp}L_{n-1}\| < \delta \quad$ and $\quad \|Q_nL_n^{\perp}\| < \delta.$

The Q_i in $\mathscr{Q}$ tend to I in the strong operator topology, so $s - \lim Q_i^{\perp} = 0$. If we have L_{n-1}, then since it is compact,

$$\lim_{i\to\infty} \|Q_i^{\perp}L_{n-1}\| = 0.$$

So we can choose $Q_n > Q_{n-1}$ satisfying (1). Similarly, if we have Q_n, we can choose L_n satisfying (1).

We will construct partial isometries T_n on $(L_n - L_{n-1})\mathscr{H}$ which belong to alg $\mathscr{L}$ and satisfy $\|Q_n^{\perp}T_nQ_n\| \geqq 3\delta$. Assuming this has been done, let $T = \oplus \sum T_n$. Then T is a partial isometry in alg $\mathscr{L}$. Also

$$T = L_{n-1}T + (L_n - L_{n-1})T + L_n^{\perp}T = L_{n-1}T + T_n + TL_n^{\perp}.$$

Hence

$$\|Q_n^{\perp}TQ_n\| \geqq \|Q_n^{\perp}T_nQ_n\| - \|Q_n^{\perp}L_{n-1}\| - \|L_n^{\perp}Q_n\| > \delta.$$

It follows that T is not in $\mathscr{Q}\mathscr{T}(\mathscr{Q})$, contradicting the hypothesis that Q alg $\mathscr{L} \subseteq \mathscr{Q}\mathscr{T}(\mathscr{Q})$.

Now fix n, and let $\mathscr{M} = \{M_i\}$ be the set of minimal projections in the discrete abelian von Neumann algebra $\mathscr{L}''$ such that $M_i \leqq L_n - L_{n-1}$. Let $\mathscr{A}$ be the set of M_i in $\mathscr{M}$ for which $\|Q_n^{\perp}M_i\| \geqq \sqrt{3\delta}$, and let $\mathscr{B}$ be the set for which $\|Q_nM_i\| \geqq \sqrt{3\delta}$. If there is a pair M_1 in $\mathscr{A}$ and M_2 in $\mathscr{B}$ with $M_1 \prec_{\mathscr{L}} M_2$, we can find a partial isometry T_n in $M_1\mathscr{B}(\mathscr{H})M_2$, and a fortiori in alg $\mathscr{L}$, such that

$$\|Q_n{}^\perp T_n Q_n\| = \|Q_n{}^\perp M_1 T_n M_2 Q_n\| = \|Q_n{}^\perp M_1\|\|M_2 Q_n\| \geqq 3\delta.$$

If there is no such pair, we set $M = \sum\{M_i : M_i \in \mathscr{B}\}$ and $N = L_{n-1} + M$. The least projection in $\mathscr{L}$ greater than N is clearly less than L_n, so consists of the span of N and those M_i in $\mathscr{M}$ which satisfy $M_i \prec M_j$ for some M_j in $\mathscr{B}$. By hypothesis, M_i must belong to $\mathscr{B}$, so N is in $\mathscr{L}$. Hence

$$\epsilon < d(Q_n, \mathscr{L}) \leqq \|Q_n - N\| = \|Q_n N^\perp - Q_n{}^\perp N\| \\ = \max\{\|Q_n N^\perp\|, \|Q_n{}^\perp N\|\}.$$

We will suppose that $\|Q_n{}^\perp N\| > \epsilon$ (the other case is similar). Since $Q_n{}^\perp M = Q_n{}^\perp N - Q_n{}^\perp L_{n-1}$, we get $\|Q_n{}^\perp M\| > \epsilon - \delta > \epsilon/2$.

The sets $\mathscr{A}$ and $\mathscr{B}$ are disjoint ($M_i \prec M_i$), so if M_i belongs to $\mathscr{B}$, $\|Q_n{}^\perp M_i\| < \sqrt{3\delta}$, and consequently $M_i Q_n{}^\perp M_i < 3\delta M_i$. So

$$M_i Q_n M_i > (1 - 3\delta) M_i,$$

and by adding over $\mathscr{B}$,

$$M Q_n M \geqq \sum M_i Q_n M_i > (1 - 3\delta) M.$$

Hence

$$\|Q_n{}^\perp M Q_n\|^2 = \|Q_n{}^\perp M Q_n M Q_n{}^\perp\| > (1 - 3\delta)\|Q_n{}^\perp M Q_n{}^\perp\| \\ > (1 - 3\delta)(\epsilon/2)^2 > (3\delta)^2.$$

We set $T_n = M$. Then T_n belongs to $\mathscr{L}''$ and thus to alg $\mathscr{L}$.

LEMMA 2.3. *Let $\mathscr{L}$ be a commutative* AF *lattice and let $\mathscr{M}$ be an* AF *lattice for which Q alg $\mathscr{L} \subseteq Q$ alg $\mathscr{M}$. Then for all $\epsilon > 0$, there is a finite rank M_ϵ in $\mathscr{M}$ so that for all M in $\mathscr{M}$, $M \geqq M_\epsilon$ implies $d(M, \mathscr{L}) < \epsilon$.*

Proof. If the lemma is false for some $\epsilon > 0$, we will construct a chain $\mathscr{P}$ in $\mathscr{M}$ such that $d(P, \mathscr{L}) > \epsilon$ for all P in $\mathscr{P}$, contradicting Lemma 2.2. We proceed by induction. Let K_i be a sequence of finite rank elements of $\mathscr{M}$ tending to I in the strong operator topology. If $P_1, \ldots, P_n$ have been defined, choose an M in $\mathscr{M}$ with $M \geqq P_n \vee K_n$ for which $d(M, \mathscr{L}) > \epsilon$. If $\{M_k\}$ is a sequence of finite rank projections $\leqq M$ converging to M in the strong operator topology, then so is $M_k' = M_k \vee P_n \vee K_n$. Because of the lower semi-continuity of the norm in this topology, we can find some k for which $d(M_k', \mathscr{L}) > \epsilon$. Set $P_{n+1} = M_k'$. Clearly $P_{n+1} > P_n$, and $P_{n+1} > K_n$ implies that P_n tends to I, so $\mathscr{P} = \{P_n\}$ is AF.

