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Maximal order Abelian subgroups of Coxeter groups

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Abstract
In this note, we give a classification of the maximal order Abelian subgroups of finite irreducible Coxeter groups. We also prove a Weyl group analog of Cartan's theorem that all maximal tori in a connected compact Lie group are conjugate.

1. Introduction

Some years ago, colleagues working in the area of statistical mechanics asked what the maximal order Abelian subgroups of the symmetric group S_n looked like. Their question arose from consideration of reducible representations constructed from tensor products of unitary representations arising in the statistical mechanics of systems of n quantum spins. In particular, they wanted to understand the situation as $n \to \infty$. A complete classification can be derived from general results in [17], and a classification was given in a more general setting in [9]. An elementary classification was given in [6] using Lagrange multipliers. This method indicates that in order to maximize the product $\prod m_i$ of the prime powers m_i (the Abelian invariants) subject to the constraint $\sum m_i \leq n$ (because it is an Abelian subgroup of S_n), all or as many as possible of the integers m_i should be chosen equal (to m say). The problem then amounts to maximizing $m^{\frac{n}{m}}$ and regarding this as a function of a real variable having a maximum at e, we would expect that the solution to the integer-valued problem (and therefore the maximal order of an Abelian subgroup of S_n) is of the form 3^k, since $2^{\frac{n}{2}} < 3^{\frac{n}{3}}$. This is essentially the case (see Theorem 1.1 below). In this note, we give a complete classification of the maximal order Abelian subgroups M for all finite irreducible Coxeter groups. We also determine the number of conjugacy classes of maximal order Abelian subgroups, and viewing a distinguished class of these subgroups as discrete analogs of maximal tori in compact Lie groups, we obtain a Weyl group analog of Cartan's theorem that all maximal tori in a connected compact Lie group G are conjugate, namely:

Theorem. *Let M and M' be discrete maximal tori of W, then $M' = w^{-1}Mw$ for some $w \in W$.*

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We now recall the precise solution for S_n (i.e. W of type A_{n-1}) and then state the general result.

Theorem 1.1 *Let M be an Abelian subgroup of maximal order in the symmetric group S_n, $n \geq 2$. Then*

(i) $M \simeq \mathbb{Z}_3^k$ *if* $n = 3k$,
(ii) $M \simeq \mathbb{Z}_3^k \times \mathbb{Z}_2$ *if* $n = 3k + 2$,
(iii) *either* $M \simeq \mathbb{Z}_3^{k-1} \times \mathbb{Z}_4$ *or* $M \simeq \mathbb{Z}_3^{k-1} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *if* $n = 3k + 1$.

A natural representative for M in S_n is generated by a collection of disjoint 3-cycles, plus a 2-cycle or a 4-cycle if appropriate.

Since the case of the root system of type A_r is settled by the above theorem, we will exclude this case from the statement of the general result.

Theorem 1.2 *Let M be an Abelian subgroup of maximal order in a finite irreducible Coxeter group W of rank r, then:*

(a) *(W crystallographic)*

(i) *For W of type B_r or C_r, we have $M \simeq \mathbb{Z}_2^s \times \mathbb{Z}_4^t$ where $0 \le s,\ \ t$ with $s + 2t = r$ and $|M| = 2^r$.*
(ii) *For W of type D_{2k} $(r = 2k)$, we have $M \simeq \mathbb{Z}_2^{2k}$ and $|M| = 2^r$.*
(iii) *For W of type D_{2k+1} $(r = 2k + 1)$, we have $M \simeq \mathbb{Z}_2^s \times \mathbb{Z}_4^t$ where $0 \le s,\ \ t$ with $s + 2t = r - 1$ and $|M| = 2^{r-1}$.*
(iv) *For W of type E_6, we have $M \simeq \mathbb{Z}_3^3$, and $|M| = 3^3$.*
(v) *For W of type E_r, $r = 7, 8$, we have $M \simeq \mathbb{Z}_2^r$ and $|M| = 2^r$.*
(vi) *For W of type F_4, we have $M \simeq \mathbb{Z}_2 \times \mathbb{Z}_3^2$, and $|M| = 2 \cdot 3^2$.*
(vii) *For W of type G_2, we have $M \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$, and $|M| = 2 \cdot 3$.*

