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Ramanujan Type Buildings

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Abstract. We will construct a finite union of finite quotients of the affine building of the group GL_3 over the field of p-adic numbers $\mathbb{Q}_p$. We will view this object as a hypergraph and estimate the spectrum of its underlying graph.

1 Introduction

For a finite regular graph, the eigenvalue λ of the adjacency matrix which has the second largest absolute value is of particular importance in estimating different invariants of the graph such as the girth, the independence number and the expansion coefficient. A large expansion coefficient is determined by a small $|\lambda|$ as shown in [18]. Lubotsky, Phillips and Sarnak, in [18], have constructed a family of expander graphs called Ramanujan graphs. Asymptotically, their graphs have the smallest possible $|\lambda|$. They are called Ramanujan graphs because all eigenvalues, except the largest (in absolute value), satisfy Ramanujan's conjecture.

The purpose of this article is to construct yet another family of graphs with small $|\lambda|$. We start with the affine building of the general linear group in three variables GL_3 over the field of p-adic numbers $\mathbb{Q}_p$, where p is an odd prime. Then we find a finite number of discrete co-compact arithmetic subgroups $\Gamma_{i,p}$ of $\mathrm{GL}_3(\mathbb{Q}_p)$ which act without fixed points on the vertices of the building of $\mathrm{GL}_3(\mathbb{Q}_p)$. The finite building quotient of $\Gamma_{i,p}\backslash \mathrm{GL}_3(\mathbb{Q}_p)$, for each i, is a hypergraph whose underlying graph is finite and regular. The adjacency operator of each of these graphs is the sum of the generators of the Hecke algebra of $\mathrm{GL}_3(\mathbb{Q}_p)$ with respect to $\mathrm{GL}_3(\mathbb{Z}_p)$ as shown in [3].

We consider the union of these graphs and give an estimate of its spectrum via the representation theory of the Hecke algebra. The one-dimensional representations of the Hecke algebra correspond to the unramified representations of $\mathrm{GL}_3(\mathbb{Q}_p)$. We consider a $\mathbb{Q}$-form G' of the unitary group in three variables $U(3)$ which is isomorphic to $\mathrm{GL}_3(\mathbb{Q}_p)$ at the place p. The local components of the automorphic representations of G' are unramified at almost all places. Rogawski [19] partitions the set of local representations of G' into finite sets called L-packets. This partition is obtained based on properties of characters. From the local packets, Rogawski defines global packets. Our estimation of the eigenvalues of the Hecke algebra depends on the different kinds of global packets of $U(3)$.

Rogawski's partition of the set of representations of $U(3)$ into finite packets matches Arthur's conjectural parameterization of the packets by parameters of the hypothetical Langlands group [1], [16]. We view Rogawski's results in the language of the Arthur parameters and use them to prove the main theorem which gives the estimation for the spectrum of the constructed hypergraphs.

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2 Combinatorics

The basic notations and definitions about graphs are taken from [4]. A *graph* G is a pair of sets $\big(V(G), E(G)\big)$ such that $E(G) \subset \{Y : Y \subset V(G), |Y| = 2\}$ and $V(G) \neq \varnothing$. The set $V(G)$ is the set of *vertices* of G and $E(G)$ is the set of *edges* of G. The vertices x and y are said to be *adjacent* if $\{x, y\}$ is an edge. The number of vertices adjacent to x is denoted by $d(x)$ and is said to be the *degree* of x. If every vertex of G has degree s, then G is said to be *s-regular*. If G is a graph with a finite number of vertices $\{x_1, \dots, x_n\}$, the *adjacency matrix* $\delta = (\delta_{ij})$ of G is the $n \times n$ matrix with entries δ_{ij} equal to 1 if x_i is adjacent to x_j and 0 otherwise. We will also denote by δ the adjacency operator on $L^2(V(G))$ whose matrix is the adjacency matrix.

A *hypergraph* X is a set V together with a family Σ of subsets of V. The elements of V and Σ are called respectively the *vertices* and the *faces* of the hypergraph. If $S \in \Sigma$, the *rank* of S is the cardinality $|S|$ of S and the *dimension* of S is given by $|S| - 1$. Vertices are faces of dimension 0.

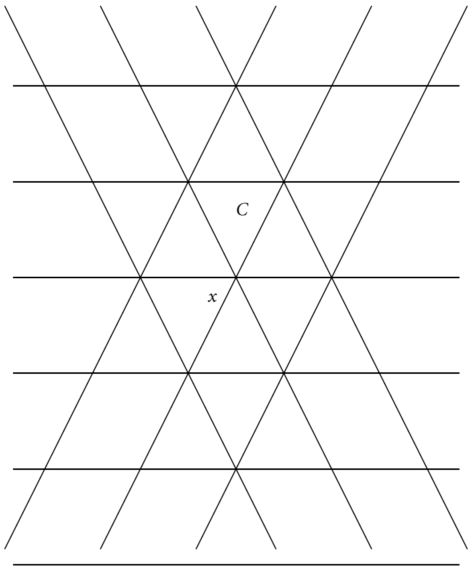
In particular, simplicial complexes are hypergraphs. Two simplices A and B will be called *joinable* if there exists another simplex C such that A and B are both faces of C. We say that two simplices are *disjoint* if their intersection is the empty simplex. The *link* of a simplex A, denoted lk_A, is a subcomplex of Σ consisting of the simplices B which are disjoint from A and which are joinable to A. We will say that a hypergraph X is *labellable* if it is a chamber complex and there exists a set I and a function which assigns to each vertex of X an element of I in such a way that the vertices of every chamber are mapped bijectively onto I. By omitting all faces of dimension higher or equal to 2 from a hypergraph X we obtain the *underlying graph* of X, denoted by $\underline{X}$.

3 Notation

Let p be an odd prime of $\mathbb{Q}$, the field of rational numbers. Denote by $\mathbb{Q}_p$ the field of p-adic numbers and by $\mathbb{Z}_p$ the ring of p-adic integers. Let $\mathbb{F}_p$ be the residue field of $\mathbb{Q}_p$. It is a finite field with p elements. For every group G we denote by $G(\mathbb{Q}_p)$ the group of $\mathbb{Q}_p$-rational points in G. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, $\mathbb{A}_f$ the ring of finite adeles defined as the restricted direct product that defines $\mathbb{A}$ but without the infinite factor, and $\mathbb{A}_f^p$ the ring of finite adeles at all places except p defined as the restricted direct product that defines $\mathbb{A}_f$ but without the term at the place p. Now let G represent the unitary group in three variables $U(3)$. For the precise definition of the group $U(3)$ see [19]. In a few instances G will denote the unitary group in two variables $U(2)$. We will clarify explicitly whenever this will be the case. Let G' be a $\mathbb{Q}$-form of $U(3)$ such that $G'(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)$ and $G'(\mathbb{R})$ is compact. Denote by K_p the maximal compact subgroup $G'(\mathbb{Z}_p) \cong \mathrm{GL}_3(\mathbb{Z}_p)$ of $G'(\mathbb{Q}_p)$. We denote by T a maximal $\mathbb{Q}_p$-split torus of GL_3, by $\mathcal{N}$ (resp. $\mathcal{Z}$) the normalizer (resp. the centralizer) of T in G. We will denote the group of characters (resp. co-characters) of T by $X^* = X^*(T) = \mathrm{Hom}_F(T, \mathrm{Mult})$ (resp. $X_* = X_*(T) = \mathrm{Hom}_F(\mathrm{Mult}, T)$). We will denote by B the Borel subgroup of GL_3 consisting of upper triangular matrices.

4 Affine Buildings and Quotients of Buildings

The basic reference for affine buildings is [26]. The reader can find a brief discussion of the building attached to GL_n in [3]. Consider the Bruhat-Tits building $\mathcal{B}$ attached to the group $\mathrm{GL}_3(\mathbb{Q}_p)$. For the definition and properties of buildings we refer the reader to [26]. The building $\mathcal{B}$ is the direct product of the building of $\mathrm{SL}_3(\mathbb{Q}_p)$ and an affine line [26]. The building of $\mathrm{SL}_3(\mathbb{Q}_p)$ is a 2-dimensional simplicial complex whose simplices are triangles. The apartments are Euclidean planes triangulated in the usual way. Below we give a picture of the apartment of $\mathrm{SL}_3(\mathbb{Q}_p)$.



The building is obtained by ramifying along every edge, each edge belonging to $p + 1$ triangles. All vertices of the building of $\mathrm{SL}_3(\mathbb{Q}_p)$ are special. Thus all vertices of $\mathcal{B}$ are special. As shown in [3], the set of vertices of $\mathcal{B}$ can be identified with the quotient group $\mathrm{GL}_3(\mathbb{Q}_p)/\,\mathrm{GL}_3(\mathbb{Z}_p)$.

In what follows we will make use of the action of the Weyl group $W = \mathcal{N}(\mathbb{Q}_p)/\mathcal{Z}(\mathbb{Q}_p)$ on $\mathcal{B}$, and in particular on the set of vertices. The group W acts trivially on the affine line and for the remainder of the discussion we will ignore the affine line when we refer to the building $\mathcal{B}$.

Now we would like to consider a suitable discrete, co-compact, arithmetic subgroup Γ_p of $G'(\mathbb{Q}_p)$ that acts totally discontinuously on $G'(\mathbb{Q}_p)/K_p$ and for which $\Gamma_p \setminus G'(\mathbb{Q}_p)/G'(\mathbb{Z}_p)$ corresponds to the vertices of a finite quotient of the building attached to $G'(\mathbb{Q}_p)$. In fact, we will find a finite number of such subgroups $\Gamma_{i,p}$ and thus we will obtain a finite disconnected union of finite quotients of the building attached to

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$G'(\mathbb{Q}_p)$. The object obtained in this way will be a finite simplicial complex with a finite number of connected components and, in particular, a finite hypergraph.

For each prime $v \neq p$, let K_v be a compact open subgroup of $G'(\mathbb{Q}_v)$ chosen to be small and such that the group

$$K_f = \prod_{q \text{ finite}} K_q$$

is a compact open subgroup of $G'(\mathbb{A}_f)$. Then the group

$$K_f^p = \prod_{\substack{q \neq p \\ q \text{ finite}}} K_q$$

is a compact open subgroup of $G'(\mathbb{A}_f^p)$. By [6, Theorem 5.1], the number of double cosets in

$$G'(\mathbb{Q}) \setminus G'(\mathbb{A})/G'(\mathbb{R})K_f$$

is finite and thus the number of double cosets in

$$G'(\mathbb{Q}) \setminus G'(\mathbb{A})/G'(\mathbb{R})G'(\mathbb{Q}_p)K_f^p$$

is finite. Let $\{x_1, \dots, x_{k'}\}$ be a set of representatives of these cosets. Then we have

$$G'(\mathbb{A}) = \bigcup_{i=1}^{k'} G'(\mathbb{Q})x_i\big(G'(\mathbb{R})G'(\mathbb{Q}_p)K_f^p\big).$$

Consider the group $\Gamma'_{i,p} = G'(\mathbb{R})G'(\mathbb{Q}_p)K_f^p \cap x_i G'(\mathbb{Q})x_i^{-1}$. It is a discrete co-compact subgroup of $G'(\mathbb{R})G'(\mathbb{Q}_p)$. Since $G'(\mathbb{R})$ is compact, the projection of $\Gamma'_{i,p}$ on $G'(\mathbb{Q}_p)$, which we also denote by $\Gamma'_{i,p}$, remains a discrete subgroup. It is not difficult to see that $\Gamma'_{i,p}$ is finitely generated. Then, according to [22, Lemma 8], $\Gamma'_{i,p}$ has a normal subgroup $\Gamma_{i,p}$ of finite index which has no nontrivial element of finite order. This implies that any element of $\Gamma_{i,p}$ different from the identity acts on $G'(\mathbb{Q}_p)/K_p$ without fixed points.