Definition 2.4. We will say that two AF lattices $\mathscr{L}$ and $\mathscr{M}$ are *asymptotic* if there is a lattice isomorphism φ from $\mathscr{L}$ to $\mathscr{M}$ such that $\lim \|\varphi(L) - L\| = 0$ in the sense that for every $\epsilon > 0$, there is a finite rank element L_0 in $\mathscr{L}$ so that every $L \geqq L_0$ in $\mathscr{L}$ satisfies $\|\varphi(L) - L\| < \epsilon$.

For commutative lattices, proximity implies a lattice isomorphism which is spacially implemented by a unitary operator.

LEMMA 2.5. *If* $\mathscr{L}$ *and* $\mathscr{M}$ *are commutative* AF *lattices such that* dist $(M, \mathscr{L}) < 1/8$ *for every* M *in* $\mathscr{M}$, *then there is a unitary operator* U *such that* $U\mathscr{M}U^{-1}$ *is a sublattice of* $\mathscr{L}$.

Proof. Since $\|L_1 - L_2\| = 1$ if L_1 and L_2 are distinct elements of $\mathscr{L}$, there is a unique element L_M of $\mathscr{L}$ satisfying $\|M - L_M\| < 1/8$. If M_1 and M_2 belong to $\mathscr{M}$, then

$$M_1 \wedge M_2 = M_1M_2 \quad \text{and} \quad M_1 \vee M_2 = M_1 + M_2 - M_1M_2.$$

So we compute that

$$\|L_{M_1}L_{M_2} - L_{M_1M_2}\| < 1 \quad \text{and} \quad \|L_{M_1} \vee L_{M_2} - L_{M_1 \vee M_2}\| < 1.$$

The first remark of the proof now implies that

$$L_{M_1}L_{M_2} = L_{M_1M_2} \quad \text{and} \quad L_{M_1} \vee L_{M_2} = L_{M_1 \vee M_2},$$

so $\varphi(M) = L_M$ is a lattice isomorphism. Furthermore, $\dim M = \dim L_M$ since $\|M - L_M\| < 1$. Consequently, there is a unitary operator U such that $UMU^{-1} = L_M$ for M in $\mathscr{M}$. This unitary is easily constructed by mapping the minimal (finite rank) projections in $\mathscr{M}''$ to the corresponding projections in $\varphi(\mathscr{M})''$.

It seems natural that $Q \operatorname{alg} \mathscr{L}$ should not depend on the behaviour of $\mathscr{L}$ restricted to any finite dimensional subspace. That is the content of the following lemma. This lemma does not hold for non-commutative lattices.

LEMMA 2.6. *If* $\mathscr{L}$ *is a commutative lattice and* L_0 *is a finite rank projection in* $\mathscr{L}'$, *then* $Q \operatorname{alg} \mathscr{L} = Q \operatorname{alg} (\mathscr{L} \vee L_0)$.

Proof. Let A be an element of $\operatorname{alg} \mathscr{L} \vee L_0$. Then since L_0 commutes with $\mathscr{L} \vee L_0$, $L_0^\perp A L_0^\perp$ is in $\operatorname{alg} \mathscr{L} \vee L_0$, and is a compact perturbation of A. If L belongs to $\mathscr{L}$,

$$L^\perp(L_0^\perp A L_0^\perp)L = (L \vee L_0)^\perp(L_0^\perp A L_0^\perp)(L \vee L_0) = 0.$$

So A belongs to $Q \operatorname{alg} \mathscr{L}$.

Conversely, if A belongs to $\operatorname{alg} \mathscr{L}$, then $L_0^\perp A L_0^\perp$ belongs to $\operatorname{alg} \mathscr{L} \vee L_0$.

Remark. If $\mathscr{L}$ is a commutative AF lattice, $Q \operatorname{alg} \mathscr{L}$ is closed. This follows from [**4**] which shows that $\mathscr{A} + \mathscr{K}(\mathscr{H})$ is closed for any norm-closed algebra such that $\mathscr{A} \cap \mathscr{K}(\mathscr{H})$ is weak* dense in $\mathscr{A}$. This condition is easily seen to hold for $\operatorname{alg} \mathscr{L}$, because if L_n are increasing finite rank projections tending strongly to the identity, then L_nAL_n tends weak* to A for all A in $\operatorname{alg} \mathscr{L}$.

THEOREM 2.7. *Let* $\mathscr{L}$ *and* $\mathscr{M}$ *be commutative* AF *lattices with* $Q \operatorname{alg} \mathscr{L} \subseteq Q \operatorname{alg} \mathscr{M}$. *Then there is a finite rank* M_0 *in* $\mathscr{M}$ *such that* $\mathscr{M} \vee M_0$ *is asymptotic and unitarily equivalent to a sublattice of* $\mathscr{L}$. *Further,*

$$Q \operatorname{alg} \mathscr{M} = Q \operatorname{alg} \mathscr{M} \vee M_0.$$

Proof. By Lemma 2.3, there is a finite rank M_0 in $\mathscr{M}$ such that $d(M, \mathscr{L}) < 1/8$ for $M \geqq M_0$ in $\mathscr{M}$. Lemma 2.5 implies that $\mathscr{M}_0$ is unitarily equivalent to a sublattice $\mathscr{L}_1$ of $\mathscr{L}$. Lemma 2.3 implies that $\mathscr{M}_0$ is asymptotic to $\mathscr{L}_1$. The last claim follows from Lemma 2.6.

COROLLARY 2.8. *If* $\mathscr{L}$ *and* M *are commutative* AF *lattices and* $Q \operatorname{alg} \mathscr{L}$ *is unitarily equivalent to a subalgebra of* $Q \operatorname{alg} \mathscr{M}$, *then there is a finite rank projection* M_0 *in* $\mathscr{M}$ *such that* $\mathscr{M} \vee M_0$ *is unitarily equivalent to a sublattice of* $\mathscr{L}$.