(b) *(W noncrystallographic)*

(i) *For W of type H_3, we have $M \simeq \mathbb{Z}_2 \times \mathbb{Z}_5$, and $|M| = 2 \cdot 5$.*
(ii) *For W of type H_4, we have $M \simeq \mathbb{Z}_2 \times \mathbb{Z}_5^2$, and $|M| = 2 \cdot 5^2$.*
(iii) *For W of type $I_2(m)$, $m \ge 5$, we have $M \simeq \mathbb{Z}_m$ and $|M| = m$.*

For W of type B_n, a natural representative for M is generated by a collection of m negative 2-cycles (having order 4) and a negative 1-cycle if n is odd.

For W of type D_{2k+1}, a natural representative for M comes from a subgroup of type $W(B_{2k})$ contained in $W(D_{2k+1})$.

For W of type D_{2k}, a natural representative for M is a direct product of k groups of type $W(A_1) \times W(A_1)$.

2. Basic facts and definitions

All basic facts and definitions used can be found in [4] or [16]. Let (W,S) be an irreducible finite Coxeter system of rank r with $S = \{s_{\alpha_1}, \dots, s_{\alpha_r}\}$ its set of simple reflections. When W is a Weyl group (W crystallographic), we have an associated connected compact Lie group G (with Lie algebra $\mathfrak{g}$), containing (a fixed) maximal torus T (with Lie algebra $\mathfrak{t}$) so that the Weyl group $W = N_G(T)/T$. If $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^{\alpha}$ is the root space decomposition of the complexification of $\mathfrak{g}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ (the complexification of $\mathfrak{t}$), then a root α is an element of the dual spaces $\mathfrak{t}^*$ (pure imaginary-valued) or $i\mathfrak{t}^*$ (real-valued). Since G is compact, the Killing form is negative definite on $\mathfrak{t}$ and gives an $(\mathrm{Ad}(G)$ invariant) real inner product $\langle\ ,\ \rangle$ on the real vector spaces $i\mathfrak{t}$ and $i\mathfrak{t}^*$. For $w \in N_G(T)$, $H \in \mathfrak{t}$ and $\alpha \in \mathfrak{t}^*$ we define $w(H) = \mathrm{Ad}(w)H = wHw^{-1}$ and $w(\alpha)(H) = \alpha(\mathrm{Ad}(w^{-1})H)$, and since $\mathrm{Ad}(T)$ acts trivially on $\mathfrak{t}$ we obtain (faithful) induced actions of W. Choosing a fundamental Weyl chamber in $i\mathfrak{t}$, we can define positive roots Φ^+ and $\{\alpha_1, \dots, \alpha_r\}$ a basis of positive simple roots whose simple reflections generate W. The fundamental weights $\{\omega_1, \dots, \omega_r\}$ are defined by the conditions that $\langle\omega_i, 2\alpha_j\rangle := \langle\alpha_j, \alpha_j\rangle\, \delta_{ij}$ for all i, j. We will normalize $\langle\ ,\ \rangle$ so that the highest root $\tilde{\alpha}$ has length squared equal to 2. For $\alpha, \beta \in \Phi$, we define (integers) $n(\alpha, \beta) = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$.