We have

$$L^2\big(G'(\mathbb{Q}) \setminus G'(\mathbb{A})/G'(\mathbb{R})K_f^p\big) \cong \bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)\big). \tag{1}$$

Denote by $\mathcal{B}_i$ the finite quotient of the building of $G'(\mathbb{Q}_p)$ by $\Gamma_{i,p}$ and by $\mathbb{B}$ the disjoint union of the $\mathcal{B}_i$'s, $i = 1, \dots, k$. For each i, the set of vertices of $\mathcal{B}_i$ can be identified with the group $\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p$.

The building $\mathcal{B}$ attached to $\mathrm{GL}_3(\mathbb{Q}_p)$ is a labellable hypergraph [7, IV.1] with labelling $\{0, 1, 2\}$. For every vertex x of $\mathcal{B}$ the link of x in $\mathcal{B}$ is canonically isomorphic to the spherical building of $\mathrm{GL}_3(\mathbb{F}_p)$ [26, 3.5.4] and the number of chambers in $\mathcal{B}$ containing x is given by

$$\frac{|\,\mathrm{GL}_3(\mathbb{F}_p)|}{|B(\mathbb{F}_p)|} = \frac{(p^3-1)(p^3-p)(p^3-p^2)}{p^3(p-1)^3} = (p^2+p+1)(p+1).$$

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Since every edge belongs to $p+1$ chambers and each of the $(p^2+p+1)(p+1)$ chambers containing x has two vertices distinct from x, it follows that x has $2(p^2+p+1)$ distinct neighbours. Therefore, the finite union of building quotients $\mathbb{B}$ is a labellable hypergraph with labelling $\{0,1,2\}$. Every vertex in $\mathbb{B}$ is special and the underlying graph of $\mathbb{B}$ is $2(p^2+p+1)$-regular.

5 The Hecke Algebra

Consider the vector space V of continuous functions on the set of vertices of $\mathbb{B}$, $V = \bigoplus_{i=1}^{k} C\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$. We denote by $\mathcal{H}_p$ the Hecke algebra of $G'(\mathbb{Q}_p)$ with respect to K_p. It is the set of complex valued, compactly supported functions on $G'(\mathbb{Q}_p)$ which are bi-invariant under K_p, endowed with the convolution given by

$$(f_1 * f_2)(g) = \int_{G'(\mathbb{Q}_p)} f_1(x) f_2(x^{-1}g)\, dx, \quad f_1, f_2 \in \mathcal{H}_p \text{ and } g \in G'(\mathbb{Q}_p).$$

The Hecke algebra $\mathcal{H}_p$ acts on V by the induced algebra representation attached to the right regular representation. We denote this action by $\star$. For $\varphi \in \mathcal{H}_p$, $f \in V$ and x a representative in $G'(\mathbb{Q}_p)$ of a coset in $\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p$, $1 \le i \le k$, the action $\star$ is given by

$$(f \star \varphi)(x) = \int_{G'(\mathbb{Q}_p)} \varphi(y) f(xy)\, dy.$$

Let $d(a_1,a_2,a_3)$ denote the 3×3 diagonal matrix with entries a_1,a_2,a_3. The Hecke algebra $\mathcal{H}_p$ is generated by the fundamental Hecke functions φ_i, $i = 0,1,2,3$, which are the characteristic functions of the double cosets $K_p t_i K_p$ with $t_0 = d(1,1,1)$, $t_1 = d(1,1,p)$, $t_2 = d(1,p,p)$, $t_3 = d(p,p,p)$.

6 The Statement of the Main Theorem

First let us introduce the following definition.

Definition 1 A building $\mathcal{B}$ with all vertices special and whose underlying graph $\underline{\mathcal{B}}$ is l-regular will be called a *Ramanujan type building with bound c* if every eigenvalue λ of the adjacency matrix of $\underline{\mathcal{B}}$ is either $\lambda = \pm l$ or $|\lambda| \le c$.

The goal of this work is to prove the following theorem.

Main Theorem

a) Each connected component of $\mathbb{B}$ is a Ramanujan type building quotient with bound $2p(p^{1/2}+1+p^{-1/2})$.
b) If G' is a compact form of $U(3)$ arising from a division algebra with an involution of the second kind [19], each connected component of $\mathbb{B}$ is a Ramanujan type building quotient with bound $6p$.

Note Even in the case in which G' arises from a split algebra, most eigenvalues of the adjacency matrix of $\underline{\mathbb{B}}$ satisfy the stronger inequality $|\lambda| \le 6p$. The eigenvalues $\lambda \neq$

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$\pm 2(p^2+p+1)$ such that $|\lambda| > 6p$ are called *exceptional eigenvalues*. They give the failure for each connected component of $\mathbb{B}$ to be a Ramanujan type building quotient with bound $6p$.

By the theorem of [3], the adjacency operator δ on the vertices of $\mathbb{B}$ is the sum of the fundamental Hecke functions φ_1 and φ_2. Since the Hecke algebra is commutative [8], in order to prove the Main Theorem we have to estimate the eigenvalues of φ_i, $i = 1, 2$.

7 The decomposition of $L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$

The group $G'(\mathbb{Q}_p)$ acts on $\bigoplus_{i=1}^k L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)\big)$ by right translations. This action gives a representation of $G'(\mathbb{Q}_p)$ which we decompose into a direct sum of irreducible representations.

Recall that the Hecke algebra acts on the vector space

$$V = \bigoplus_{i=1}^{k} C\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$$

of continuous functions defined on the set of vertices of $\mathbb{B}$. Hence, in the above decomposition, we need only observe irreducible representations that restricted to K_p have a K_p-fixed vector. These representations are the irreducible unramified representations of $G'(\mathbb{Q}_p)$ [19, 4.5]. The Hecke algebra $\mathcal{H}_p$ acts on $\bigoplus_{i=1}^k L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$ by the action $\star$ defined previously. This action gives a representation of $\mathcal{H}_p$ on this space which is decomposable into a direct sum of irreducible representations. The one-dimensional representations of $\mathcal{H}_p$ are in one-to-one correspondence with the irreducible unramified representations of $G'(\mathbb{Q}_p)$ [8, Corollary 4.2]. An eigenvalue of the Hecke algebra $\mathcal{H}_p$ is a one-dimensional representation λ of $\mathcal{H}_p$ and an eigenvalue of a particular Hecke function φ is given by the value of λ at φ. Thus in order to estimate the eigenvalues of the fundamental Hecke functions φ_1 and φ_2 we have to study the irreducible unramified representations of $G'(\mathbb{Q}_p)$ to which they correspond. We will do this by passing to the global situation and examining the automorphic representations of $G'(\mathbb{A})$.

For the present purpose, an automorphic representation of $G'(\mathbb{A})$ can be considered as an irreducible unitary representation of $G'(\mathbb{A})$ which occurs in the decomposition of the right regular representation on $L^2\big(G'(\mathbb{Q}) \setminus G'(\mathbb{A})\big)$. For the proper definition see [14]. Let π be an automorphic representation of $G'(\mathbb{A})$. It decomposes into a tensor product $\pi = \bigotimes \pi_v$ with π_v an irreducible unitary representation of $G'(\mathbb{Q}_v)$ for each finite v and $\pi_\infty = \pi_\mathbb{R}$ a irreducible unitary representation of $G'(\mathbb{R})$ [10]. The representation π_v is unramified for almost all v [14] and we can assume that the component π_p at the place p is unramified. The automorphic representations of the three-dimensional unitary group have been classified by Rogawski [19], [20].

Let us use the following notation. For a representation ρ of a group H, if V_ρ is the space of ρ and L is a subgroup of H, we denote by V_ρ^L the space of L-fixed vectors in V_ρ. If π is an automorphic representation of $G'(\mathbb{A})$, let $m(\pi)$ be the multiplicity of π in the right regular representation of $G'(\mathbb{A})$ on $L^2\big(G'(\mathbb{Q}) \setminus G'(\mathbb{A})\big)$. We set

$$\pi^p = \bigotimes_{\substack{v \neq p \\ v \text{ finite}}} \pi_v.$$

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We have the spectral decomposition

$$L^2\big(G'(\mathbb{Q})\setminus G'(\mathbb{A})/G'(\mathbb{R})K_f^p\big) = \bigoplus_{\pi} m(\pi)(V_{\pi_p} \otimes V_{\pi_{\mathbb{R}}\pi^p}^{G'(\mathbb{R})K_f^p})$$
$$= \bigoplus_{\{\pi=\pi_{\mathbb{R}}\pi_p\pi^p:\pi_{\mathbb{R}}=1\}} m(\pi)(V_{\pi_p} \otimes V_{\pi^p}^{K_f^p}).$$

The space $V_{\pi^p}^{K_f^p}$ is finite dimensional and from the spectral decomposition

$$L^2\big(\Gamma_{i,p}\setminus G'(\mathbb{Q}_p)\big) = \bigoplus_{\pi_p} m(\pi_p,\Gamma_{i,p})V_{\pi_p}$$

it follows that

$$\sum_{i=1}^{k} m(\pi_p,\Gamma_{i,p}) = \sum_{\{\pi=\pi_{\mathbb{R}}\pi_p\pi^p:\pi_{\mathbb{R}}=1\}} m(\pi)\cdot \dim(V_{\pi^p}^{K_f^p}).$$

The observations above lead to the following theorem.

Theorem 1 *We have*

$$\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p}\setminus G'(\mathbb{Q}_p)/K_p\big) \cong \bigoplus_{\pi_p \text{ unramified}} \Big(\sum_{\{\pi=\pi_{\mathbb{R}}\pi_p\pi^p:\pi_{\mathbb{R}}=1\}} m(\pi)\cdot \dim(V_{\pi^p}^{K_f^p})\Big) V_{\pi_p}^{K_p}. \qquad \blacksquare$$

8 Spherical Functions and Unramified Representations

The decomposition of Theorem 1 provides a decomposition of $\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p}\setminus G'(\mathbb{Q}_p)/K_p\big)$ into eigenspaces of the Hecke algebra $\mathcal{H}_p$ as we shall explain in this section. First we have the following lemma.

Lemma 1 *For an unramified representation of $G'(\mathbb{Q}_p)$ the subspace invariant under $K_p = G'(\mathbb{Z}_p)$ is at most one-dimensional.*

Proof [8, 4.4]. $\blacksquare$

Let now $I_{K_pgK_p}$ be the characteristic function of K_pgK_p, with $g \in G'(\mathbb{Q}_p)$. For every function α on $G'(\mathbb{Q}_p)$ which is bi-invariant under K_p,

$$\omega(f) = \int_{G'(\mathbb{Q}_p)} f(g)\alpha(g)\,dg \quad \text{for } f \in \mathcal{H}_p$$

defines an element in the dual to $\mathcal{H}_p$. Conversely, for every element ω in the dual to $\mathcal{H}_p$,

$$\alpha(g) = \omega(I_{K_pgK_p})\Big/\int_{K_pgK_p} dg' \quad \text{for } g \in G'(\mathbb{Q}_p)$$

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is a function on $G'(\mathbb{Q}_p)$, bi-invariant under K_p. This defines an isomorphism of the dual to the space $\mathcal{H}_p$ onto the space of functions on $G'(\mathbb{Q}_p)$, bi-invariant under K_p [26].

Definition 2 A zonal spherical function of $G'(\mathbb{Q}_p)$ with respect to K_p is a function α on $G'(\mathbb{Q}_p)$, bi-invariant under K_p, such that $\alpha(1) = 1$ and satisfying the equivalent properties:

(a) ω is a homomorphism of algebras from $\mathcal{H}_p$ to $\mathbb{C}$.
(b) One has $\alpha(g_1)\alpha(g_2) = \int_{K_p} \alpha(g_1 k g_2)\, dk$, for $g_1, g_2 \in G'$.
(c) For any function f in $\mathcal{H}_p$, there exists a constant $\lambda_\alpha(f)$ such that $f * \alpha = \alpha * f = \lambda_\alpha(f) \cdot \alpha$.