COROLLARY 2.9. *If* $\mathscr{L}$ *and* $\mathscr{M}$ *are commutative* AF *lattices and* $Q \operatorname{alg} \mathscr{L}$ *is unitarily equivalent to* $Q \operatorname{alg} \mathscr{M}$, *then there are finite rank projections* L_0 *in* $\mathscr{L}$ *and* M_0 *in* $\mathscr{M}$ *such that* $\mathscr{L} \vee L_0$ *is unitarily equivalent to* $\mathscr{M} \vee M_0$.

If we specialize to the case in which $\mathscr{M}$ is a chain, we need only assume similarity.

COROLLARY 2.10. *If* $\mathscr{L}$ *is a commutative* AF *lattice, then* $\operatorname{alg} \mathscr{L}$ *is similar to a subalgebra of* $\mathscr{Q}\mathscr{T}(\mathscr{P})$ *if and only if* $\mathscr{L}$ *contains a chain* $\{L_n : n \geqq N\}$ *such that* $\dim L_n = \dim P_n$ *for* $n \geqq N$.

Proof. An algebra similar to $\operatorname{alg} \mathscr{P}$ is unitarily equivalent to $\operatorname{alg} \mathscr{P}$ since $S(P_n \mathscr{H})S^{-1}$ are nested subspaces of dimension $\dim P_n$ (See [**1**]). The corollary now follows immediately from Theorem 2.7.

COROLLARY 2.11. $\mathscr{Q}\mathscr{T}(\mathscr{L})$ *is similar to* $\mathscr{Q}\mathscr{T}(\mathscr{P})$ *if and only if there is an* L_0 *in* $\mathscr{L}$ *for which* $\mathscr{L} \vee L_0 = \{L_n : n \geqq N\}$ *is a chain and* $\dim L_n = \dim P_n$ *for* $n \geqq N$.

The Corollary 2.11 for the case in which $\mathscr{L}$ is a priori a chain is proved in [**8**]. Corollary 2.9 is proved for complemented AF lattices in [**9**]. J. Plastiras has informed me that she had also independently proved Corollary 2.10 for $\mathscr{L}$ a chain. I would like to thank her for pointing out that the unitary U in Theorem 2.7 need not be a compact perturbation of the identity.

Unfortunately, we do not know if the converse of Theorem 2.7 holds. That is, if $\mathscr{L}$ and $\mathscr{M}$ are asymptotic, are $Q \operatorname{alg} \mathscr{L}$ and $Q \operatorname{alg} \mathscr{M}$ equal? When $\mathscr{M}$ is a chain, the converse does hold because of the characterization of quasitriangular algebras mentioned in the introduction. It is also true if $\mathscr{M}$ is complemented. In this case, [**7**] implies that

$$Q \operatorname{alg} \mathscr{M} = \{A : AM - MA \in \mathscr{K}(\mathscr{H}) \text{ for all } M \text{ in } \mathscr{M}\}.$$

If $\mathcal{L}$ is asymptotic to $\mathcal{M}$, it is readily verified that $L - \varphi(L)$ is compact for every L in $\mathcal{L}$ so $Q \operatorname{alg} \mathcal{L} = Q \operatorname{alg} \mathcal{M}$.

3. The AF **condition for commutative lattices.** The following theorem shows for commutative lattices that containment in $Q \operatorname{alg} \mathcal{M}$ for some AF lattice $\mathcal{M}$ essentially implies the AF condition. This theorem will enable us to reformulate most of the results of this paper, but for the sake of clarity, this will not be explicitly carried out.

THEOREM 3.1. *If $\mathcal{L}$ is a commutative lattice and $Q \operatorname{alg} \mathcal{L} \subseteq \mathcal{Q}\mathcal{T}(\mathcal{P})$, then there is a finite rank L_0 in $\mathcal{L}''$ such that $\mathcal{L} \vee L_0$ is* AF. *Further,*

$$Q \operatorname{alg} \mathcal{L} \vee L_0 = Q \operatorname{alg} \mathcal{L}.$$

Proof. $\mathcal{L}''$ is an abelian von Neumann algebra. Either $\mathcal{L}''$ is AF, or $\mathcal{L}''$ contains a projection M with no nonzero finite rank sub-projection in $\mathcal{L}''$. In the first case, Theorem 2.7 (applied to $\operatorname{alg} \mathcal{L}'' \subset \mathcal{Q}\mathcal{T}(\mathcal{P})$) implies that $\mathcal{P}$ is asymptotic to a sublattice $\mathcal{L}_1$ of $\mathcal{L}''$. Since $\mathcal{Q}\mathcal{T}(\mathcal{P}) = \mathcal{Q}\mathcal{T}(\mathcal{L}_1)$, we can assume that $\mathcal{P} \subseteq \mathcal{L}''$ after this change. Denote by $\prec_{\mathcal{L}}$ and $\prec_{\mathcal{P}}$ the partial orders on the minimal projections in $\mathcal{L}''$ induced by $\mathcal{L}$ and $\mathcal{P}$ respectively. Notice that $\mathcal{P} \subseteq \mathcal{L}$ if and only if $N \prec_{\mathcal{L}} M \Rightarrow N \prec_{\mathcal{P}} M$. We claim that, except for finitely many M_i in $\mathcal{L}''$, this relation holds. Otherwise choose distinct $\{N_k, M_k, k \geqq 1\}$ so that

$$N_k \mathcal{B}(\mathcal{H}) M_k \subseteq \operatorname{alg} \mathcal{L}$$

but

$$N_k \mathcal{B}(\mathcal{H}) M_k \nsubseteq \operatorname{alg} \mathcal{P}.$$

Since M_k and N_k are minimal, there is a projection P_{n_k} in $\mathcal{P}$ with $M_k P_{n_k} = M_k$ and $P_{n_k}{}^{\perp} N_k = N_k$. Let U_k be non-zero partial isometries with domain in $M_k \mathcal{H}$ and range in $N_k \mathcal{H}$. Then $U = \oplus \sum U_k$ belongs to $\operatorname{alg} \mathcal{L}$, and

$$\|P_{n_k}{}^{\perp} U P_{n_k}\| \geqq \|N_k U M_k\| = 1.$$

Since P_n is finite rank only finitely many M_k satisfy $M_k P_n = M_k$, so $n_k \to \infty$ as $k \to \infty$. But then U is not in $\mathcal{Q}\mathcal{T}(\mathcal{P})$ contradicting the hypothesis. Hence we have $N \prec_{\mathcal{L}} M \Rightarrow N \prec_{\mathcal{P}} M$ except for a finite set $M_1, \ldots, M_n$. Let P_0 be a projection in $\mathcal{P}$ greater than all these M_i. Then

$$N \prec_{\mathcal{L} \vee P_0} M \Rightarrow N \prec_{\mathcal{P} \vee P_0} M,$$

so $\mathcal{P} \vee P_0 \subseteq \mathcal{L} \vee P_0$. In particular, $\mathcal{L} \vee P_0$ is AF.

