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ASYMPTOTIC BEHAVIOUR OF IDEALS RELATIVE TO INJECTIVE MODULES OVER COMMUTATIVE NOETHERIAN RINGS

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Let E be an injective module over the commutative Noetherian ring A, and let $\mathfrak{a}$ be an ideal of A. The A-module $(0:_E\mathfrak{a})$ has a secondary representation, and the finite set $\mathrm{Att}_A(0:_E\mathfrak{a})$ of its attached prime ideals can be formed. One of the main results of this note is that the sequence of sets $(\mathrm{Att}_A(0:_E\mathfrak{a}^n))_{n\in\mathbb{N}}$ is ultimately constant. This result is analogous to a theorem of M. Brodmann that, if M is a finitely generated A-module, then the sequence of sets $(\mathrm{Ass}_A(M/\mathfrak{a}^nM))_{n\in\mathbb{N}}$ is ultimately constant.

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1. Introduction

It is a well-known result of M. Brodmann [1] that, if M is a finitely generated module over the commutative Noetherian ring A (with identity) and $\mathfrak{a}$ is an ideal of A, then the sequence of sets $(\mathrm{Ass}_A(M/\mathfrak{a}^nM))_{n\in\mathbb{N}}$ is ultimately constant, that is $\mathrm{Ass}_A(M/\mathfrak{a}^nM)$ is, for all sufficiently large n, independent of n. (We use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers.) It is a very easy consequence of this result of Brodmann that the same conclusion holds of we relax the hypotheses and assume only that A is a commutative ring (with identity) and M is a Noetherian A-module.

In [**8**], the present second author established a dual result for an Artinian module N over a commutative ring R (with identity). This dual result was phrased in terms of the notions of secondary representations and attached prime ideals of Artinian R-modules (see [**3**, **2** or **6**]), and showed that, if I is an ideal of R, then the sequence of sets $(\mathrm{Att}_R(0:_N I^n))_{n\in\mathbb{N}}$ is ultimately constant.

Every Artinian R-module possesses a secondary representation. (It is convenient to take the view that the zero R-module is the sum of the empty family of its secondary submodules.) Now it was shown in [7] that the class of all R-modules which possess secondary representations can be more extensive than the class of all Artinian R-modules: indeed, when we take the commutative Noetherian ring A for R, every injective A-module possesses a secondary representation [7, Theorem 2.3]. We shall see below that the arguments of that paper can easily be modified to show that, when A is Noetherian, E is an injective A-module and $\mathfrak{a}$ is an ideal of A, the submodule $(0:_E\mathfrak{a})$ of E has a secondary representation, and so we can form the finite set $\mathrm{Att}_A(0:_E\mathfrak{a})$ of its attached prime ideals. One of the main results of this note is that, in these

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circumstances, the sequence of sets $(\mathrm{Att}_A(0:_E \mathfrak{a}^n))_{n\in\mathbb{N}}$ is ultimately constant; in addition, we shall obtain other results which are reminiscent of facts concerning the asymptotic behaviour of an ideal I of R relative to an Artinian module N over R.

Throughout the remainder of this paper, A will denote a (non-trivial) commutative Noetherian ring with identity; the symbol R will denote a commutative ring (with identity) which is not necessarily Noetherian. We shall use the notation and terminology of [7] and [8] concerning secondary representations and attached prime ideals. We shall also use the notation $\mathrm{Occ}(E)$ of [7, Section 2] in connection with an injective A-module E: this is explained as follows. By well-known work of Matlis and Gabriel, there is a family $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda}$ of prime ideals of A for which $E\cong\bigoplus_{\alpha\in\Lambda} E(A/\mathfrak{p}_\alpha)$ (we use $E(L)$ to denote the injective envelope of an A-module L), and the set $\{\mathfrak{p}_\alpha:\alpha\in\Lambda\}$ is uniquely determined by E: we denote it by $\mathrm{Occ}(E)$.

2. Secondary representations for certain modules over a commutative Noetherian ring

The purpose of this section is to show that, if E is an injective A-module and M is a finitely generated A-module, then $\mathrm{Hom}_A(M,E)$ always has a secondary representation; we shall also give a precise description of the set $\mathrm{Att}_A(\mathrm{Hom}_A(M,E))$ in terms of $\mathrm{Ass}_A(M)$ and $\mathrm{Occ}(E)$. The results of this section follow easily from work in [7].