For the proof of the equivalence of the properties (a), (b), and (c) we refer the reader to [8]. Property (b) gives a functional equation for zonal spherical functions.

If α is a zonal spherical function on $G'(\mathbb{Q}_p)$ with respect to K_p, denote by V_α the space of functions f on $G'(\mathbb{Q}_p)$ of the form

$$f(g) = \sum_{i=1}^{n} c_i \alpha(g g_i)$$

for $c_1, \dots, c_n \in \mathbb{C}$ and $g_1, \dots, g_n \in G'(\mathbb{Q}_p)$. Let $G'(\mathbb{Q}_p)$ operate on V_α by right translations:

$$\big(\pi_\alpha(g) f\big)(g_1) = f(g_1 g), \quad \text{for } f \in V_\alpha \text{ and } g, g_1 \in G'(\mathbb{Q}_p).$$

Then (π_α, V_α) is an irreducible unramified representation of $G'(\mathbb{Q}_p)$ [8, 4.4].

By [8, Theorem 4.3], given an irreducible unramified representation (π, V) of $G'(\mathbb{Q}_p)$, there exists a unique zonal spherical function α on $G'(\mathbb{Q}_p)$ with respect to K_p such that (π, V) is isomorphic to (π_α, V_α). Thus we have a one-to-one correspondence between the set of irreducible unramified representations of $G'(\mathbb{Q}_p)$ and the set of zonal spherical functions of $G'(\mathbb{Q}_p)$ with respect to K_p.

Given an irreducible unramified representation (π_p, V_{π_p}) of $G'(\mathbb{Q}_p)$, we denote by α_{π_p} the corresponding zonal spherical function. The zonal spherical function α_{π_p} satisfies the following property:

$$\alpha_{\pi_p}(g) = \big(\pi_p(g) v, v\big) \quad \text{for } v \in V_{\pi_p}^{K_p} \text{ with } (v, v) = 1.$$

By condition (c) in the definition of zonal spherical functions, for every $\varphi \in \mathcal{H}_p$ there exists a constant $\lambda_{\alpha_{\pi_p}}(\varphi)$ such that $\alpha_{\pi_p} * \varphi = \lambda_{\alpha_{\pi_p}}(\varphi)\alpha_{\pi_p}$. Thus α_{π_p} is an eigenfunction for φ with eigenvalue $\lambda_{\alpha_{\pi_p}}(\varphi)$. Both α_{π_p} and $\lambda_{\alpha_{\pi_p}}(\varphi)$ are determined by π_p. By Lemma 1, the space $V_{\alpha_{\pi_p}}^{K_p}$ is one-dimensional and thus $V_{\alpha_{\pi_p}}^{K_p}$ is an eigenspace for $\mathcal{H}_p$ with eigenvalue given by the one-dimensional representation

$$\varphi \mapsto \lambda_{\alpha_{\pi_p}}(\varphi), \quad \varphi \in \mathcal{H}_p.$$

Thus, from the decomposition of $\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$ into a direct sum of irreducible unramified representations in Theorem 1, we obtain a decomposition of this space into a direct sum of eigenspaces of the Hecke algebra $\mathcal{H}_p$.

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Our goal is now to estimate the eigenvalues $\lambda_{\alpha_{\pi_p}}(\varphi_i)$ of the fundamental Hecke functions φ_i, $i = 1, 2$, for each unramified representation π_p in the decomposition of

$$\bigoplus_{i=1}^{k} \left(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\right).$$

9 The Eigenvalues of the Fundamental Hecke Functions

In the case of GL(n), Tate [24] calculates the Fourier transform for the generators of the Hecke algebra and uses the Fourier transforms to calculate, for each generator, the eigenvalues determined by the different unramified representations. We restrict our attention to the group GL(3).

Let A be the subgroup of diagonal matrices, B the subgroup of upper triangular matrices, N the subgroup of unipotent upper triangular matrices and K a maximal compact subgroup of $\mathrm{GL}_3(\mathbb{Q}_p)$. The Iwasawa decomposition, $\mathrm{GL}_3(\mathbb{Q}_p) = BK = ANK$, holds. Let $a = d(a_1, a_2, a_3) \in A$ and set $\Delta(a) = \prod_{i<j}(1 - \frac{a_j}{a_i})$, $\delta(a) = \prod_{i<j} \frac{a_j}{a_i}$, and $D(a) = \|\delta(a)\|^{-1/2}\|\Delta(a)\|$. Let f be a continuous function on $\mathrm{GL}_3(\mathbb{Q}_p)$, compactly supported and bi-invariant under K.

Definition 3 The Harish transform Hf of f is defined by

$$Hf(a) = D(a) \int_{A\setminus G} f(g^{-1}ag)\, d\bar{g} = \|\delta(a)\|^{-1/2} \int_N f(an)\, dn.$$

Thus H maps compactly supported functions on $K \setminus G/K$ to functions on A. The second expression is valid for all a, the first only for regular a.

Note Elsewhere in the literature the Harish transform is referred to as the Satake transformation.

Proposition 1 *The Harish transform satisfies the following properties:*

(a) Hf has compact support.
(b) $H(f * g) = (Hf) * (Hg)$.
(c) Hf is invariant under conjugation by the Weyl group $W \cong S_3$.

Proof [24]. ■

Recall that an uniformizer of $\mathbb{Q}_p$ is equal to p. In view of Proposition 1, the Harish transform H gives a homomorphism of the Hecke algebra $\mathcal{H}_p$ into the algebra of symmetric Laurent polynomials in 3 variables,

$$f \longrightarrow \hat{f}(\underline{x}) = \sum_{\underline{m}\in\mathbb{Z}^3} (Hf)([p^{\underline{m}}])\underline{x}^{\underline{m}} = \sum_{m_1\in\mathbb{Z}} \sum_{m_2\in\mathbb{Z}} \sum_{m_3\in\mathbb{Z}} (Hf)([p^{\underline{m}}])x_1^{m_1} \cdot x_2^{m_2} \cdot x_3^{m_3},$$

where $[p^{\underline{m}}]$ denotes the diagonal matrix $d(p^{m_1}, p^{m_2}, p^{m_3})$.

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Let σ_i be the i-th symmetric function. Tate proves the following two results:

Theorem 2 *The homomorphism $f \to \hat{f}$ is an isomorphism of the Hecke algebra $\mathcal{H}_p$ onto the subalgebra*

$$\mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]^W = \mathbb{C}[\sigma_1, \sigma_2, \sigma_3, \sigma_3^{-1}]$$

of $\mathbb{C}[x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}]$ consisting of the invariants of the Weyl group.

Theorem 3 *Let $\chi = (\chi_1, \chi_2, \chi_3)$ be a 3-tuple of unramified characters of $\mathbb{Q}_p^*$ and let*

$$V_\chi = \mathrm{Ind}_B^{\mathrm{GL}_3(\mathbb{Q}_p)}(\chi_1|\ |, \chi_2, \chi_3|\ |^{-1}).$$

Then $(V_\chi)^K$ is of dimension 1 and $\mathcal{H}_p$ acts on it by

$$f \longrightarrow \lambda_f = \hat{f}\big(\chi_1(p), \chi_2(p), \chi_3(p)\big).$$

In particular, the i-th fundamental Hecke function φ_i, $i = 0, 1, 2, 3$, has eigenvalue

$$p^{\frac{1}{2}i(3-i)}\sigma_i\big(\chi_1(p), \chi_2(p), \chi_3(p)\big).$$

We will refer to Theorem 3 as Tate's Theorem.

It is the last part of the latter theorem that will be especially useful to us. Given an unramified representation of $\mathrm{GL}_3(\mathbb{Q}_p)$, which is induced from a character $\chi = (\chi_1, \chi_2, \chi_3)$, Theorem 3 will provide us with the corresponding eigenvalue of φ_i, $i = 1, 2$.

10 Characters of the Hecke Algebra and Semisimple Conjugacy Classes

Again, let p be a place at which $G'(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)$. We denote by Γ the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, where $\bar{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. The Frobenius automorphism of $\bar{\mathbb{F}}_p$ is given by $f(x) = x^p$, $x \in \bar{\mathbb{F}}_p$, where $\bar{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. Define the Weil group $W_{\mathbb{F}_p}$ for $\mathbb{F}_p$ to be the infinite cyclic subgroup of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ generated by f. We would like to introduce the Weil group for $\mathbb{Q}_p$. For the detailed definition of the Weil group see [25]. The maximal ideal of the ring of integers $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is a principal ideal generated by p. We have $\mathbb{F}_p = \mathbb{Z}_p/(p)$. Viewed as an ideal in the ring of integers $\bar{\mathbb{Z}}_p$ of $\bar{\mathbb{Q}}_p$, (p) is also a maximal $\bar{\mathbb{Z}}_p$-ideal [9]. There is a short exact sequence

$$1 \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{un}) \longrightarrow \Gamma \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1$$

where $\mathbb{Q}_p^{un}$ is the compositum of all finite unramified extensions of $\mathbb{Q}_p$. The map

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$$

is defined in the usual way: $\sigma \longmapsto \bar{\sigma}$, where

$$\bar{\sigma}\big(x + (p)\big) = \sigma(x) + (p), \quad x \in \bar{\mathbb{Z}}_p.$$

Note that the extension of the p-adic valuation to a valuation on $\bar{\mathbb{Q}}_p$ is unique and thus $|\sigma(x)|_p = |x|_p$. As an abstract group, the Weil group $W_{\mathbb{Q}_p}$ for $\mathbb{Q}_p$ is defined as the inverse image of $W_{\mathbb{F}_p}$ in $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ under the map defined above. We will call an element Φ of the Weil group $W_{\mathbb{Q}_p}$ a *Frobenius element* if it maps onto f^{-1}.

For any group $\underline{G}$, we denote by $\hat{\underline{G}}$ the dual group of $\underline{G}$ and by ${}^L\underline{G}$ the L-group of G. For the exact definitions we refer the reader to [2], [5]. The dual group $\hat{\underline{G}}$ is a complex, connected, reductive group. The L-group ${}^L\underline{G}$ is the semidirect product

$$ {}^L\underline{G} = \hat{\underline{G}} \rtimes W_{\mathbb{Q}_p}. $$

The group $W_{\mathbb{Q}_p}$ acts on $\hat{\underline{G}}$ through its projection on $\mathrm{Gal}(L/\mathbb{Q}_p) \cong \Gamma/\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$, where $L/\mathbb{Q}_p$ is any finite Galois extension over which $\underline{G}$ splits. The action of $\mathrm{Gal}(L/\mathbb{Q}_p)$ on $\hat{\underline{G}}$ is determined in a canonical way, up to inner automorphism, from the action of the Galois group on the Dynkin diagram of $\underline{G}$.

Let T denote the maximal $\mathbb{Q}_p$-split torus of $\mathrm{GL}_3(\mathbb{Q}_p)$ consisting of the diagonal matrices in $\mathrm{GL}_3(\mathbb{Q}_p)$. Let $\hat{T}$ be the dual torus to T. Let us identify the Weyl group W of $\mathrm{GL}_3(\mathbb{Q}_p)$ with the Weyl group $W\big(\widehat{\mathrm{GL}}_3(\mathbb{Q}_p), \hat{T}\big)$ of the dual group. The Harish transform provides a canonical identification

$$ \mathcal{H}_p \xrightarrow{\sim} \mathbb{C}[\hat{T}]^W $$

by means of Theorem 2. By [5, 6.6], $\mathbb{C}[\hat{T}]^W$ is equal to the group algebra $\mathbb{C}[\hat{T}/W]$ of the quotient $\hat{T}/W$. Therefore, we have a canonical identification of $\hat{T}/W$ with the characters of $\mathcal{H}_p$. By [5, 6.7] we obtain a canonical bijection between the characters of $\mathcal{H}_p$ and the semisimple conjugacy classes in ${}^L\mathrm{GL}_3(\mathbb{Q}_p)$ of the form $g \rtimes \Phi$ with $g \in \widehat{\mathrm{GL}}_3(\mathbb{Q}_p)$. Furthermore, each such class can be represented by an element of the form (t, Φ), with $t \in \hat{T}$.