In the second case, we have M in $\mathcal{L}''$ with no finite rank minimal projections. So we can extend $M\mathcal{L}''$ to a maximal abelian non-atomic von Neumann algebra $\mathcal{N}$ on $M\mathcal{H}$. By induction we will construct pairwise orthogonal projections M_n in $\mathcal{N}$, projections P_n in $\mathcal{P}$, and partial isometries U_n in $\mathcal{N}$ supported on $M_n \mathcal{H}$ such that

1) $\|P_n^\perp U_n P_n\| > 1/2$

2) $\left\|P_n^\perp \sum_{k<n} U_k P_n\right\| < 1/8$ and

3) $\left\|\sum_{k>n} M_k P_n\right\| < 1/8.$

Assuming this is possible, then $U = \oplus \sum U_n$ is a partial isometry in $\mathcal{N} \subseteq \mathcal{L}'$ for which $\|P_n^\perp U P_n\| > 1/4$, contradicting the containment $\mathcal{L}' \subseteq \mathcal{Q}\mathcal{T}(\mathcal{P})$.

Now assume that M_k, P_k, and U_k have been chosen for $k < n$, and that N_n is a given non-zero projection in $\mathcal{N}$ orthogonal to $\sum_{k<n} M_k$. We can choose a sequence of projections in $\mathcal{N}$ less than N_n which tend to zero in the strong operator topology. Then since P_{n-1} is compact, it follows that for some R_n in this sequence $\|P_{n-1}R_n\| < 1/8$. Also, P_n tends to I in the strong operator topology, so we can choose P_n such that $\|R_n P_n\| > 3/4$. Since $\sum_{k<n} U_k$ belongs to alg $\mathcal{L}$ and hence $\mathcal{Q}\mathcal{T}(\mathcal{P})$, we can also choose P_n to satisfy

$$\left\|P_n^\perp \sum_{k<n} U_k P_n\right\| < 1/8.$$

Again using the properties of $\mathcal{N}$, we can choose M_n in $\mathcal{N}$ strictly less than R_n for which $\|M_n P_n\| > 3/4$. We set $N_{n+1} = R_n - M_n$. We now have 2) for P_n and since by construction we will have $\sum_{k\geqq n} M_k \leqq R_n$, we have satisfied 3) for P_{n-1}.

Let $m = \dim P_n$ and fix a unit vector x for which $\|M_n P_n x\| > 3/4$. Since $\mathcal{N}$ is maximal abelian non-atomic, we can find 2^{4m} pairwise orthogonal projections $Q_i \leqq M_n$ such that

$$\|Q_i P_n x\| = 3/4\, 2^{-2m}.$$

For $l = 0, 1, \ldots, 4m - 1$, let $\sigma_l:\{1, 2, \ldots, 2^{4m}\} \rightarrow \{1, -1\}$ be the function taking the value $+1$ and -1 on alternate blocks of length 2^l. Let

$$x_l = \sum \sigma_l(i) Q_i P_n x.$$

Then $\|x_l\| = 3/4$ and $(x_k, x_l) = 0$ if $k \neq l$. Since the Hilbert-Schmidt norm of P_n is $\sqrt{m}$,

$$\sum_{l=0}^{4m-1} \left\|P_n x_l\right\|^2 \leqq m.$$

Choose an l for which $\|P_n x_l\|^2 \leqq 1/4$. Now define

$$U_n = \sum_{i=1}^{2^{4m}} \sigma_l(i) Q_i.$$

Then

$$\|P_n^{\perp}U_nP_nx\|^2 = \|P_n^{\perp}x_l\|^2 \geqq (3/4)^2 - 1/4 > 1/4,$$

proving 1).

4. The non-commutative case. If $\mathscr{M}$ is not commutative, the situation is less clear because the structure of $\mathscr{M}$ is less rigidly defined. In particular, we cannot hope for a unitary equivalence as in Theorem 2.7. The best one could expect is a similarity by an invertible operator, but we do not know if this is possible. Also, if M_0 is a finite rank element of $\mathscr{M}$, it may happen that Q alg $M \vee M_0$ is not equal to Q alg $\mathscr{M}$.

LEMMA 4.1. *If A and C are projections on a finite dimensional Hilbert space $\mathscr{H}$ satisfying $A \wedge C = 0$, $A \vee C = I_{\mathscr{H}}$, and $\|A^{\perp} - C\| > 3/4$, then*

$$d(A^{\perp}, \text{alg}\,\{A, C\}) > 1/4.$$

Proof. Since $\dim A^{\perp} = \dim C < \infty$, we have

$$\|A^{\perp} - C\| = \sin \theta > 3/4$$

where θ is the greatest angle between $A^{\perp}\mathscr{H}$ and $C\mathscr{H}$. So there is a unit vector x such that $Cx = x$, and

$$\|A^{\perp}x\| = \cos \theta < 1/\sqrt{2}.$$

(To verify this, choose a unit vector y with $\|A^{\perp} - C\| = \|A^{\perp} - Cy\|$. Note that $\mathscr{K} = \text{span}\,\{A^{\perp}y, Cy\}$ is invariant for both $A^{\perp}$ and C. Compute $A^{\perp}$ and C on $\mathscr{K}$.) Let U be a partial isometry of $A^{\perp}\mathscr{H}$ onto $C\mathscr{H}$. Decomposing $\mathscr{H} = A\mathscr{H} \oplus A^{\perp}\mathscr{H}$, the matrix of U has the form

$$\begin{bmatrix} 0 & X \\ 0 & W \end{bmatrix}.$$

Since $A \wedge C = 0$, W is invertible in $A^{\perp}\mathscr{H}$. Let $y = U^*x$. Then $x = Uy = Wy + Xy$, so $A^{\perp}x = Wy \neq 0$ and $\|Wy\| < 1/\sqrt{2}$. Now $A^{\perp} = U^*U = W^*W + X^*X$, so $1 = \|A^{\perp}y\|^2 = \|Wy\|^2 + \|Xy\|^2$. Thus

$$\|Xy\|^2 > 1/2 \quad \text{and} \quad \|XW^{-1}\| \geqq \|XW^{-1}(Wy)\|\,\|Wy\|^{-1} \geqq 1.$$