The Dynkin diagram D is the (multi) graph with r vertices (labeled by the positive simple roots), and $c_{ij}c_{ji}$ edges joining α_i to α_j where $c_{ij} = n(\alpha_i, \alpha_j)$. The extended Dynkin diagram $\tilde{D}$ (always labeled as in [4]) is the graph constructed from D by adding a new vertex $\alpha_0 = -\tilde{\alpha}$ (the affine vertex or node) and joining it to any vertex α_i by (the old rule of) $n(\alpha_i, \tilde{\alpha}) \cdot n(\tilde{\alpha}, \alpha_i)$ edges. We then write the coefficient n_i over the vertex α_i and $n_0 = 1$ over α_0, where $\tilde{\alpha} = \sum_{i=1}^{r} n_i\alpha_i$. Deletion of any vertex from $\tilde{D}$ and the edges connected to it produces a new (typically non-connected) Dynkin diagram D_1 (with the same number

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of vertices as D) of a semisimple Lie subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}_1$ is said to obtained from $\mathfrak{g}$ by an elementary operation. Of course, we can perform a new elementary operation on any of the connected components of D_1. Continuing this process, we obtain a chain of subalgebras $\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \ldots \supseteq \mathfrak{g}_m$, each obtained from its predecessor by an elementary operation, and any semi-simple Lie subalgebra of maximal rank is obtained by a finite number of elementary operations (see [12,18]). Note that when a diagram of type A_n occurs, an elementary operation does not change the algebra. Among the maximal rank Lie subalgebras are those corresponding to maximal subgroups of maximal rank in G, and we recall from [20] the following for later use:

The fundamental simplex

$$\mathfrak{D}_0 = \{h \in i\mathfrak{t} : \alpha_i(h) \geq 0 \,\forall\, i, \tilde{\alpha}(h) \leq 1\}$$

has vertices $\{v_0, v_1, \ldots, v_r\}$ where $v_0 = 0$, $\alpha_i(v_j) = \frac{1}{n_i}\delta_{ij}$, and it has the property that every element of G (connected and centerless) is conjugate to an element of $\exp(2\pi i\mathfrak{D}_0)$. The conjugacy classes of maximal connected subgroups of maximal rank in G are obtained from it by a theorem of Borel and de Siebenthal which we now recall.

Theorem 2.1 *([2,20], p. 278) Let G be a compact centerless simple Lie group with fundamental simplex* $\mathfrak{D}_0 = \{v_0, v_1, \ldots, v_r\}$ *and let* $1 \leq i \leq r$.

(i) *Suppose that* $n_i = 1$, *then the centralizer of the circle group* $\{\exp(2\pi i t v_i) : t \in \mathbb{R}\}$ *is a maximal connected subgroup of maximal rank in G with*

$$\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$$

as a system of simple roots.

(ii) *Suppose that* n_i *is a prime* $p > 1$, *then the centralizer of the element* $\exp(2\pi i v_i)$ *(of order p) is a maximal connected subgroup of maximal rank in G with*

$$\{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$$

as a system of simple roots.

(iii) *Every maximal connected subgroup of maximal rank in G is conjugate to one of the above groups.*

Finally, the *trace* of a finite Abelian group $A = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_k}$ is the integer $\mathrm{Tr}(A) = \sum_{i=1}^{k} m_i$ (see [15]).

3. Proof of Theorem 1.2

For W of each possible type, we first prove the existence of an Abelian subgroup of the required order and isomorphism type. We then check that W contains no Abelian subgroups of larger order, or other isomorphism types of Abelian subgroups of maximal order. For the exceptional crystallographic types (except G_2) and the noncrystallographic type H_4, the check involves computer calculations using the computer package CHEVIE for GAP [14]. In order to prove the existence, we observe that if K is a connected subgroup of G of maximal rank (necessarily equal to that of G), then any maximal torus of K is also a maximal torus of G. The Weyl group $W(K)$ of K can therefore be identified with a subgroup of W the Weyl group of G.