Given the unramified representation (π, V_π) of $\mathrm{GL}_3(\mathbb{Q}_p)$, which is induced from the unramified character χ of T, by Theorem 3, $\mathcal{H}_p$ acts on the one-dimensional space V_π^K via the character λ of $\mathcal{H}_p$ given by $f \mapsto \lambda_f = \hat{f}\big(\chi_1(p), \chi_2(p), \chi_3(p)\big)$. To this character of $\mathcal{H}_p$ we assign a semisimple conjugacy class in $\widehat{\mathrm{GL}}_3(\mathbb{Q}_p) \rtimes \Phi$, as explained previously. We obtain in this way a correspondence between the set of unramified representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ and the set of semisimple conjugacy classes in $\widehat{\mathrm{GL}}_3(\mathbb{Q}_p) \rtimes \Phi$.

The semisimple conjugacy class represented by (t, Φ) with $t = d(t_1, t_2, t_3)$ determines an unramified character χ_t of T by

$$ \chi_t\big(d(a_1, a_2, a_3)\big) = t_1^{\mathrm{ord}_p(a_1)} t_2^{\mathrm{ord}_p(a_2)} t_3^{\mathrm{ord}_p(a_3)}, $$

for $d(a_1, a_2, a_3) \in T$.

11 L-Packets on the Unitary Group in Three Variables

At this point we will briefly present the theory of L-packets on the unitary group in three variables following Rogawski's book [19].

Notation We keep the notation of the previous section. As before, p is a place of $\mathbb{Q}$ at which $G'(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)$. If v is a place of $\mathbb{Q}$, we set $G_v = G(\mathbb{Q}_v)$. In addition, $\mathbf{G}$ will denote the group of $\mathbb{A}$-points of G. Let $Z(G)$ be the center of G. As before, $W_{\mathbb{Q}_p}$ denotes the

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absolute Weil group of $\mathbb{Q}_p$, and if $L/\mathbb{Q}_p$ is a finite Galois extension, $W_{L/\mathbb{Q}_p}$ will denote the Weil group of $L/\mathbb{Q}_p$ [25]. If $L/\mathbb{Q}_p$ is a Galois extension, let $\Gamma(L/\mathbb{Q}_p)$ be the Galois group.

Let $E/\mathbb{Q}_p$ be the quadratic extension defining the unitary group [19]. If $x \in E$, we write $\bar{x}$ for the conjugate of x in E with respect to $\mathbb{Q}_p$. Let $I_{\mathbb{Q}_p}$ denote the ideles of $\mathbb{Q}_p$. The character of order two of $I_{\mathbb{Q}_p}$ associated to $E/\mathbb{Q}_p$ by class field theory will be denoted by $\omega_{E/\mathbb{Q}_p}$. Denote by C_E and $C_{\mathbb{Q}_p}$ the idele classes of E and $\mathbb{Q}_p$, respectively. Fix a character μ of C_E whose restriction to $C_{\mathbb{Q}_p}$ is $\omega_{E/\mathbb{Q}_p}$.

All measures on groups are assumed to be Haar measures.

Representations All local representations of G are assumed to be admissible and we identify a representation with its isomorphism class. If π is a representation, let $JH(\pi)$ be the set of irreducible constituents of π and let $E(G)$ be the set of irreducible admissible representations of G. Rogawski [19] defines a packet structure on the set $E(G)$ in terms of properties of characters, and this structure conjecturally matches that given by the Langlands conjecture. He partitions $E(G)$ into finite subsets called *L-packets*. The set of L-packets on G will be denoted by $\Pi(G)$. An L-packet Π will be called square integrable (resp. supercuspidal, tempered, unitary) if each element of Π is square integrable (resp. supercuspidal, tempered, unitary). The set of square integrable L-packets will be denoted by $\Pi^2(G)$.

Endoscopic Groups The following definition can be introduced for any linear algebraic group $\underline{G}$ and any local or global field F of characteristic 0. For the definition only, let $G = \underline{G}(F)$. To the group G one associates a set of auxiliary groups called *endoscopic groups* in the following way.

An element $s \in \hat{G}$ will be called *semisimple* if the endomorphism $\mathrm{ad}(s)$ of $\hat{G}$ fixes a Borel pair $(\hat{B}, \hat{T})$. Here $\hat{B}$ is a Borel subgroup of $\hat{G}$ and $\hat{T}$ is a maximal torus in $\hat{B}$. Denote by $\hat{G}(s)$ the centralizer of s in $\hat{G}$ and by $\hat{G}(s)^\circ$ its connected component. The group $\hat{G}(s)^\circ$ is a connected reductive subgroup of $\hat{G}$ [23]. A map between L-groups is called an *L-map* if it commutes with the natural projections to the Weil group.

Definition 4 **[19]** An *endoscopic triple* is a triple (H, s, η) consisting of a quasi-split group H, a semisimple element $s \in \hat{G}$ and an L-map $\eta\colon {}^LH \longrightarrow {}^LG$, which satisfies the following two conditions:

(I) η restricts to an isomorphism of complex groups from $\hat{H}$ to $\hat{G}(s)^\circ$.
Define $\lambda(\omega) = s\eta(\omega)s^{-1}\eta(\omega)^{-1}$ for $\omega \in W_F$.

(II) λ takes values in $Z(\hat{G})$ (in which case λ defines a cocycle with values in $Z(\hat{G})$) and the class of λ in $H^1\big(W_F, Z(\hat{G})\big)$ is locally trivial (resp. trivial) if F is global (resp. local).

The *endoscopic group* is the quasi-split connected reductive group H. The endoscopic group H is called *elliptic* if $\eta\Big(\big(Z(\hat{H})^\Gamma\big)^\circ\Big) \subset Z(\hat{G})^\Gamma$.

If $G = U(3)$, a proper elliptic endoscopic triple (H, s, η) for G has the property that H must be isomorphic to $U(2)\times U(1)$ [19, Prop. 4.6.1]. (For the definition of an isomorphism between endoscopic groups we refer the reader to [19].) Thus, up to isomorphism, the only elliptic endoscopic groups for G are $H = U(2) \times U(1)$ and G. The dual group of G is $\hat{G} = \mathrm{GL}(3, \mathbb{C})$ and the dual group of H is $\hat{H} = \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$.

Similarly, if $G = U(2)$, up to isomorphism, $H = U(1) \times U(1)$ is the only proper elliptic endoscopic group [19, Prop. 4.6.1]. The dual group of G is $\hat{G} = \mathrm{GL}(2,\mathbb{C})$ and the dual group of H is $\hat{H} = \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. The embeddings η for both cases above are described in [19].

Local Packets Let first G be the group $U(2)$ and $H = U(1) \times U(1)$. Then the derived group of $G(\mathbb{Q}_p)$ is isomorphic to $\mathrm{SL}(2)/_{\mathbb{Q}_p}$. The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on G by conjugation and hence on $E(G)$. Rogawski notes that the methods of [15] can be used and he defines an L-packet on $G(\mathbb{Q}_p)$ as a $\mathrm{PGL}_2(\mathbb{Q}_p)$-orbit in $E(G)$ [19, 11.1].

Let $\theta = \theta_1 \otimes \theta_2$ be a character of H. We call θ *singular* if $\theta_1 = \theta_2$ and *regular* if $\theta_1 \neq \theta_2$. Two characters $\theta = \theta_1 \otimes \theta_2$ and $\theta' = \theta_1' \otimes \theta_2'$ of H are said to be equivalent if the sets $\{\theta_1, \theta_2\}$ and $\{\theta_1', \theta_2'\}$ coincide. To every character θ of H there is associated an L-packet $\rho(\theta)$ on G with two elements which depend only on the equivalence class of θ. This transfer corresponds to functoriality with respect to the embedding $\eta\colon {}^LH \longrightarrow {}^LG$ from the definition of the endoscopic triple.

Now let G be the group $U(3)$, $H = U(2) \times U(1)$ and $C = U(1) \times U(1) \times U(1)$. To each character θ of C there is associated an L-packet $\rho(\theta)$ on H [19].

The character $\theta = \theta_1 \otimes \theta_2 \otimes \theta_3$ of C will be called *regular* if θ_j are distinct ($j = 1, 2, 3$), *semi-regular* if $\theta_1 \neq \theta_3$ and $\theta_2 = \theta_1$ or θ_3, and *singular* if $\theta_1 = \theta_2 = \theta_3$.

Let now B be the Borel subgroup of upper triangular matrices in $G(\mathbb{Q}_p)$ and N the unipotent radical of B consisting of upper triangular unipotent matrices. The Levi component M of B has the following form:

$$M = \{d(\alpha, \beta, \bar{\alpha}^{-1}) : \alpha \in E^*, \beta \in E^1\},$$

where E^1 is the set of elements of norm 1 in E.

Let χ be a character of M regarded as a character of B on which N acts trivially. We denote by $i_G(\chi)$ (resp. $i_H(\chi)$) the representation of G (resp. H) unitarily induced from χ. Let χ_1 be a character of E^* and χ_2 a character of E^1. We denote by $\chi = (\chi_1, \chi_2)$ the character of M defined by $\chi\bigl(d(\alpha, \beta, \bar{\alpha}^{-1})\bigr) = \chi_1(\alpha)\chi_2(\alpha\bar{\alpha}^{-1}\beta)$.

Let $\tilde{\omega}$ be a character of a subgroup Z' of the center of $G(\mathbb{Q}_p)$ and denote by $C\bigl(G(\mathbb{Q}_p), \tilde{\omega}\bigr)$ the space of smooth functions f on $G(\mathbb{Q}_p)$ such that $\mathrm{supp}(f)$ is compact modulo Z' and $f(zg) = \tilde{\omega}^{-1}(z) f(g)$ for $z \in Z'$ and $g \in G(\mathbb{Q}_p)$. If $\pi \in E(G)$ is such that the restriction of π to Z' is $\tilde{\omega}$ and $f \in C\bigl(G(\mathbb{Q}_p), \tilde{\omega}\bigr)$, then we let

$$\pi(f) = \int_{Z' \backslash G(\mathbb{Q}_p)} f(g)\pi(g)\, dg$$

and denote by $\chi_\pi(f)$ the trace $\mathrm{Tr}\bigl(\pi(f)\bigr)$.

Let $f^H \in C\bigl(H(\mathbb{Q}_p), \tilde{\omega}\bigr)$ be the unique function on H whose orbital integrals match those of f [19, 4.3]. Rogawski proves the following theorem.

Theorem 4 **[19]** *There exists a unique partition of $E(G)$ into L-packets and a map*

$$\xi_H \colon \Pi(H) \longrightarrow \Pi(G)$$

such that:

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(1) *An L-packet* $\Pi \in \Pi(G)$ *satisfies* $\mathrm{Card}(\Pi) > 1$ *if and only if* $\Pi = \xi_H(\rho)$ *for some* $\rho \in \Pi^2(H)$.