All operators in alg $\{A, C\}$ have the form

$$T = \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$$

with respect to the (non-orthogonal) decomposition $\mathscr{H} = A\mathscr{H} + C\mathscr{H}$. The invertible operator $S = A + U$ carries $A\mathscr{H}$ onto $A\mathscr{H}$ and $A^{\perp}\mathscr{H}$ onto $C\mathscr{H}$, so with respect to $\mathscr{H} = A\mathscr{H} \oplus A^{\perp}\mathscr{H}$;

$$S = \begin{bmatrix} I & X \\ 0 & W \end{bmatrix}, \qquad S^{-1} = \begin{bmatrix} I & -XW^{-1} \\ 0 & W^{-1} \end{bmatrix}$$

and T has the form

$$\begin{bmatrix} I & X \\ 0 & W \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I & -XW^{-1} \\ 0 & W^{-1} \end{bmatrix} = \begin{bmatrix} Y & (XW^{-1})Z' - Y(XW^{-1}) \\ 0 & Z' \end{bmatrix}$$

where $Z' = W^{-1}ZW$.

If such an operator satisfies $\|T - A^{\perp}\| < 1/4$, then $\|Y\| < 1/4$ and $\|I - Z'\| < 1/4$. But then

$$\|XW^{-1}Z' - YXW^{-1}\| \geqq \|XW^{-1}\| - \|XW^{-1}\|(\|I - Z'\| + \|Y\|) \geqq 1/2\|XW^{-1}\| > 1/2.$$

Hence if T is in alg $\{A, C\}$, then $\|T - A^{\perp}\| \geqq 1/4$.

THEOREM 4.2. *Let $\mathscr{L}$ be a commutative* AF *lattice and let $\mathscr{M}$ be an* AF *lattice for which Q alg $\mathscr{L} \subseteq Q$ alg $\mathscr{M}$. Then there is a finite rank M_1 in $\mathscr{M}$ such that $\mathscr{M} \vee M_1$ is asymptotic to a sublattice of $\mathscr{L}$.*

Proof. By Lemma 2.3, there is a finite rank M_0 in $\mathscr{M}$ such that $d(M, \mathscr{L}) < 1/24$ for $M \geqq M_0$. So to each $M \geqq M_0$, there corresponds a unique element L_M in $\mathscr{L}$ with $\|M - L_M\| < 1/24$. If $M > N$, then $MN = N$, so $L_M > L_N$ as in the proof of Lemma 2.5. Also since

$$L_P \wedge L_R = L_P L_R \quad \text{and} \quad P \wedge R = \lim_{n\to\infty} (PRP)^n,$$

a simple estimate shows that $L_{P\wedge R} \leqq L_P L_R$. Similarly, $L_{P\vee R} \geqq L_P \vee L_R$. We will show that there is a finite rank M_1 in $\mathscr{M}$ so that if P and R belong to $\mathscr{M} \vee M_1$, then equality actually holds in both these relations.

If this is not the case, we can inductively choose finite rank projections M_n, P_n, R_n and N_n in $\mathscr{M} \vee M_0$ so that $P_n \wedge R_n = M_n$, $P_n \vee R_n = N_n$, $M_{n+1} \geqq N_n$ and either $L_{P_n}L_{R_n} > L_{M_n}$ or $L_{P_n} \vee L_{R_n} < L_{N_n}$. We achieve this as follows: Given N_{n-1}, we can find P and R greater than N_{n-1} so that either $L_P L_R > L_{P\wedge R}$ or $L_P \vee L_R < L_{P\vee R}$. We must ensure that P_n and R_n are finite rank. So take a chain C_n of finite rank elements of $\mathscr{M}$ with $C_0 = N_n$, $C_n \xrightarrow{s} P \wedge R$ in the strong operator topology. Then select chains P_n and R_n of finite rank projections in $\mathscr{M}$ converging to P and R respectively such that $P_n \geqq C_n$ and $R_n \geqq C_n$. This ensures that

$$P_n \wedge R_n \xrightarrow{s} P \wedge R \quad \text{and} \quad P_n \vee R_n \xrightarrow{s} P \vee R.$$

For any chain $M_n \xrightarrow{s} M$, the lower semi-continuity of the norm in the strong operator topology, and the fact that $\|M - L\| < 1/8$ uniquely determines L as L_M implies that $L_{M_n} \xrightarrow{s} L_M$. Hence

$$L_{P_n\wedge R_n} \xrightarrow{s} L_{P\wedge R} \quad \text{and} \quad L_{P_n\vee R_n} \xrightarrow{s} L_{P\vee R}.$$

Since $\mathscr{L}$ is commutative, it also follows that

$$L_{P_n}L_{R_n} \xrightarrow{s} L_P L_R \quad \text{and} \quad L_{P_n} \vee L_{R_n} \xrightarrow{s} L_P \vee L_R.$$

https://doi.org/10.4153/CJM-1981-054-0 Published online by Cambridge University Press

Thus, if $L_{P_n \vee R_n} \neq L_{P_n} \vee L_{R_n}$,

$$\limsup \|L_{P_n \vee R_n} - L_{P_n} \vee L_{R_n}\| \geqq \|L_{P \vee R} - L_P \vee L_R\| = 1$$

and consequently $L_{P_n \vee R_n} \neq L_{P_n} \vee L_{R_n}$ for some n. The case for intersections is identical.