Theorem 2.1. *Let E be an injective module over our commutative Noetherian ring A, and let M be a finitely generated A-module. Then $\mathrm{Hom}_A(M,E)$ has a secondary representation, and, furthermore,*

$$\mathrm{Att}_A(\mathrm{Hom}_A(M,E))=\{\mathfrak{p}'\in\mathrm{Ass}_A(M):\mathfrak{p}'\subseteq\mathfrak{p}\text{ for some }\mathfrak{p}\in\mathrm{Occ}(E)\}.$$

Proof. In view of our conventions, the claims are clear when $M=0$, and so we shall assume that $M\neq 0$. Let $0=\bigcap_{i=1}^n Q_i$ be a minimal (that is normal) primary decomposition for the zero submodule of M, and, for each $i=1,\dots,n$, let $\pi_i\colon M\to M/Q_i$ denote the natural epimorphism. Denote $\sqrt{(Q_i:M)}$ by $\mathfrak{p}_i$ (for $i=1,\dots,n$), so that $\mathrm{Ass}_A(M)=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$.

Let T denote the exact, additive, contravariant, A-linear functor $\mathrm{Hom}_A(\ ,E)$ from the category of all A-modules and A-homomorphisms to itself. For each $i=1,\dots,n$, set

$$S_i=T(\pi_i)T(M/Q_i)=\mathrm{Hom}_A(\pi_i,E)\,\mathrm{Hom}_A(M/Q_i,E),$$

a submodule of $T(M)=\mathrm{Hom}_A(M,E)$. It follows from [7, 3.2] that S_i is either zero or $\mathfrak{p}_i$-secondary (for $1\leqq i\leqq n$), and that

$$T(M)=\mathrm{Hom}_A(M,E)=S_1+S_2+\cdots+S_n.$$

Thus $\mathrm{Hom}_A(M,E)$ has a secondary representation, and, if it is non-zero, we can arrive at a reduced secondary representation for it by first deleting any zero terms from the right

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hand side of the above equation, and then deleting redundant terms one at a time. Thus our proof will be complete once we have shown, for an integer j between 1 and n, that (i) if, for every $\mathfrak{p}\in\mathrm{Occ}(E)$, we have $\mathfrak{p}_j\nsubseteq\mathfrak{p}$, then $\mathrm{Hom}_A(M/Q_j,E)=0$, and (ii) if $\mathfrak{p}_j\subseteq\mathfrak{p}'$ for some $\mathfrak{p}'\in\mathrm{Occ}(E)$, then

$$\sum_{\substack{i=1\\ i\neq j}}^{n} S_i\neq \mathrm{Hom}_A(M,E).$$

We now prove these two claims in turn. We let $(\mathfrak{p}_\alpha)_{\alpha\in\Lambda}$ be a family of prime ideals of A for which $E\cong\bigoplus_{\alpha\in\Lambda}E(A/\mathfrak{p}_\alpha)$.

(i) Since M/Q_j is finitely generated,

$$\mathrm{Hom}_A(M/Q_j,E)\cong\bigoplus_{\alpha\in\Lambda}\mathrm{Hom}_A(M/Q_j,E(A/\mathfrak{p}_\alpha)).$$

We therefore show that $\mathrm{Hom}_A(M/Q_j,E(A/\mathfrak{p}_\alpha))=0$ for each $\alpha\in\Lambda$. Now, for such an α, we have $\mathfrak{p}_j\nsubseteq\mathfrak{p}_\alpha$: choose $a\in\mathfrak{p}_j\backslash\mathfrak{p}_\alpha$. Then there exists $h\in\mathbb{N}$ such that $a^h\in(0:M/Q_j)$. But multiplication by a^h provides an automorphism of $E(A/\mathfrak{p}_\alpha)$, by [**4**, Lemma 3.2(2)]. It therefore follows from the A-linearity of $\mathrm{Hom}_A(\ ,\ )$ that $\mathrm{Hom}_A(M/Q_j,E(A/\mathfrak{p}_\alpha))=0$. Thus point (i) has been established.

(ii) Set $K_j=\bigcap_{i\in J}Q_i$, where $J=\{1,\dots,j-1,j+1,\dots,n\}$. It follows from [**7**, 3.4] that $\mathrm{Hom}_A(M,E)=\sum_{i\in J}S_i$ if and only if $\mathrm{Hom}_A(K_j,E)=0$, and so it is enough for us to show that, if $\mathfrak{p}_j\subseteq\mathfrak{p}_\alpha$ for some $\alpha\in\Lambda$, then $\mathrm{Hom}_A(K_j,E(A/\mathfrak{p}_\alpha))\neq0$.