(2) *Let* $\rho \in \Pi(H)$ *such that* $\dim(\rho) \neq 1$ *and* ρ *is not of the form* $i_H(\chi\mu^{-1})$ *with* $\chi_1(\alpha) = \|\alpha\|$ *or* $\|\alpha\|^{-1}$. *Then there is a unique map* $\pi \longrightarrow \langle \rho, \pi \rangle$ *from* $\xi_H(\rho)$ *to* $\{\pm 1\}$ *such that*

$$\chi_\rho(f^H) = \sum_{\pi \in \xi_H(\rho)} \langle \rho, \pi \rangle \chi_\pi(f). \tag{2}$$

(3) *If* $\dim(\rho) = 1$, *then* $\xi_H(\rho) = \pi^n(\rho)$, *where* $\pi^n(\rho)$ *is the nontempered representation defined in [19, 12.2].*

If $\dim(\rho) \neq 1$, denote the image of ρ under ξ_H by $\Pi(\rho)$. If ξ is a one-dimensional character of H, denote by $\mathrm{St}_H(\xi)$ the Steinberg representation of H [19]. Then there is a supercuspidal representation in the packet determined by $\mathrm{St}_H(\xi)$. We denote this representation by $\pi^s(\xi)$. Thus, if $\xi \in \Pi(H)$ is one-dimensional, Rogawski defines the packet $\Pi(\xi)$ to be $\{\pi^n(\xi), \pi^s(\xi)\}$. He refers to $\Pi(\xi)$ as an *A-packet* and defines $\langle \xi, \ \rangle$ to be the function with constant value 1 on $\Pi(\xi)$. If $\xi \in \Pi(H)$ and $\dim(\xi) = 1$, the relation (2) holds [19, Prop. 13.1.4].

Following [19], let $\Pi_e(G)$ be the set of L-packets of the form $\Pi(\rho)$ where $\rho \in \Pi^2(H)$. Since each L-packet in $\Pi_e(G)$ comes from a square integrable L-packet of H, all L-packets in $\Pi_e(G)$ have cardinality greater than one. Let $\Pi_s(G)$ be the set of L-packets of cardinality one which are not of the form $\{\pi^n(\xi)\}$ for any one-dimensional representation ξ of H. Let $\Pi_a(G)$ be the set of A-packets $\Pi(\xi)$. Let $\Pi'(G) = \Pi_e(G) \cup \Pi_s(G) \cup \Pi_a(G)$. Each representation π of G lies in at least one local packet in $\Pi'(G)$ and in at most one unless $\pi = \pi^s(\xi)$ for some ξ.

Global Packets on $U(3)$ Let π be an automorphic representation of $G = U(3)$. Then, by [19, Theorem 13.3.1], the multiplicity $m(\pi)$ of π in the discrete spectrum of G is equal to 1.

To define a global packet [19] choose a local packet $\Pi_v \in \Pi'(G_v)$ for each place v in such a way that for almost all finite v the local packet Π_v contains an unramified representation π_v°. Note that if v splits in E, then G_v is isomorphic to $\mathrm{GL}_3(\mathbb{Q}_v)$ and an L-packet consists of a single irreducible representation. Define a global packet $\Pi = \bigotimes \Pi_v$ on G to be the set of representations $\pi = \bigotimes \pi_v$ such that $\pi_v \in \Pi_v$ for all v and $\pi_v = \pi_v^\circ$ for almost all v. A global packet Π will be called *discrete* if some member of Π occurs in the discrete spectrum. The discrete global packet Π will be called *cuspidal* if each member of Π which occurs discretely occurs in the space of cusp forms. Let $\Pi(\mathbf{G})$ and $\Pi(\mathbf{H})$ be the set of discrete global packets on G and H respectively, and let $\Pi_\circ(\mathbf{H})$ be the set of cuspidal global packets on H. If $\rho = \bigotimes \rho_v \in \Pi(\mathbf{H})$, let $\Pi(\rho) = \bigotimes \Pi(\rho_v)$.

If $G = U(2)$ and $H = U(1) \times U(1)$ and if $\theta = \bigotimes \theta_v$ is a regular character of $H \setminus \mathbf{H}$, then $\rho(\theta) = \bigotimes \rho(\theta_v)$ is a cuspidal global packet which depends only on the equivalence class of θ. If θ is singular, then $\rho(\theta)$ does not occur in the discrete spectrum.

Let G be again the group $U(3)$ and H the group $U(2) \times U(1)$. If $\rho \in \Pi(\mathbf{H})$, then [19, Theorem 13.3.2] shows that $\Pi(\rho)$ is discrete if and only if ρ is not of the form $\rho(\theta)$ with $\theta = \bigotimes \theta_v$ semi-regular character of $C \setminus \mathbf{C}$. It also shows that if $\Pi(\rho)$ is discrete and

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$\dim(\rho) \neq 1$, then $\Pi(\rho)$ is cuspidal. Define:

$$\Pi_a(\mathbf{G}) = \{\Pi(\xi) : \xi \in \Pi(\mathbf{H}) \text{ and } \dim(\xi) = 1\}$$
$$\Pi_e(\mathbf{G}) = \{\Pi(\rho) : \rho \in \Pi_\circ(\mathbf{H}), \rho \neq \rho(\theta) \text{ for } \theta \text{ semi-regular}\}.$$

The sets $\Pi_a(\mathbf{G})$ and $\Pi_e(\mathbf{G})$ are disjoint. Let $\Pi_s(\mathbf{G})$ be the set of discrete global packets $\Pi = \bigotimes \Pi_v$ on $\mathbf{G}$ such that there is no $\rho \in \Pi(\mathbf{H})$ with $\Pi_v = \Pi(\rho_v)$ for almost all v.

Any two elements of any global packet Π have the same local components almost everywhere. However, they may occur in the discrete spectrum with different multiplicities. A global packet Π is called *stable* if all representations in Π have the same multiplicity. By [19, Theorem 13.3.3 (c)], if $\Pi \in \Pi_s(\mathbf{G})$, then $m(\pi) = 1$ for all $\pi \in \Pi$. For obvious reasons, Rogawski calls the global packets in $\Pi_e(\mathbf{G})$ *endoscopic* and the global packets in $\Pi_s(\mathbf{G})$ *stable*.

To each global packet on G we assign an L-group which will be called the *endoscopic support* of the packet. We define the endoscopic support of the global packets in $\Pi_s(\mathbf{G})$ to be ${}^LG$. The endoscopic support of the global packets in $\Pi_a(\mathbf{G})$ is defined to be ${}^LH$. For global packets $\Pi = \Pi(\rho)$ in $\Pi_e(\mathbf{G})$ (with $\rho \in \Pi_\circ(\mathbf{H})$) such that there is no $\xi \in \Pi(\mathbf{C})$ with $\rho_v = \rho_v(\xi_v)$ for almost all v, the endoscopic support is defined to be ${}^LH$. For the global packets Π in $\Pi_e(\mathbf{G})$ such that $\Pi = \Pi(\rho)$ with $\rho = \rho(\theta)$ for a character θ of $\mathbf{C}$ which is not semi-regular, the endoscopic support is defined to be ${}^LC$.

The set of discrete global packets $\Pi(\mathbf{G})$ is the disjoint union of $\Pi_s(\mathbf{G})$, $\Pi_e(\mathbf{G})$ and $\Pi_a(\mathbf{G})$ [19]. In [19] and [20] Rogawski establishes multiplicity formulas for the representations in packets in $\Pi_e(\mathbf{G})$ and $\Pi_a(\mathbf{G})$.

Eigenvalue Packages Before we discuss the situation for an inner form of $U(3)$, we need to introduce the notion of an eigenvalue package.

Let S be a finite set of places of $\mathbb{Q}$, containing the infinite places, such that v is unramified in E for all $v \notin S$. An *eigenvalue package* (e.v.p.) will be a collection

$$t = t_S = \{t_v : v \notin S\}$$

where t_v is a homomorphism of the Hecke algebra $\mathcal{H}_v = \mathcal{H}\big(G(\mathbb{Q}_v), G(\mathbb{Z}_v)\big)$ into $\mathbb{C}$.

Let T_v be a maximal torus of G_v and denote by $\Pi^u(T_v)$ the set of unramified characters of T_v. We will regard t_v as an orbit of unramified characters in $\Pi^u(T_v)$ modulo the action of the Weyl group $W(T, G)$ as in Section 10.

If $\pi = \bigotimes \pi_v$ is a representation such that π_v is unramified for $v \notin S$, then π defines an eigenvalue package $t(\pi) = t_S(\pi) = \{t_v(\pi)\}_{v \notin S}$ as follows. The homomorphism $t_v(\pi)$ is the orbit in $\Pi^u(T_v)$ such that π_v is isomorphic to the unique unramified constituent of $i_G(\chi)$ for any χ in the orbit $t_v(\pi)$.

A map ψ of L-groups defines a transfer $t \longrightarrow \psi(t)$ of e.v.p.'s on the source group to e.v.p.'s on the target group. By the Strong Multiplicity One theorem for GL_n [11], two cuspidal representations π and π' of GL_n coincide if $t_v(\pi) = t_v(\pi')$ for almost all v. Two e.v.p.'s are said to be equivalent if they are equal almost everywhere. If Π is a global packet in $\Pi(\mathbf{G})$ then the eigenvalue package attached to Π, denoted $t_S(\Pi)$, will be the equivalence class of $t_S(\pi)$ for a representation $\pi \in \Pi$.

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Packets on Inner Forms of $U(3)$ Let $G = U(3)$ and let G' be an inner form of G defined by a pair (D, α), where D is a finite dimensional semisimple algebra over E and α is an antiautomorphism [19]. Assume that G' is not isomorphic to G over $\mathbb{Q}_p$. Fix an inner isomorphism $\psi\colon G'(\mathbb{Q}) \longrightarrow G(\mathbb{Q})$.

Let v be a place of $\mathbb{Q}$. Then α extends to an involution of the second kind on $D_v = D \otimes_{\mathbb{Q}} \mathbb{Q}_v$. If v remains prime in E, then α defines an isomorphism of D_v with its opposite algebra and Rogawski notes that in this case D_v is a split algebra. If v splits in E, then $D_v = D_w \oplus D_{w'}$, where w and w' are the places of E lying above v, and α interchanges the two factors [19, 14.2]. Then G'_v is isomorphic to one of the following [19]:

(i) G_v, if v is finite and does not split in E
(ii) G_v or the compact real group $U(\mathbb{R})$, if $E_v/\mathbb{Q}_v$ is $\mathbb{C}/\mathbb{R}$
(iii) D_w^*, if v splits in E, where w is a place of E dividing v.

Let $S_\circ$ be the set of infinite places v of $\mathbb{Q}$ such that G'_v is isomorphic to $U(\mathbb{R})$ and let S be the set of places v such that D_w is ramified for $w|v$. If $v \notin S \cup S_\circ$, one defines local packets on G'_v by means of ψ and $\Pi(G'_v)$ will be identified with $\Pi(G_v)$. If $v \in S \cup S_\circ$, a local packet on G'_v is defined to be a set consisting of a single irreducible, admissible representation of G'_v. This representation is necessarily finite dimensional since G'_v is compact modulo Z_v. In this case Rogawski shows that there is a bijection $\psi'_v\colon \Pi(G'_v) \longrightarrow \Pi^2(G_v)$ [19, 14.4]. The global packets on G' are then introduced in the same manner as the global packets for G as tensor products of local packets. Let $\Pi(\mathbf{G}')$ denote the set of global packets Π' such that π occurs discretely for some $\pi \in \Pi'$.

Let S' be a finite set of places of $\mathbb{Q}$ containing $S \cup S_\circ$. For $v \notin S'$ we have identified G'_v and G_v, and a collection $t_{S'} = \{t_v\}_{v \notin S'}$, where $t_v \in \operatorname{Hom}(\mathcal{H}_v, \mathbb{C})$, can be regarded as an eigenvalue package on either G' or G. Rogawski shows in [19, 14.6] that $t_{S'}$ is of the form $t_{S'}(\pi)$ for a discrete representation π of G' if and only if $t_{S'} = t_{S'}(\Pi)$ for some $\Pi \in \Pi(\mathbf{G})$. Furthermore, the packet Π is shown to be unique.