We now apply Lemma 4.1 to the Hilbert space $\mathscr{K} = (N_n - M_n)\mathscr{H}$ with $I = N_n - M_n$, $A = P_n - M_n$ and $C = R_n - M_n$. Let $I' = L_{N_n} - L_{M_n}$, $A' = L_{P_n} - L_{M_n}$ and $C' = L_{R_n} - L_{M_n}$. Then $\|A - A'\| < 1/12$, $\|C - C'\| < 1/12$ and $\|I - I'\| < 1/12$. By construction, either $A'C' \neq 0$ or $A' \vee C' \neq I'$, so $C' \neq I' - A'$ and hence $\|C' + A' - I'\| = 1$. Thus,

$$\|C - A^{\perp}\| = \|C + A - I\| \geqq 3/4.$$

So Lemma 4.1 shows that

$$d(A^{\perp}, \text{alg}\,\{A, C\}) > 1/4.$$

Therefore

$$d(N_n - P_n, \text{alg}\mathscr{M}) \geqq d(N_n - P_n, \text{alg}\,\{P_n, R_n\}) > 1/4.$$

Drop to a subsequence if necessary to ensure that

$$\sum \|(N_n - P_n) - (L_{N_n} - L_{P_n})\| < \infty.$$

Let $B = \sum \oplus (N_n - P_n)$ and $B' = \sum \oplus L_{N_n} - L_{P_n}$. Then B' belongs to $\text{alg}\,\mathscr{L}$ and $B - B'$ is compact. If T is any operator in $\text{alg}\mathscr{M}$, the operator $(N_n - M_n)T(N_n - M_n)$ belongs to $\text{alg}\,\{P_n, R_n\}$. So

$$\|(N_n - M_n)(T - B)(N_n - M_n)\| \geqq d(\text{alg}\,\{P_n, R_n\}, N_n - P_n) \geqq 1/4.$$

Consequently, B is not a compact perturbation of T so neither is B'. This contradicts $\text{alg}\,\mathscr{L} \subseteq Q\,\text{alg}\mathscr{M}$.

So we must have $\mathscr{M} \vee M_1$ lattice isomorphic to a sublattice $\mathscr{L}_1$ of $\mathscr{L}$. Finally Lemma 2.3 shows that $\mathscr{M} \vee M_1$ is asymptotic to $\mathscr{L}_1$.

5. Quasitriangular algebras. Suppose that $\mathscr{L}$ is commutative and $Q\,\text{alg}\,\mathscr{L} \subseteq Q\,\text{alg}\mathscr{M}$. Then for every M in $\mathscr{M}$ and T in $\text{alg}\,\mathscr{L}$, $M^{\perp}TM$ is compact. In other words, M is an essentially invariant subspace of $\text{alg}\,\mathscr{L}$. The following theorem due to the author [**2**] makes it seem likely that the AF condition of Theorem 3.2 is unnecessary.

THEOREM 5.1. *If $\mathscr{L}$ is commutative and M is essentially invariant for* $\text{alg}\,\mathscr{L}$, *then there is an L in $\mathscr{L}$ for which $M - L$ is compact.*

This theorem can be applied to give variants of Theorem 3.2, but we will restrict ourselves to the following application.

https://doi.org/10.4153/CJM-1981-054-0 Published online by Cambridge University Press

LEMMA 5.2. *Suppose* Q alg$\mathcal{M}$ *contains a quasitriangular algebra* $\mathcal{QT}(\mathcal{P})$ *and is not all of* $\mathcal{B}(\mathcal{H})$. *Then there is a finite dimensional projection* $M_0^\perp$ *in* $\mathcal{M}^\perp$ *such that* $\mathcal{M} \vee M_0^\perp$ *is* AF.

Proof. By Theorem 5.1, every projection in $\mathcal{M}$ is either finite or cofinite. Let M_0 (M_1) be the supremum (infimum) of all finite (cofinite) rank projections in $\mathcal{M}$. Then M_0 and M_1 are either finite or cofinite. If M_1 is finite, there is a decreasing chain $\mathcal{R}$ of cofinite projections in $\mathcal{M}$ with infimum M_1. So

$$\mathcal{QT}(\mathcal{P}^\perp) = \mathcal{QT}(\mathcal{P})^* \subseteq (Q \operatorname{alg}\mathcal{M})^* \subseteq (Q \operatorname{alg}\mathcal{R})^* = Q \operatorname{alg}\mathcal{R}^\perp = Q \operatorname{alg}(\mathcal{R}^\perp \vee M_1).$$

But Q alg $(\mathcal{R}^\perp \vee M_1)$ is quasitriangular, so by Theorem 3.1, $\mathcal{P}^\perp$ is AF which is absurd. So M_1 is cofinite. So if M_0 were finite, Q alg$\mathcal{M}$ would equal $\mathcal{B}(\mathcal{H})$. Thus M_0 is cofinite and $\mathcal{M} \vee M_0^\perp$ is AF.

THEOREM 5.3. *If* $\mathcal{L}$ *is a commutative lattice such that* Q alg $\mathcal{L}$ *contains a quasitriangular algebra, then* Q alg $\mathcal{L}$ *is quasitriangular.*

Proof. By Lemma 5.2, $\mathcal{L} \vee L_0^\perp$ is AF. So by Theorem 2.7, there is a finite dimension L_1 in $\mathcal{L} \vee L_0^\perp$ so that $\mathcal{L} \vee L_0^\perp \vee L_1$ is a chain. Hence

$$Q \operatorname{alg}\mathcal{L} = Q \operatorname{alg}(\mathcal{L} \vee L_0^\perp \vee L_1)$$

is quasitriangular.

LEMMA 5.4. *If* Q alg$\mathcal{M}$ *contains a quasitriangular algebra, then there is a finite rank projection* M *in* $\mathcal{M} \vee \mathcal{M}^\perp$ *for which* Q alg $(\mathcal{M} \vee M)$ *is quasitriangular.*

Proof. Substitute Theorem 4.3 for Theorem 2.7 in the proof of Theorem 5.3.