For W of type B_r or C_r, the elementary operation (in the extended Dynkin diagram of C_r) corresponding to deletion of the vertex connected to the $\alpha_0 = -\tilde{\alpha}$ vertex of $\tilde{D}$ (the vertex α_1) gives the Dynkin diagram D_1 of a semisimple subalgebra $\mathfrak{g}_1$ with corresponding maximal rank subgroup of G of type type $A_1 \times C_{r-1}$. Repeating this process in the component of D_1 corresponding to C_{r-1}, we eventually obtain a maximal rank subgroup of type $A_1 \times A_1 \times \cdots \times A_1$ (r copies) and hence a subgroup $M \simeq \mathbb{Z}_2^r$ of W. This sequence of elementary operations (i.e. successive deletion of the vertex connected to $-\tilde{\alpha}$, in successive extended Dynkin diagrams) we will call the Wolf sequence (on account of its connection to Wolf spaces,

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see [13,21]), and it also produces a maximal rank subgroup of type $A_1 \times A_1 \times \ldots \times A_1$ (r copies) for types D_{2k} and E_r, $r = 7, 8$ and hence a subgroup $M \simeq \mathbb{Z}_2^r$ of W.

The maximal order Abelian subgroups of W for type B_r or C_r containing direct factors isomorphic to $\mathbb{Z}_4$ are also realized in the extended Dynkin diagram of C_r (recalling that the Weyl group for root systems of type B_2 or C_2 is isomorphic to the dihedral group of order eight) as follows: as our first elementary operation, we delete the vertex α_2 from $\tilde{D}$ to obtain the Dynkin diagram D_1 of a semisimple algebra $\mathfrak{g}_1$ with corresponding maximal rank subgroup of G of type $C_2 \times C_{n-2}$. Repeating either this process or taking the Wolf sequence, in the component of D_1 corresponding to C_{n-2} we can eventually obtain any subgroup $M \simeq \mathbb{Z}_4^t \times \mathbb{Z}_2^s$ where $0 \le s$, t with $s + 2t = r$.

In the case of W of type D_{2k+1}, we note that not all subgroups listed arise from subgroups of maximal rank, for example, the maximal order Abelian subgroup $M \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ when W is of type D_5. However, the maximal order Abelian subgroups $M \simeq \mathbb{Z}_2^2 \times \mathbb{Z}_4$ and $M \simeq \mathbb{Z}_2^4$ both arise from a maximal rank subgroup of type $A_1 \times A_1 \times A_3$. The result follows, however, from the B_{2k} case by folding the D_{2k+1} diagram which comes from a regular embedding (taking a torus to a torus) of Lie groups (see [5], p. 265).

For W of type E_6, F_4 or G_2, the elementary operation of deletion of the vertex α_i such that $n_i = 3$, where $\tilde{\alpha} = \sum_{i=1}^{r} n_i\alpha_i$, gives a maximal rank subgroup of type $A_2 \times A_2 \times A_2$, $A_2 \times A_2$ or A_2, respectively, giving rise to an Abelian subgroup M of W with $M \simeq \mathbb{Z}_3^3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_3$, respectively. Since $-1 \in W$ for F_4 and G_2 and the center $Z(W) \simeq \mathbb{Z}_2 \simeq \langle -1 \rangle$ ([11]), we can extend these groups by $Z(W)$ in both these cases.

We now consider the noncrystallographic cases. For W of type $I_2(m)$, $m \ge 5$ (the Dihedral groups), the result is clear. For W of type H_3 and H_4, the classification of their maximal proper subroot systems in [8,10] gives rise to Abelian subgroups M of W with $M \simeq \mathbb{Z}_5$ (from a maximal subroot system of type $I_2(5)$), and $\mathbb{Z}_5^2$ (from a maximal subroot system of type $I_2(5) \times I_2(5)$) in H_3 and H_4, respectively. Extending these groups by their centers $Z(W) \simeq \mathbb{Z}_2$ gives the required Abelian subgroups M.