Now

$$K_j=K_j/0=K_j/(K_j\cap Q_j)\cong(K_j+Q_j)/Q_j,$$

so that $\mathrm{Ass}(K_j)=\{\mathfrak{p}_j\}$. Hence K_j has a submodule isomorphic to $A/\mathfrak{p}_j$, and it is easy to use the injective property of $E(A/\mathfrak{p}_\alpha)$ in conjunction with the natural, non-zero, epimorphism $A/\mathfrak{p}_j\to A/\mathfrak{p}_\alpha$ to deduce that $\mathrm{Hom}_A(K_j,E(A/\mathfrak{p}_\alpha))\neq0$. This completes the proof of point (ii).

The result is now proved.

Corollary 2.2. *Let E be an injective module over our commutative Noetherian ring A, and let $\mathfrak{a}$ be a proper ideal of A. Then $(0:_E\mathfrak{a})$ has a secondary representation, and*

$$\mathrm{Att}_A(0:_E\mathfrak{a})=\{\mathfrak{p}'\in\mathrm{ass}\,\mathfrak{a}:\mathfrak{p}'\subseteq\mathfrak{p}\ \textit{for some}\ \mathfrak{p}\in\mathrm{Occ}(E)\}.$$

Proof. This is an immediate consequence of 2.1 because $\mathrm{Hom}_A(A/\mathfrak{a},E)\cong(0:_E\mathfrak{a})$ and $\mathrm{Ass}_A(A/\mathfrak{a})=\mathrm{ass}\,\mathfrak{a}$.

3. The results

To provide some motivation for our results, let us first consider an Artinian module N over the commutative ring R, and let I be an ideal of R. Every Artinian R-module possesses a secondary representation. In [**8**, Theorem 3.1], it was shown that both the sequences of sets

$$(\mathrm{Att}_R(0:_N I^n))_{n\in\mathbb{N}} \quad \text{and} \quad (\mathrm{Att}_R((0:_N I^{n+1})/(0:_N I^n)))_{n\in\mathbb{N}}$$

are ultimately constant; let $\mathrm{At}^*(I,N)$ and $\mathrm{Bt}^*(I,N)$ (respectively) denote their ultimate constant values. Those results are, in a sense, dual to results which follow quickly from the work of Brodmann which was mentioned in the Introduction. It is easy to see (by, for example, [3, (2.4)]) that $\mathrm{Bt}^*(I,N)\subseteq \mathrm{At}^*(I,N)$. The second author proved in [**9**, Theorem (4.3)] that

$$\mathrm{At}^*(I,N)\backslash \mathrm{Bt}^*(I,N)\subseteq \mathrm{Att}_R(N),$$

and this result can be viewed as dual to a (natural generalization of a) result of McAdam and Eakin [**5**, Corollary 13]: see [**9**, Theorem (4.2)].

Now consider an injective module E over our commutative Noetherian ring A, and let $\mathfrak{a}$ be an ideal of A. It follows from 2.2 that $(0:_E \mathfrak{a}^n)$ has a secondary representation (for every $n\in\mathbb{N}$), and so, by [**3**, (2.4)], the A-module

$$(0:_E \mathfrak{a}^{n+1})/(0:_E \mathfrak{a}^n)$$

has a secondary representation (for every $n\in\mathbb{N}$); we show below that the obvious analogues of the above-mentioned results for N and I hold for E and $\mathfrak{a}$.

We shall use the following additional notation. For a finitely generated A-module M, we shall use $\mathrm{As}^*(\mathfrak{a},M)$ and $\mathrm{Bs}^*(\mathfrak{a},M)$ to denote the ultimate constant values of the sequences

$$(\mathrm{Ass}_A(M/\mathfrak{a}^n M))_{n\in\mathbb{N}} \quad \text{and} \quad (\mathrm{Ass}_A(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M))_{n\in\mathbb{N}}$$

respectively: see [**1**].