In view of this lemma, if Q alg$\mathcal{M}$ contains a quasitriangular algebra $\mathcal{QT}(\mathcal{R})$, then $\mathcal{M} \vee M$ is asymptotic to a subset of $\mathcal{R}$. So we may assume that $\mathcal{M} \vee M$ is contained in $\mathcal{R} = \{R_n : n \geqq 0\}$. For definiteness we can assume that $R_0 = M$ and $\mathcal{R}$ is maximal (dim $R_{n+1}R_n^\perp = 1$ for all n, dim $R_0 = n_0 < \infty$). Choose a basis for $R_0^\perp\mathcal{H}$ so that

$$R_n = R_0 \oplus [e_k : 1 \leqq k \leqq n] \quad \text{for each } n.$$

Let Σ be the subset of $\mathbf{N}$ for which

$$\mathcal{M} \vee R_0 = \{R_n : n \in \Sigma\} = \mathcal{R}_\Sigma.$$

We now restrict ourselves to the following special case. Let f_k be a sequence of unit vectors in $R_0\mathcal{H}$, and let a_n be an arbitrary sequence of complex numbers. Define

$$M_n = [e_k + a_k f_k, 1 \leqq k \leqq n]$$

for n belonging to a subset Λ of Σ. Then $M_n \vee R_0 = R_n$ and $M_n \wedge R_0 = 0$. Let $M_\infty = \vee_{n \geqq 1} M_n$. Then

$$\mathscr{M} = \{0, \mathscr{R}_\Sigma, M_n, n \in \Lambda, M_\infty, I\}$$

is a lattice. By the remarks of the preceding paragraph, we see that $Q \operatorname{alg}\mathscr{M}$ represents a large class of those algberas containing a quasi-triangular algebra. Let $\mathscr{D}$ be the algebra of diagonal operators on the basis $\{e_n\}$.

THEOREM 5.5. *The following are equivalent.*

1) $\mathscr{D} \subseteq Q \operatorname{alg}\mathscr{M}$

2) $\mathscr{Q}\mathscr{T}(\mathscr{R}_\Sigma) = Q \operatorname{alg}\mathscr{M}$

3) $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

In this case, there is an idempotent E for which the map $A \rightarrow EA$ is a projection of $\mathscr{T}(\mathscr{R}_\Sigma)$ into $\operatorname{alg}\mathscr{M}$ *with kernel $R_0\mathscr{B}(\mathscr{H})$.*

Proof. 2) implies 1) is clear, and the inclusion $Q \operatorname{alg}\mathscr{M} \subseteq \mathscr{Q}\mathscr{T}(\mathscr{R}_\Sigma)$ is also obvious. Assume that 3) holds.

We will denote by $T_{x \otimes y}$ the operator $T_{x \otimes y} h = (h, y)x$. Let

$$E = R_0^{\perp} + \sum_{k \leqq 1} a_k T_{f_k \otimes e_k}.$$

The Hilbert-Schmidt norm of $R_0 E$ is

$$\|R_0 E\|_2^{\,2} = \sum_{k=1}^{\infty} \|R_0 E e_k\|^2 = \sum_{k=1}^{\infty} |a_k|^2 < \infty.$$

Hence $E = R_0^{\perp} + R_0 E = E R_0^{\perp}$ is a bounded operator, and these relations readily verify that $E = E^2$ with kernel $R_0\mathscr{H}$. Now if A belongs to $\mathscr{T}(\mathscr{R}_\Sigma)$, then EA does also because $R_0^{\perp}A$ and $R_0\mathscr{B}(\mathscr{H})$ belong to $\mathscr{T}(\mathscr{R}_\Sigma)$. If M_p is in $\mathscr{M}$ and $n \leqq p$, then $Af_n \in R_0\mathscr{H}$ and $Ae_n \in R_p\mathscr{H}$, so

$$\begin{aligned} EA(e_n + a_n f_n) &= R_0^{\perp} A e_n + \sum_{k=1}^{\infty} a_k (A(e_n + a_n f_n), e_k) f_k \\ &= \sum_{k=1}^{p} (Ae_n, e_k)(e_k + a_k f_k). \end{aligned}$$

Thus M_p is invariant for EA. Consequently,

$$E\mathscr{T}(\mathscr{R}_\Sigma) \subseteq \operatorname{alg}\mathscr{M},$$

and since $I - E$ is compact,

$$\mathscr{Q}\mathscr{T}(\mathscr{R}_\Sigma) = Q \operatorname{alg}\mathscr{M}.$$

Thus, 3) implies 2).

Now suppose that $\mathscr{D} \subseteq Q \operatorname{alg}\mathscr{M}$ but $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. We need a

https://doi.org/10.4153/CJM-1981-054-0 Published online by Cambridge University Press

certain subset Δ of $\mathbf{N}$, but as its construction is delicate, we will delay it temporarily. Once given a subset Δ, we define the projection D onto $[e_n : n \in \Delta]$. Then D is a diagonal operator with matrix (d_n) where $d_n = 1$ for n in Δ, $d_n = 0$ otherwise. Since D is in $\mathscr{D}$, there is a compact operator K such that $A = D + K$ lies in alg$\mathscr{M}$. Since D is in $\mathscr{T}(\mathscr{R}_\Sigma)$, so is K.

We define Hilbert-Schmidt operators

$$H_p = -R_0 + \sum_{k=1}^{p} a_k T_{f_k \otimes e_k}.$$

We compute

$$\|H_p\|_2^2 = n_0 + \sum_{k=1}^{p} |a_k|^2$$

where $n_0 = \dim R_0\mathscr{H}$. If p is in Λ and $n \leqq p$, then $R_p \in \mathscr{M}$, hence

$$Ae_n = r_0 + \sum_{k=1}^{p} b_k e_k$$

where $r_0 = R_0Ke_n$, $b_n = d_n + (Ke_n, e_n)$, and $b_k = (Ke_n, e_k)$, $k \neq n$. Since M_p is in $\mathscr{M}$,

$$A(e_n + a_n f_n) = \sum_{k=1}^{p} b_k'(e_k + a_k f_k).$$

Also since Kf_n is in $R_0\mathscr{H}$, a comparison of the coefficients shows that $b_k = b_k'$. So by projecting onto $R_0\mathscr{H}$, we get

$$R_0Ke_n + a_nKf_n = \sum_{k=1}^{p} (Ke_n, e_k)a_k f_k + d_n a_n f_n$$

$$a_n(K - d_n)f_n = -R_0Ke_n + \sum_{k=1}^{p} (Ke_n, e_k)a_k f_k = H_pKe_n \quad \text{(E1)}.$$

By assumption, $\|H_p\|_2 \to \infty$ as p increases, so

$$\lim_{p\to\infty} \|H_p\|_2^{-1}H_pe_n = \lim_{p\to\infty} \|H_p\|_2^{-1}a_nf_n = 0.$$

Since the sequence $\|H_p\|_2^{-1}H_p$ has norm bounded by one, it tends to zero in the strong operator topology.