We now show that the obtained lower bounds on $|M|$ are also upper bounds, and that there are no other isomorphism types of maximal order Abelian subgroups. Before doing so, we remark (see [3], p. 134) that a subgroup of W isomorphic to $\mathbb{Z}_p^s$ (p a prime) admits a faithful real representation of dimension r (on $\mathfrak{t}$) and therefore $s \le r$ if $p = 2$ and $s \le \frac{r}{2}$ if $p \ne 2$. There is therefore no larger order elementary Abelian 2-group in $W(B_r)$, $W(D_{2k})$, $W(E_7)$, or $W(E_8)$, and no larger order elementary Abelian 3-group in $W(E_6)$ than obtained above. In ruling out other possibilities, we begin with the infinite families (i.e. classical types). We embed M in a symmetric group S_N and use the fact that we must then have $\mathrm{Tr}(M) \le N$. Here, the vertices of the extended Dynkin diagram $\tilde{D}$ with $n_i = 1$ and the maximal subgroup of maximal rank K corresponding to part (i) of Theorem 2.1. play a role. This subgroup is the isotropy subgroup of an Hermitian symmetric space $H = G/K$. Taking a maximal torus T of G to lie in K, the Weyl group W acts transitively and faithfully on the fixed point set $F(T,H)$ of the action of T on H. This set has cardinality equal to the Euler number $\chi(H)$ of H which is equal to $2r$ when W is of type B_r or D_r (and its elements are pairwise antipodal on totally geodesic two-dimensional spheres in H) see [19], so that $\mathrm{Tr}(M) \le 2r$. Alternatively, instead of $F(T,H)$, we can take the weights $\{\pm\lambda_1, \ldots, \pm\lambda_r\}$ of the vector representation of $\mathfrak{g}$ for the simple Lie algebras of type C_r and D_r. Now for B_r and D_r (r even), the center $Z(W) \simeq \langle -1 \rangle$ is contained in M so that the orbits Ω_k of M have even cardinality and M is the direct product of its restrictions to the orbits Ω_k. Since a transitive Abelian permutation group has order equal to its degree, we can rule out elements of order three in M since they must contribute at least six to $\mathrm{Tr}(M)$ and $|M|$, whereas $\mathbb{Z}_2^3$ contributes six to $\mathrm{Tr}(M)$ and eight to $|M|$. The argument for ruling out higher torsion elements of M other than four is similar, so that 2^r is the maximal order of an Abelian subgroup in these cases. Again, the case of D_{2k+1} follows from folding to B_{2k}. The remaining large order cases are in the exceptional families and were verified by computer calculations.

Definition 3.1.

(i) *The 2-rank of W is equal to the integer r_2 such that the maximal order of an elementary Abelian 2-subgroup of W is 2^{r_2}.*

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(ii) *A pair of roots α and $\beta \in \Phi$ are said to be strongly orthogonal (s.o.) if $\alpha + \beta$ is not a root and $\alpha - \beta$ is not a root. A subset consisting of pairwise s.o. roots will be called a s.o. set of roots.*

We now have the following corollary to Theorem 1.2.

Corollary 3.1. *Let W be the Weyl group of an irreducible root system, then the 2-rank (r_2) of W is equal to the maximal cardinality of a set of strongly orthogonal roots and $r_2 = r$ if and only if $-1 \in W$.*

Proof: As in the proof of Theorem 1.2, the Wolf sequence of elementary operations corresponding to successive deletion of the vertex connected to $-\tilde{\alpha}$ (in successive extended Dynkin diagrams) produces a maximal rank subgroup of G of type $A_1 \times A_1 \times \ldots \times A_1$ (with r copies), and corresponding maximal order elementary 2-subgroup $M \simeq \mathbb{Z}_2^r$ of W, unless we start with or encounter a diagram of type A_s with $2 \le s$, in which case $r_2 < r$. This will occur only in diagrams of type A_r, D_r, r odd, or E_6, namely in those cases where $-1 \notin W$, and then r_2 takes the values $\lfloor (r+1)/2 \rfloor$, $r-1$ and 4, respectively. That the corresponding elementary 2-groups are of maximal order was checked by computer for E_6, follows from Theorem 1.2 for D_r, r odd and by induction for A_r. That these procedures also produce a set of s.o. roots of maximal cardinality r_2 follows from orthogonality in the simply laced cases and from the classification of maximal sets of strongly orthogonal roots in [1], p. 121 and p. 127 otherwise.