Theorem 3.1. *Let E be an injective module over our commutative Noetherian ring A, and let $\mathfrak{a}$ be an ideal of A. Then the sequence of sets $(\mathrm{Att}_A(0:_E \mathfrak{a}^n))_{n\in\mathbb{N}}$ is ultimately constant. We denote its ultimate constant value by $\mathrm{At}^*(\mathfrak{a},E)$. In fact,*

$$\mathrm{At}^*(\mathfrak{a},E)=\{\mathfrak{p}'\in \mathrm{As}^*(\mathfrak{a},A):\mathfrak{p}'\subseteq\mathfrak{p} \textit{ for some } \mathfrak{p}\in \mathrm{Occ}(E)\}.$$

Proof. By 2.2,

$$\mathrm{Att}_A(0:_E \mathfrak{a}^n)=\{\mathfrak{p}'\in \mathrm{ass}\,\mathfrak{a}^n:\mathfrak{p}'\subseteq\mathfrak{p} \text{ for some } \mathfrak{p}\in \mathrm{Occ}(E)\}.$$

However, by Brodmann's work in [1], we have $\operatorname{ass} \mathfrak{a}^n = \operatorname{As}^*(\mathfrak{a}, A)$ for all sufficiently large integers n, and the result follows immediately.

Theorem 3.2. *Let E be an injective module over our commutative Noetherian ring A, and let $\mathfrak{a}$ be an ideal of A. Then the sequence of sets*

$$(\operatorname{Att}_A((0:_E \mathfrak{a}^{n+1})/(0:_E \mathfrak{a}^n)))_{n\in\mathbb{N}}$$

is ultimately constant. We denote its ultimate constant value by $\operatorname{Bt}^*(\mathfrak{a}, E)$. In fact,

$$\operatorname{Bt}^*(\mathfrak{a}, E) = \{\mathfrak{p}' \in \operatorname{Bs}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \textit{ for some } \mathfrak{p} \in \operatorname{Occ}(E)\}.$$

Proof. We use a similar method to that employed above for 3.1. This time, we use, for $n \in \mathbb{N}$, the canonical short exact sequence

$$0 \to \mathfrak{a}^n/\mathfrak{a}^{n+1} \to A/\mathfrak{a}^{n+1} \to A/\mathfrak{a}^n \to 0,$$

the injective property of E, and the natural isomorphisms

$$\operatorname{Hom}_A(A/\mathfrak{a}^j, E) \xrightarrow{\cong} (0:_E \mathfrak{a}^j) \quad (j = n, n+1)$$

to deduce that $(0:_E \mathfrak{a}^{n+1})/(0:_E \mathfrak{a}^n) \cong \operatorname{Hom}_A(\mathfrak{a}^n/\mathfrak{a}^{n+1}, E)$. It therefore follows from [1] and 2.1 that, for all sufficiently large integers n,

$$\operatorname{Att}_A((0:_E \mathfrak{a}^{n+1})/(0:_E \mathfrak{a}^n)) = \{\mathfrak{p}' \in \operatorname{Bs}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \operatorname{Occ}(E)\}.$$

The claims of the theorem now follow immediately.

Corollary 3.3. *Let E be an injective module over our commutative Noetherian ring A, and let $\mathfrak{a}$ be an ideal of A. Then, with the notation of* 3.1 *and* 3.2,

$$\operatorname{Bt}^*(\mathfrak{a}, E) \subseteq \operatorname{At}^*(\mathfrak{a}, E) \quad \textit{and} \quad \operatorname{At}^*(\mathfrak{a}, E) \backslash \operatorname{Bt}^*(\mathfrak{a}, E) \subseteq \operatorname{Att}_A(E).$$

Proof. It follows from [3, (2.4)] that $\operatorname{Bt}^*(\mathfrak{a}, E) \subseteq \operatorname{At}^*(\mathfrak{a}, E)$. It is immediate from 3.1 and 3.2 that $\operatorname{At}^*(\mathfrak{a}, E) \backslash \operatorname{Bt}^*(\mathfrak{a}, E)$ is equal to the set

$$\{\mathfrak{p}' \in \operatorname{As}^*(\mathfrak{a}, A) \backslash \operatorname{Bs}^*(\mathfrak{a}, A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \operatorname{Occ}(E)\}.$$

But, by [5, Corollary 13], $\operatorname{As}^*(\mathfrak{a}, A) \backslash \operatorname{Bs}^*(\mathfrak{a}, A) \subseteq \operatorname{Ass}_A(A)$. Hence

$$\operatorname{At}^*(\mathfrak{a}, E) \backslash \operatorname{Bt}^*(\mathfrak{a}, E) \subseteq \{\mathfrak{p}' \in \operatorname{Ass}_A(A) : \mathfrak{p}' \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \operatorname{Occ}(E)\}.$$

However, the right hand set in the above display is, by [7, Theorem 2.6], just $\operatorname{Att}_A(E)$, and so the proof is complete.

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