Let $\Pi_s(\mathbf{G}')$, $\Pi_e(\mathbf{G}')$, and $\Pi_a(\mathbf{G}')$ denote the sets of global packets Π' in $\Pi(\mathbf{G}')$ such that the e.v.p. $t(\Pi')$ coincides with $t(\Pi)$ for some Π belonging to $\Pi_s(\mathbf{G})$, $\Pi_e(\mathbf{G})$, and $\Pi_a(\mathbf{G})$, respectively. These sets are disjoint and every discrete representation of $\mathbf{G}'$ belongs to a global packet in one of them. Thus $\Pi(\mathbf{G}') = \Pi_s(\mathbf{G}') \cup \Pi_e(\mathbf{G}') \cup \Pi_a(\mathbf{G}')$.

It is important to notice that the eigenvalue packages transfer global packets from one group to the other but they do not preserve the structure inside the packet. However, in proving the Main Theorem, we are only interested in the eigenvalue packages and thus we can work with either $\mathbf{G}$ or $\mathbf{G}'$.

By [19, Proposition 14.6.2] there is a bijection between $\Pi_s(\mathbf{G}')$ and

$$\{\Pi \in \Pi_s(\mathbf{G}) : \dim(\Pi) = 1 \text{ or } \Pi_v \in \Pi^2(G_v) \text{ for all } v \in S \cup S_\circ\}.$$

If $D \neq M_3(E)$, then $\Pi(\mathbf{G}') = \Pi_s(\mathbf{G}')$ [19, Theorem 14.6.3] and therefore, in this case, all discrete representations of $\mathbf{G}'$ are automorphic.

12 The Decomposition of $\bigoplus_{i=1}^k L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$ Revisited

Recall that in Theorem 1 we decomposed $\bigoplus_{i=1}^k L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$ into a direct sum of one-dimensional subspaces of unramified representations consisting of vectors that are

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invariant under K_p,

$$\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big) \cong \bigoplus_{\pi_p \text{ unramified}} \Big(\sum_{\{\pi=\pi_{\mathbb{R}}\pi_p\pi^p:\pi_{\mathbb{R}}=1\}} m(\pi) \cdot \dim(V_{\pi^p}^{K_f^p}) \Big) \cdot V_{\pi_p}^{K_p}.$$

In Section 8 we obtained a decomposition of $\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big)$ into a direct sum of eigenspaces of the Hecke algebra $\mathcal{H}_p$,

$$\bigoplus_{i=1}^{k} L^2\big(\Gamma_{i,p} \setminus G'(\mathbb{Q}_p)/K_p\big) = \bigoplus_{\pi_p \text{ unramified}} V_{\alpha_{\pi_p}}^{K_p},$$

where α_{π_p} is the zonal spherical function corresponding to π_p. Then the space $V_{\alpha_{\pi_p}}^{K_p}$ is equal to

$$\bigoplus_{\{\Pi\in\Pi(\mathbf{G}'):t_p(\Pi)=t(\alpha_{\pi_p})\}} \Big(\sum_{\{\pi\in\Pi:\pi=\pi_{\mathbb{R}}\pi_p\pi^p,\pi_{\mathbb{R}}=1\}} m(\pi) \cdot \dim(V_{\pi^p}^{K_f^p}) \Big) \cdot V_{\pi_p}^{K_p},$$

where $t_p(\Pi)$ is the p-component of the eigenvalue package $t(\Pi)$ and $t(\alpha_{\pi_p}) = \lambda_{\alpha_{\pi_p}}$ is the eigenvalue of $\mathcal{H}_p$ corresponding to the eigenspace $V_{\alpha_{\pi_p}}$.

13 Parameters and L-Packets

Rogawski's results presented in the previous section are very striking. They are even more so when compared to Arthur's conjectures [1]. Rogawski's partition of the set of representations for the unitary group in three variables into finite packets matches Arthur's conjectural parameterization of the packets by parameters of the hypothetical Langlands group [16]. One can see that the Hecke eigenvalues determined by Rogawski's packets are the same as the eigenvalues determined by the parameters [5, 10.4].

We would like to stress the fact that the Langlands group is hypothetical and, thus, the parameters are conjectural objects. However, in this chapter we will view Rogawski's results in the language of the parameters and use them to provide the structure of the proof of the Main Theorem.

Following Arthur's discussion in [1], let us assume the existence of the hypothetical locally compact Langlands group $L_{\mathbb{Q}}$. It is to be an extension of $W_{\mathbb{Q}}$ by a compact group. For every place v of $\mathbb{Q}$ there is a homomorphism

$$L_{\mathbb{Q}_v} \longrightarrow L_{\mathbb{Q}},$$

where

$$L_{\mathbb{Q}_v} = \begin{cases} W_{\mathbb{Q}_v} & \text{if } v \text{ is archimedean} \\ W_{\mathbb{Q}_v} \times \mathrm{SL}(2,\mathbb{C}) & \text{if } v \text{ is non-archimedean.} \end{cases}$$

If G is a connected reductive group over $\mathbb{Q}$, Arthur conjectured [1] that the global packets on G are parameterized by $\hat{G}$-conjugacy classes of *admissible* homomorphisms

$$\psi\colon L_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^L G$$

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such that the projection of $\psi(L_{\mathbb{Q}})$ onto $\hat{G}$ is bounded. Here admissible means that the projection of each of the elements $\psi(\omega)$, $\omega \in L_{\mathbb{Q}}$, onto $\hat{G}$ is semisimple and if the image $\psi(L_{\mathbb{Q}})$ of $L_{\mathbb{Q}}$ is contained in a parabolic subgroup of ${}^{L}G$ then the corresponding parabolic subgroup of G must be defined over $\mathbb{Q}$ [5, 8.2]. If G is the unitary group in three variables $U(3)$, Rogawski's partition into global packets is compatible with Arthur's conjectural parameterization [16].

As before, G' will be a form of $G = U(3)$ such that $G'(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)$ and $G'(\mathbb{R})$ is compact. The local L-packets of G' at the place p consist of a single irreducible admissible representation. Since we are interested in the eigenvalues of the fundamental Hecke functions, we can work with $G = U(3)$ instead of G'.

Let S_ψ be the centralizer in $\hat{G} = \mathrm{GL}(3, \mathbb{C})$ of the image of ψ,

$$S_\psi = \mathrm{Cent}\big(\mathrm{Im}(\psi), \hat{G}\big).$$

We impose the additional condition on the parameter ψ that the group S_ψ be nontrivial. Let $\mathcal{S}_\psi = S_\psi / S_\psi^\circ$, where S_ψ° is the connected component of the identity in S_ψ. We are only interested in the global representations that occur in the discrete spectrum. They correspond to parameters ψ such that S_ψ is finite and therefore $S_\psi^\circ = \{1\}$ [1]. We would like to compute S_ψ for all parameters ψ such that S_ψ is finite.

Let I_{tr} denote the set of elements in $W_{\mathbb{Q}}$ that act trivially on $\hat{G}$ and let $A_{\mathbb{Q}}$ denote the subgroup of $L_{\mathbb{Q}}$ such that

$$\psi\big(A_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C})\big) \subseteq \hat{G} \rtimes I_{\mathrm{tr}}.$$

Then $A_{\mathbb{Q}}$ is a normal subgroup of $L_{\mathbb{Q}}$ of index two. If $\psi|_{A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C})}$ is the restriction of ψ to $A_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C})$, let ρ denote the projection of $\psi|_{A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C})}$ onto $\hat{G}$. Thus,

$$\rho\colon A_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow \mathrm{GL}(3, \mathbb{C})$$

is a three-dimensional representation of $A_{\mathbb{Q}} \times \mathrm{SL}(2, \mathbb{C})$. We will refer to ρ as the representation attached to ψ.

As before, E will denote the quadratic extension of $\mathbb{Q}$ that defines the unitary group. In the definition of the L-group ${}^{L}G$, the absolute Weil group $W_{\mathbb{Q}}$ acts on $\hat{G}$ through its projection on the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, where L is a finite Galois extension of $\mathbb{Q}$ containing E. Since the Weil group W_E acts trivially on $\hat{G}$, we can replace $W_{\mathbb{Q}}$ by $W_{E/\mathbb{Q}}$ and $\mathrm{Gal}(L/\mathbb{Q})$ by $\mathrm{Gal}(E/\mathbb{Q})$. The Galois group $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore

$$S_\psi = \mathrm{Cent}\big(\mathrm{Im}(\psi), \hat{G}\big) \cong \mathrm{Cent}\big(\mathrm{Im}(\rho), \hat{G}\big)^{\mathbb{Z}/2\mathbb{Z}}.$$

Denote by Π_ψ the global packet conjecturally corresponding to the parameter ψ. Denote by ${}^{L}A_\psi$ the centralizer of S_ψ in ${}^{L}G$, ${}^{L}A_\psi = \mathrm{Cent}(S_\psi, {}^{L}G) \in \{{}^{L}G, {}^{L}H, {}^{L}C\}$. Then $\mathrm{Im}(\psi) \subset {}^{L}A_\psi$.

Let $g \rtimes \sigma \in {}^{L}A_\psi = \mathrm{Cent}(S_\psi, {}^{L}G)$, where $g \in \hat{G}$ and σ is the element of $W_{E/\mathbb{Q}}$ whose projection on Γ is the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q})$. Then $s(g \rtimes \sigma) = (g \rtimes \sigma)s$ for $s \in S_\psi$. This implies that $sg \rtimes \sigma = g \cdot {}^{\sigma}s \rtimes \sigma$. Since $s \in S_\psi$, s is stabilized by σ and it follows that $g \rtimes \sigma \in {}^{L}A_\psi$ if and only if $sg \rtimes \sigma = gs \rtimes \sigma$ for any $s \in S_\psi$. Therefore

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${}^LA_\psi \cong \mathrm{Cent}(S_\psi, \hat{G}) \rtimes W_{E/\mathbb{Q}}$. The group ${}^LA_\psi$ would then be the endoscopic support of the packet Π_ψ as defined in Section 11.

The following definition as well as the correspondence between packets and homomorphisms of $L_\mathbb{Q} \times \mathrm{SL}(2,\mathbb{C})$ discussed in the remainder of the chapter depend on the assumption of the existence of the Langlands group.

Definition 5 An automorphic representation π of G will be called *primitive* for G (resp. for H) if it lies in a global packet Π_ψ with endoscopic support ${}^LA_\psi$ satisfying ${}^LA_\psi \not\subset {}^LH$ (resp. ${}^LA_\psi \not\subset {}^LC$).

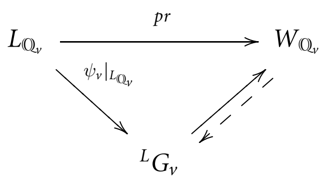
For every discrete global packet Π on G the representations $\pi \in \Pi$ are primitive for some uniquely determined smaller group.

An automorphic representation π of $\mathbf{G}$ is primitive for G if π belongs to a global packet $\Pi \in \Pi_s(\mathbf{G})$. Thus, in the previous section, we divided the global packets on $\mathbf{G}$ into global packets $\Pi \in \Pi_s(\mathbf{G})$ consisting of primitive representations for G and the other global packets $\Pi \in \Pi_e(\mathbf{G}) \cup \Pi_a(\mathbf{G})$. We would like to divide the global packets of $\mathbf{H}$ in a similar manner. Define the set

$$\Pi_s(\mathbf{H}) = \Big\{\Pi = \bigotimes \Pi_v \in \Pi(\mathbf{H}) : \text{ there is no } \xi \in \Pi(\mathbf{C}) \text{ with } \Pi_v = \Pi(\xi_v) \text{ for a. a. } v\Big\}.$$

The endoscopic support of the global packets in $\Pi_s(\mathbf{H})$ is ${}^LH$ and thus $\Pi_s(\mathbf{H})$ consists of representations that are primitive for H. We denote the set of the remaining global packets in $\Pi(\mathbf{H})$ by $\Pi_e(\mathbf{H})$. The global packets in $\Pi_e(\mathbf{H})$ come from global packets on $\mathbf{C}$.