Definition 3.2. *A maximal order Abelian subgroup M (of a finite irreducible Coxeter group W) with minimal number of Abelian invariants is called a discrete maximal torus of W.*

Remarks and Examples: The Weyl group of Type A_3 is the symmetric group S_4, and it already hints at the definition of a maximal torus (of W). S_4 has three conjugacy classes of maximal order Abelian subgroups, those of $M_1 = \langle (1234) \rangle$, $M_2 = \langle (12), (34) \rangle$, and $M_3 = \langle (12)(34), (13)(24) \rangle$. Whereas M_2 and M_3 are isomorphic as abstract groups they are not as permutation groups, that is, they are not conjugate in S_4. On the other hand, the cycle (1234) is a Coxeter element and it generates (for W of any type) a maximal Abelian subgroup of W (in this case also of maximal order) and a distinguished conjugacy class. Similarly, the Weyl group of Type B_2 has three conjugacy classes of maximal order Abelian subgroups, two of which are isomorphic to $\mathbb{Z}_2^2$ and the other (isomorphic to $\mathbb{Z}_4$) is corresponding to the Coxeter element. Whereas the Abelian subgroup generated by the Coxeter element (although maximal) is no longer of maximal order in higher rank, by Theorem 1.2. the above conjugacy class phenomenon persists for classical types (excluding D_{2k}).

We now prove an analog (for W) of Cartan's theorem that all maximal tori of a compact connected Lie group G are conjugate in G.

Theorem 3.1. *Let M and M' be discrete maximal tori of W, then $M' = w^{-1}Mw$ for some $w \in W$.*

Proof: For W of type A_r, we note that Theorem 1.1 and the definition of a maximal torus M of W imply that

(i) $M \simeq \mathbb{Z}_3^k$ if $r + 1 = 3k$,
(ii) $M \simeq \mathbb{Z}_3^k \times \mathbb{Z}_2$ if $r + 1 = 3k + 2$, and
(iii) $M \simeq \mathbb{Z}_3^{k-1} \times \mathbb{Z}_4$ if $r + 1 = 3k + 1$.

Since all direct factors correspond to disjoint cycles of length 2, 3, or 4, (with the sum of all lengths equal to $r + 1$) and at most one transposition occurring, the result follows from the fact that permutations of the same cycle type are conjugate in S_n.

For W of type B_r, viewed as all signed permutations of $\{1, 2, \ldots, r\}$, that is, injective maps from $\{1, 2, \ldots, r\}$ to $\{\pm 1, \pm 2, \ldots, \pm r\}$, with either i or $-i$ in the image, elements can again be expressed in cyclic form, and the above argument generalizes. Cycles either contain both i and $-i$ (called negative

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cycles) and are of the form $(i_1i_2\ldots i_k - i_1 - i_2\cdots - i_k)$ or do not contain both i and $-i$ for any i (called positive cycles), and they occur in pairs of the form $(i_1i_2\ldots i_k)(-i_1 - i_2\cdots - i_k)$. Since (as with ordinary permutations) conjugation by w of a signed permutation in cyclic form sends i to $w(i)$, two signed permutations are conjugate if and only if they have the same number of positive and negative cycles of every length. We now recall that a maximal torus M of W (by Theorem 1.2.) is of the form:

(i) $M \simeq \mathbb{Z}_4^k$ if $r = 2k$ and
(ii) $M \simeq \mathbb{Z}_4^k \times \mathbb{Z}_2$ if $r = 2k + 1$.

When $r = 2k$, the k commuting $\mathbb{Z}_4$ factors are negative cycles $(i_1i_2, -i_1 - i_2)$ (Coxeter elements of a B_2 or C_2 system), and we have that all maximal tori are conjugate. When $r = 2k + 1$, the argument is the same because the additional $\mathbb{Z}_2$ factor must be a negative 1-cycle. The case of D_r (r odd) is similar.