We now make use of the following elementary lemma which does not seem to be in the literature. The proof is omitted.

LEMMA 5.6. *If K is a compact operator and h_n is a sequence of Hilbert Schmidt operators with $\|h_n\|_2$ bounded such that h_n tends to zero in the strong operator topology, then $\|h_nK\|_2$ tends to zero.*

We conclude that $\|H_p\|_2^{-1}H_pK$ tends to zero in the Hilbert-Schmidt norm. We will prove 1) implies 3) by showing that this fails.

We now return to the construction of Δ. Let $P_n = \{C_{n,1}\}$ be finite partitions of the unit sphere of $M_0\mathscr{H}$ such that the diameter of $C_{n,i}$ is less than $1/n$. For each subset α of elements of P_n, let $\sigma_{n,\alpha} = \{k: f_k \in C_{n,i} \in \alpha\}$ be the set of integers k for which f_k belongs to $C_{n,1}$ for some element of α. Let σ_j be an enumeration of the $\sigma_{n,\alpha}$ as n ranges over $\mathbf{N}$ so that each $\sigma_{n,\alpha}$ is repeated infinitely often. We inductively choose integers $n_j < n_{j+1}$ in Λ and a subset Δ of $\mathbf{N}$, such that

$$\tfrac{1}{2}\|H_{n_j}\|_2^2 \leqq \sum \{|a_k|^2 : k \leqq n_j, k \in S_j\} \tag{E2}$$

where $S_j = (\Delta \cap \sigma_j) \cup (\Delta^c \cap \sigma_j^c)$, and c denotes the complement in $\mathbf{N}$. If n_j and $\Delta \cap \{1, 2, \ldots, n_j\}$ have been chosen, choose $n_{j+1} \in \Lambda$ sufficiently large so that

$$\tfrac{1}{2}\|H_{n_{j+1}}\|_2^2 \leqq \sum \{|a_k|^2 : k \leqq n_j, k \in S_{j+1}\} + \sum \{|a_k|^2 : n_j < k \leqq n_{j+1}\}.$$

This is always possible as $\sum |a_k|^2 = \infty$. Then define

$$\Delta \cap \{n_j + 1, \ldots, n_{j+1}\} = \sigma_{j+1} \cap \{n_j + 1, \ldots, n_{j+1}\}.$$

Clearly, Δ now satisfies (E2) for σ_{j+1}. Hence for a given $\sigma_{n,\alpha}$, there are infinitely many n_j for which (E2) is satisfied.

Now take D and K corresponding to Δ as in the first paragraph. Choose an integer N for which $\|K\| \leqq N/4$. If f_k belongs to $C_{N,i}$ of P_N, either

$$\|Kf_k\| \geqq 1/2 \quad \text{or} \quad \|(K - 1)f_k\| \geqq 1/2.$$

If f_j is any other vector in $C_{n,i}$,

$$\|K(f_i - f_j)\| \leqq \|K\|/N \leqq 1/4.$$

So either $\|Kf_k\| \geqq 1/4$ for all f_k in $C_{n,i}$ or $\|(K - 1)f_k\| \geqq 1/4$ for all f_k in $C_{N,i}$. Let α be the set of $C_{N,i}$ for which the latter relation holds. Let $\sigma = \sigma_{n,\alpha}$, and let $n_1, n_2, \ldots$ be integers in Λ for which (E2) is satisfied for σ. Then using (E1) we get

$$\begin{aligned}
\|H_{n_i}K\|_2^2 &\geqq \sum_{n=1}^{n_i} |a_n|^2\|(K - d_n)f_n\|^2 \\
&= \sum \{|a_n|^2\|(K - 1)f_n\|^2 : n \leqq n_i, n \in \Delta\} \\
&\quad + \sum \{|a_n|^2\|Kf_n\|^2 : n \leqq n_i, n \notin \Delta\} \\
&\geqq \sum \{16^{-1}|a_n|^2 : n \leqq n_i, n \in \Delta \cap \sigma\} \\
&\quad + \sum \{16^{-1}|a_n|^2 : n \leqq n_i, n \in \Delta^c \cap \sigma^c\} \\
&\geqq 1/32\|H_{n_i}\|_2^2.
\end{aligned}$$

Consequently, $\|H_{n_i}\|_2^{-1}\|H_{n_i}K\|^2$ is bounded away from zero, giving the desired contradiction.

https://doi.org/10.4153/CJM-1981-054-0 Published online by Cambridge University Press

Added in proof. Since this paper was written, a paper by N. T. Andersen, *Compact perturbations of reflexive algebras*, J. Func. Anal. *38* (1980), 366–400, has appeared which contains related results.

REFERENCES

1. W. B. Arveson, *Interpolation in nest algebras*, J. Func. Anal. *20* (1972), 208–233.
2. K. R. Davidson, *Commutative subspace lattices*, Indiana Univ. Math. J. *27*, 479–490.
3. K. R. Davidson and C. K. Fong, *An operator algebra which is not closed in the Calkin algebra*, Pac. J. Math. *72*, 57–58.
4. T. Fall, W. Arveson and P. Muhly, *Perturbations of nest algebras*, J. Oper. Th. *1*.
5. P. R. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) *29* (1968), 283–293.
6. A. Hopenwasser and J. Plastiras, *Isometries of quasitriangular operator algebras*, to appear.
7. B. E. Johnson and S. K. Parrot, *Operators commuting with a von Neumann Algebra modulo the set of compact operators*, J. Func. Anal. *11* (1972), 39–61.
8. J. Plastiras, *Quasitriangular operator algebras*, Pac. J. Math. *64* (1976), 543–549.
9. ——— *Compact perturbations of certain von Neumann algebras*, Trans. AMS *234* (1977), 561–577.

University of Waterloo,
Waterloo, Ontario