Up to conjugacy, for non-archimedean v, $L_{\mathbb{Q}_v} = W_{\mathbb{Q}_v} \times \mathrm{SL}(2,\mathbb{C})$ can be embedded into $L_\mathbb{Q}$. Then $A_{\mathbb{Q}_v}$ is either a subgroup of index 1 or a subgroup of index 2 of $L_{\mathbb{Q}_v}$. Since p is a place that splits in E, we have $A_{\mathbb{Q}_p} = L_{\mathbb{Q}_p}$. For any place v, let ψ_v denote the restriction of ψ to $L_{\mathbb{Q}_v} \times \mathrm{SL}(2,\mathbb{C})$.



Let $I_{\mathbb{Q}_p}$ denote the inertia subgroup of $W_{\mathbb{Q}_p}$. The quotient $W_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$ is infinite cyclic. We say that a parameter ψ is *unramified* at v if $\psi_v|_{L_{\mathbb{Q}_v}}$ is trivial on the kernel of the projection pr of $L_{\mathbb{Q}_v}$ onto $W_{\mathbb{Q}_v}$, and on the inertia subgroup $I_{\mathbb{Q}_v} \subset W_{\mathbb{Q}_v}$.

There is also a conjectural local correspondence between local parameters

$$\psi_v \colon L_{\mathbb{Q}_v} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^LG$$

and local packets. Locally, unramified parameters correspond to local packets containing unramified representations.

Note The reason for having the parameter ψ defined on $L_\mathbb{Q} \times \mathrm{SL}(2,\mathbb{C})$ and not only on $L_\mathbb{Q}$ is to obtain a correspondence for all automorphic representations. If we omit the $\mathrm{SL}(2,\mathbb{C})$ part we would only obtain the correspondence with the global packets consisting of tempered representations. The parameters whose restriction to $\mathrm{SL}(2,\mathbb{C})$ is non-trivial correspond to global packets containing non-tempered representations [1].

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14 The Group S_ψ for $U(n)$

For the sake of generality, let us consider the case of $U(n)$ with arbitrary n. We would like to study all possible parameters $\psi\colon L_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^L G$ and calculate S_ψ in each case. Consider the n-dimensional representation $\rho\colon A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ attached to ψ. The representation ρ is decomposable into a direct sum of irreducible representations: $\rho = \bigoplus \rho_i$. Let n_i be the dimension of ρ_i. Each ρ_i is a tensor product $\rho_i = \gamma_i \otimes \sigma_i$, with γ_i an irreducible representation of $A_{\mathbb{Q}}$ and σ_i an irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. Let l_i be the dimension of γ_i and s_i the dimension of σ_i. Then

$$n_i = l_i s_i.$$

Recall that, up to equivalence, for each finite dimension there is only one irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ [13, Prop. 2.1 and Theorem 2.4].

By an extension of Schur's Lemma, $\mathrm{Cent}\big(\mathrm{Im}(\rho), \mathrm{GL}(n,\mathbb{C})\big)$ is a commuting algebra of the space of $\bigotimes \rho_i$ and therefore it is isomorphic to a direct product of general linear groups $\mathrm{GL}(t_i,\mathbb{C})$. There is one factor $\mathrm{GL}(t_i,\mathbb{C})$ for each ρ_i, with t_i equal to the multiplicity of ρ_i in ρ.

To determine $S_\psi \cong \mathrm{Cent}\big(\mathrm{Im}(\rho), \hat{G}\big)^{\mathbb{Z}/2\mathbb{Z}}$, let $g \in \mathrm{Cent}\big(\mathrm{Im}(\rho), \hat{G}\big)$ be such that ${}^\sigma g = g$, where σ is the nontrivial element of $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$. Since σ maps g to ${}^t g^{-1}$, the element g of $\mathrm{Cent}\big(\mathrm{Im}(\rho), \hat{G}\big)$ is such that $g = {}^t g^{-1}$. Therefore,

$$S_\psi = \Big\{ g \in \prod \mathrm{GL}(t_i,\mathbb{C}) \mid g \cdot {}^t g = 1 \Big\} = \prod \{ g \in \mathrm{GL}(t_i,\mathbb{C}) \mid g \cdot {}^t g = 1 \}.$$

If any of the t_i's is larger than 1, then, for that t_i, the space $\{g \in \mathrm{GL}(t_i,\mathbb{C}) \mid g \cdot {}^t g = 1\}$ has infinitely many elements and therefore $|S_\psi| = \infty$. However, we are only interested in parameters ψ such that S_ψ is finite. Therefore we only consider the parameters ψ for which the decomposition of the attached representation ρ contains no irreducible factors with multiplicity greater than 1. For such parameters ψ, $\mathrm{Cent}\big(\mathrm{Im}(\rho), \hat{G}\big) = \prod \mathrm{GL}(1,\mathbb{C})$ and S_ψ is isomorphic to the direct product of as many copies of $\mathbb{Z}/2\mathbb{Z}$ as there are irreducible constituents ρ_i in the decomposition of ρ.

15 Parameters for $U(3)$

Again let G be the unitary group in three variables $U(3)$ and let ρ be the three-dimensional representation of $A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C})$ attached to ψ,

$$\rho\colon A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(3,\mathbb{C}).$$

Let ψ_s be the restriction $\psi|_{L_{\mathbb{Q}}}$ of the parameter ψ to the group $L_{\mathbb{Q}}$. The restriction $\rho|_{A_{\mathbb{Q}}}$ of ρ to $A_{\mathbb{Q}}$ is a representation of $A_{\mathbb{Q}}$. If any constituent $\rho_i = \gamma_i \otimes \sigma_i$ in the decomposition of ρ is such that σ_i is not trivial, then $\rho|_{A_{\mathbb{Q}}}$ will contain constituents with multiplicities greater than 1 in its decomposition into a direct sum of irreducible representations. In this case $S_{\psi_s} = \mathrm{Cent}\big(\mathrm{Im}(\rho|_{A_{\mathbb{Q}}}), \hat{G}\big)^{\mathbb{Z}/2\mathbb{Z}}$ is infinite. However, we can still consider the group ${}^L A'_\psi = \mathrm{Cent}(S_{\psi_s}, {}^L G)$. The group ${}^L A'_\psi$ is then a smaller group of endoscopic type containing $\mathrm{Im}(\psi_s)$. Thus the maps

$$\psi_s : L_{\mathbb{Q}} \longrightarrow {}^L A'_\psi$$

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parameterize the primitive cuspidal representations for A'_ψ. If all σ_i's are trivial, then $\rho|_{A_\mathbb{Q}} = \rho$, ${}^LA'_\psi = {}^LA_\psi$ and $A'_\psi = A_\psi$. Thus we obtain a parameterization of the global packets in A'_ψ by primitive cuspidal representations for A'_ψ. Since there is no non-tempered part in the representations $\rho|_{A_\mathbb{Q}}$, the cuspidal representations of A'_ψ satisfy the Ramanujan conjecture. For a discussion of the generalized Ramanujan conjecture see Rogawski's Appendix to [17].

In order to return to the group G, we have to consider parameters for G. The uniformizer of $\mathbb{Q}_p$ is equal to p. There is a quasi-character $|\ \ |$ on $W_{\mathbb{Q}_p}$ such that $|\omega| = 1$ for $\omega \in I_{\mathbb{Q}_p}$ and $|\Phi| = |p|_p = 1/p$, where Φ is a Frobenius element in $W_{\mathbb{Q}_p}$. We extend the quasi-character $|\ \ |$ of $W_\mathbb{Q}$ trivially to $L_\mathbb{Q}$. Let

$$\phi_\psi \colon L_\mathbb{Q} \longrightarrow {}^LG$$

be the parameter given by

$$\omega \longrightarrow \omega \times d(|\omega|^{1/2}, |\omega|^{-1/2}) \xrightarrow{\psi} {}^LG$$

for $\omega \in L_\mathbb{Q}$. These parameters correspond to global packets in G, but the representations ρ_{ϕ_ψ} attached to them given by

$$\rho_{\phi_\psi}(\omega) = \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big), \quad \omega \in A_\mathbb{Q},$$

might not satisfy the Ramanujan conjecture.

Denote by Π_{ψ_p} (resp. $\Pi_{\phi_{\psi_p}}$) the local packet conjecturally corresponding to the local parameter ψ_p (resp. ϕ_{ψ_p}). Then we would have $\Pi_{\phi_{\psi_p}} \subset \Pi_{\psi_p}$ [2].

16 Semisimple Conjugacy Classes for Parameters

Consider the parameter ϕ_ψ defined above and let T be the subgroup of diagonal matrices in $G'(\mathbb{Q}_p)$. If the parameter ψ is unramified at the place p, so is ϕ_ψ. Following [5, 10.4], the image of ϕ_{ψ_p} may be assumed to be in ${}^LT$ and there exists a $t \in \hat{T}^\Gamma$ such that

$$\phi_{\psi_p}(\Phi) = (t, \Phi)$$

and

$$\phi_{\psi_p}(\omega) = (t, \Phi)^{\epsilon(\omega)}, \quad \omega \in L_\mathbb{Q},$$

where $\epsilon \colon L_\mathbb{Q} \longrightarrow \mathbb{Z}$ is the canonical homomorphism $L_\mathbb{Q} \to W_\mathbb{Q} \to \mathbb{Q}^* \to \mathbb{Z}$.

The semisimple conjugacy class represented by (t, Φ) with $t = d(t_1, t_2, t_3)$ determines an unramified character χ_t of T by

$$\chi_t\big(d(a_1, a_2, a_3)\big) = t_1^{\mathrm{ord}_p(a_1)} t_2^{\mathrm{ord}_p(a_2)} t_3^{\mathrm{ord}_p(a_3)}$$

for $d(a_1, a_2, a_3) \in T$. Then the local packet $\Pi_{\phi_{\psi_p}}$ of $G(\mathbb{Q}_p)$ consists of the unramified representation induced from χ_t.

17 Proof of the Main Theorem

As before G is the group $U(3)$ and p is a place that splits in E. Now we can use Theorem 3 (Tate) and the above discussion about the parameters to prove the Main Theorem. The proof relies heavily on applications of Deligne's Theorem [12, Theorem 1.6 and Lemma 1.7] that implies that for every automorphic representation $\pi \in \Pi_\psi$ whose component $\pi_\mathbb{R}$ at the archimedean place is square integrable, the restriction $\rho|_{A_\mathbb{Q}}$ of the representation ρ to $A_\mathbb{Q}$ is bounded. This, in turn, implies that the representation γ_{i_p} of $A_{\mathbb{Q}_p} = L_{\mathbb{Q}_p}$ is unitary for each i. Here $\rho = \bigoplus \rho_i = \bigoplus(\gamma_i \otimes \sigma_i)$ is the representation attached to the parameter ψ, and γ_{i_p} is the component of γ_i at the place p.

The proof of the theorem will be a case by case discussion of the parameters according to the decomposition of the attached representation ρ.

Recall that p is a place where $G'(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)$ and that the automorphic representations π occurring in the decomposition of Theorem 1 are such that the component π_p at the place p is unramified and $\pi_\mathbb{R} = 1$. Thus we need only consider parameters ψ which are unramified at the place p.