We next consider those cases where a maximal torus is of the form $M \simeq \mathbb{Z}_2^r$, that is, D_{2k} and E_r, $r \in \{7, 8\}$. Using the fact that for these cases $\langle -1\rangle = Z(W)$ must be contained in M, it is not hard to prove that M has a set of r generators, none of which is a nontrivial product of commuting reflections. These generating reflections therefore yield a maximal set of orthogonal roots that are in fact strongly orthogonal as the root systems are simply laced in the cases at hand ([1], p. 117). However, by [1], p. 119, all maximal subsets of strongly orthogonal roots are in the same Weyl group orbit for simply laced root systems (the number of such W-orbits is the number of short simple roots), and therefore the corresponding stabilizing sets of reflection generators of the discrete maximal tori are conjugate in W.

Similarly in the case of E_6, there is a unique W orbit of sets of three orthogonal roots ([7] p. 14), and therefore (as we now show) there is a unique W orbit of subroot systems of type $3A_2 = A_2 + A_2 + A_2$, and therefore all stabilizers (the M's) are conjugate. Deletion of the branch node in the extended Dynkin diagram gives a subroot system of type $3A_2$ with simple roots $\{\alpha_1, \alpha_3\}$, $\{\alpha_5, \alpha_6\}$, and $\{\tilde{\alpha}, \alpha_2\}$. Let $\{\alpha_1', \alpha_3'\}$, $\{\alpha_5', \alpha_6'\}$, and $\{\tilde{\alpha}', \alpha_2'\}$ be the simple roots of another $3A_2$, and by [7] p. 14 we may assume that $w(\alpha_1') = \alpha_1$, $w(\alpha_5') = \alpha_5$, and $w(\tilde{\alpha}') = \tilde{\alpha}$ for some $w \in W$. We now show that $w(\alpha_2') \in \{\alpha_2,\ \tilde{\alpha} - \alpha_2\}$. Since $\langle\tilde{\alpha}, w(\alpha_2')\rangle = \langle w(\tilde{\alpha}'), w(\alpha_2')\rangle = \langle\tilde{\alpha}', \alpha_2'\rangle \neq 0$, we have that $b_2 \neq 0$, where $w(\alpha_2') = \sum_{i=1}^r b_i\alpha_i$, because $\tilde{\alpha} = \omega_2$. Similarly $w(\alpha_2')$ must be orthogonal to $\alpha_1 = 2\omega_1 - \omega_3$ and $\alpha_3 = -\omega_1 + 2\omega_3 - \omega_4$ so that $2b_1 = b_3$ and $b_1 + b_4 = 2b_3 = 4b_1$ and therefore $3b_1 = b_4$. As there are only two positive roots with α_4-coefficient equal to 3, namely $\tilde{\alpha} = \omega_2$ and $s_{\alpha_2}(\omega_2) = -\omega_2 + \omega_4 = \tilde{\alpha} - \alpha_2$ (because the next highest root $s_{\alpha_4}(s_{\alpha_2}(\omega_2)) = -\omega_4 + \omega_3 + \omega_5$ has α_4-coefficient equal to 2), we have that either $b_1 = 1$ and $w(\alpha_2') = \tilde{\alpha} - \alpha_2$ or $b_1 = 0 = b_4 = b_3$ and therefore $w(\alpha_2') = \alpha_2$ as the only positive root of the system of type $A_2 + A_1 + A_2$ (as $b_4 = 0$) with nonzero α_2-coefficient. An identical argument gives $w(\alpha_6') \in \{\alpha_6,\ \alpha_5 + \alpha_6\}$ and $w(\alpha_3') \in \{\alpha_3,\ \alpha_1 + \alpha_3\}$. That all maximal order Abelian subgroups are conjugate for the cases G_2, F_4, H_3, and H_4 follows from Sylow's Second Theorem, as the groups $M/\langle -1\rangle$ are Sylow p-subgroups of $W/\langle -1\rangle$ with $p = 3$ or 5.

Remark: The definition of a discrete maximal torus as a maximal order Abelian subgroup with minimal number of Abelian invariants applies to any finite group. However, in general, there is more than one conjugacy class of them, as illustrated by Q_8. A computer search of groups of small order indicates that groups with a single conjugacy class of maximal tori are the exception rather than the rule.

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