Given the parameter ψ, denote by $\lambda_\psi^{(1)}$ (resp. $\lambda_\psi^{(2)}$) the eigenvalue of the fundamental Hecke function φ_1 (resp. φ_2) determined by the unramified representation which is the component at the place p of an automorphic representation in the global packet Π_ψ that corresponds to ψ. Then $\lambda_\psi = \lambda_\psi^{(1)} + \lambda_\psi^{(2)}$ is the corresponding eigenvalue of the adjacency matrix δ of the underlying graph of $\mathbb{B}$. Then Tate's Theorem will give an estimation of all eigenvalues of the adjacency matrix δ of the underlying graph of $\mathbb{B}$.

Case (1) ρ is a three-dimensional irreducible representation of $A_\mathbb{Q} \times \mathrm{SL}(2,\mathbb{C})$. Then $S_\psi \cong \mathbb{Z}/2\mathbb{Z}$ and ${}^LA_\psi = \mathrm{Cent}(S_\psi, {}^LG) = {}^LG$. Therefore, the group ${}^LA_\psi$ is not contained in ${}^LH$ and ψ corresponds to a global packet of G containing primitive representations for G. The parameters in this case correspond to global packets in $\Pi_s(\mathbf{G})$.

There are two subcases:

(a) $\rho = \gamma\otimes\sigma$, where γ is three-dimensional and σ is one-dimensional. The representation σ is trivial and we have

$$\rho_{\phi_\psi}(\omega) = \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big) = \gamma(\omega),$$

for $\omega \in A_\mathbb{Q}$.

Since the parameter ψ is unramified at the place p, so is ϕ_ψ and the representation $\rho_{\phi_{\psi_p}}$ of $A_{\mathbb{Q}_p} = L_{\mathbb{Q}_p}$ corresponds to the semisimple conjugacy class in $\hat{G} = \mathrm{GL}(3,\mathbb{C})$ given by

$$\rho_{\phi_{\psi_p}}(\Phi) = \gamma_p(\Phi)$$

which is represented by an element in $\hat{T}$, say

$$d\big(\gamma_p^{(1)}(\Phi), \gamma_p^{(2)}(\Phi), \gamma_p^{(3)}(\Phi)\big).$$

This semisimple conjugacy class corresponds to an unramified character $\chi = (\chi_1, \chi_2, \chi_3)$ of the subgroup of diagonal matrices of $\mathrm{GL}_3(\mathbb{Q}_p)$, where the χ_i's, $i = 1, 2, 3$, are unramified characters of $\mathbb{Q}_p^*$. The character χ is given by

$$\chi\big(d(a_1, a_2, a_3)\big) = \chi_1(a_1)\chi_2(a_2)\chi_3(a_3)$$

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with

$$\chi_i(a) = \big(\gamma_p^{(i)}(\Phi)\big)^{\mathrm{ord}_p(a)}, \quad i = 1,2,3, a \in \mathbb{Q}_p^*.$$

By Tate's Theorem the eigenvalue $\lambda_\psi^{(1)}$ of φ_1 determined by the local component at the place p of representations in the L-packet Π_ψ is given by

$$\lambda_\psi^{(1)} = p^{\frac{1}{2}\cdot 2}\big(\chi_1(p)+\chi_2(p)+\chi_3(p)\big) = p\big(\gamma_p^{(1)}(\Phi)+\gamma_p^{(2)}(\Phi)+\gamma_p^{(3)}(\Phi)\big).$$

By Deligne's Theorem the representation γ_p is unitary and hence $|\gamma_p^{(i)}(\Phi)| = 1$ for $i = 1,2,3$. Therefore

$$|\lambda_\psi^{(1)}| \le 3p.$$

Similarly, the eigenvalue $\lambda_\psi^{(2)}$ of φ_2 is given by

$$\begin{aligned}\lambda_\psi^{(2)} &= p^{\frac{1}{2}\cdot 2}\big(\chi_1(p)\chi_2(p)+\chi_1(p)\chi_3(p)+\chi_2(p)\chi_3(p)\big)\\ &= p\big(\gamma_p^{(1)}(\Phi)\gamma_p^{(2)}(\Phi)+\gamma_p^{(1)}(\Phi)\gamma_p^{(3)}(\Phi)+\gamma_p^{(2)}(\Phi)\gamma_p^{(3)}(\Phi)\big),\end{aligned}$$

and again we have

$$|\lambda_\psi^{(2)}| \le 3p.$$

Thus the corresponding eigenvalue λ_ψ of the adjacency matrix δ satisfies the inequality $|\lambda_\psi| \le 6p$.

(b) $\rho = \gamma \otimes \sigma$, where γ is one-dimensional σ is three-dimensional. In this case ${}^LA'_\psi = {}^L\big(U(1)\big)$. The parameters in this case correspond to the one-dimensional representations of $U(3)$ [2]. It follows from [13, II, Section 3] that the three-dimensional representation σ of $\mathrm{SL}(2,\mathbb{C})$ is such that it maps the element $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{C})$ to

$$\begin{pmatrix} d^2 & -dc & c^2\\ -db & ad+bc & -ac\\ b^2 & -ab & a^2\end{pmatrix}.$$

Then, for $\omega \in A_{\mathbb{Q}}$,

$$\begin{aligned}\rho_{\phi_\psi}(\omega) &= \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big) = \gamma(\omega)\sigma\big(d(|\omega|^{1/2}, |\omega|^{-1/2})\big)\\ &= \gamma(\omega)d(|\omega|^{-1}, 1, |\omega|) = d\big(\gamma(\omega)|\omega|^{-1}, \gamma(\omega), \gamma(\omega)|\omega|\big).\end{aligned}$$

At the place p, the parameter ϕ_ψ is unramified and the representation $\rho_{\phi_{\psi_p}}$ of $L_{\mathbb{Q}_p}$ corresponds to the semisimple conjugacy class in $\mathrm{GL}(3,\mathbb{C})$ given by

$$\rho_{\phi_{\psi_p}}(\Phi) = d\big(\gamma_p(\Phi)|\Phi|^{-1}, \gamma_p(\Phi), \gamma_p(\Phi)|\Phi|\big).$$

This semisimple conjugacy class corresponds to the unramified character $\chi = (\chi_1, \chi_2, \chi_3)$ of the subgroup of diagonal matrices of $\mathrm{GL}_3(\mathbb{Q}_p)$ with

$$\begin{aligned}\chi_1(a) &= \big(\gamma_p(\Phi)|\Phi|^{-1}\big)^{\mathrm{ord}_p(a)}\\ \chi_2(a) &= \big(\gamma_p(\Phi)\big)^{\mathrm{ord}_p(a)}\\ \chi_3(a) &= \big(\gamma_p(\Phi)|\Phi|\big)^{\mathrm{ord}_p(a)}\end{aligned}$$

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for $a \in \mathbb{Q}_p^*$. Then Tate's Theorem gives

$$\lambda_\psi^{(1)} = p^{\frac{1}{2}\cdot 2}\big(\chi_1(p) + \chi_2(p) + \chi_3(p)\big) = p\big(\gamma_p(\Phi)|\Phi|^{-1} + \gamma_p(\Phi) + \gamma_p(\Phi)|\Phi|\big)$$
$$= p\gamma_p(\Phi)(p + 1 + p^{-1}).$$

Since γ_p is one-dimensional and therefore $|\gamma_p(\Phi)| = 1$, we have

$$|\lambda_\psi^{(1)}| = p^2 + p + 1.$$

Similarly

$$\lambda_\psi^{(2)} = p\big(\gamma_p^2(\Phi)|\Phi|^{-1} + \gamma_p^2(\Phi) + \gamma_p^2(\Phi)|\Phi|\big) = p\gamma_p^2(\Phi)(p + 1 + p^{-1})$$

and

$$|\lambda_\psi^{(2)}| = p^2 + p + 1.$$

In this case, the corresponding eigenvalue $\lambda_\psi \neq 0$ of the adjacency matrix δ is such that $|\lambda_\psi| = 2(p^2 + p + 1)$. The underlying graph of $\mathbb{B}$ is $2(p^2 + p + 1)$-regular and thus the eigenvalues we obtain in this case are the eigenvalues with the largest absolute value.

Case (2) $\rho = \rho_1 \oplus \rho_2$, where ρ_1 is a one-dimensional irreducible representation of $A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C})$ and ρ_2 is a two-dimensional irreducible representation of $A_{\mathbb{Q}} \times \mathrm{SL}(2,\mathbb{C})$. Then $S_\psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. To determine ${}^L A_\psi = \mathrm{Cent}(S_\psi, {}^L G)$ we consider first the centralizer of S_ψ in $\hat{G}$. Since $S_\psi = \{d(a,b,a) : a = \pm 1, b = \pm 1\}$, the centralizer of S_ψ in $\hat{G}$ consists of the matrices $g \in \mathrm{GL}(3,\mathbb{C})$ of the form

$$g = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

The condition $\det g \neq 0$ is equivalent to $a_{22} \neq 0$ and $\det \left(\begin{smallmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{smallmatrix}\right) \neq 0$ and thus the centralizer of S_ψ in $\hat{G}$ is isomorphic to $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) = \hat{H}$. It follows that ${}^L A_\psi = {}^L H$ and thus the group ${}^L A_\psi$ is not contained in ${}^L C$. Hence, these parameters correspond to global packets containing primitive representations for H.

Since $\rho_1 = \gamma_1 \otimes \sigma_1$ is one-dimensional, γ_1 is one-dimensional and σ_1 is trivial. There are two possibilities for the case $\rho = \rho_1 \oplus \rho_2$:

a) $\rho_2 = \gamma_2 \otimes \sigma_2$ with γ_2 two-dimensional and σ_2 one-dimensional. Hence, σ_2 is trivial and we have

$$\rho_{\phi_\psi}(\omega) = \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big) = \gamma_1(\omega) \oplus \gamma_2(\omega).$$

At the place p, the parameter ϕ_ψ is unramified and the representation $\rho_{\phi_{\psi_p}}$ of $L_{\mathbb{Q}_p}$ corresponds to the semisimple conjugacy class in $\hat{G} = \mathrm{GL}(3,\mathbb{C})$ given by

$$\rho_{\phi_{\psi_p}}(\Phi) = \gamma_{1_p}(\Phi) \oplus \gamma_{2_p}(\Phi)$$

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which is represented by an element in $\hat{T}$, say

$$d\big(\gamma_{2_p}^{(1)}(\Phi), \gamma_{1_p}(\Phi), \gamma_{2_p}^{(2)}(\Phi)\big).$$

This semisimple conjugacy class corresponds to an unramified character $\chi = (\chi_1, \chi_2, \chi_3)$ of the subgroup of diagonal matrices of $\mathrm{GL}_3(\mathbb{Q}_p)$ with

$$\chi_1(a) = \big(\gamma_{2_p}^{(1)}(\Phi)\big)^{\mathrm{ord}_p(a)}$$
$$\chi_2(a) = \big(\gamma_{1_p}(\Phi)\big)^{\mathrm{ord}_p(a)}$$
$$\chi_3(a) = \big(\gamma_{2_p}^{(2)}(\Phi)\big)^{\mathrm{ord}_p(a)}$$

for $a \in \mathbb{Q}_p^*$. By Tate's Theorem we have

$$\lambda_\psi^{(1)} = p^{\frac{1}{2}\cdot 2}\big(\chi_1(p) + \chi_2(p) + \chi_3(p)\big) = p\big(\gamma_{2_p}^{(1)}(\Phi) + \gamma_{1_p}(\Phi) + \gamma_{2_p}^{(2)}(\Phi)\big),$$

and since γ_{1_p} and γ_{2_p} are unitary by Deligne's Theorem, we have

$$|\lambda_\psi^{(1)}| \le 3p.$$

Similarly

$$\lambda_\psi^{(2)} = p\big(\gamma_{2_p}^{(1)}(\Phi)\gamma_{1_p}(\Phi) + \gamma_{2_p}^{(1)}(\Phi)\gamma_{2_p}^{(2)}(\Phi) + \gamma_{2_p}^{(2)}(\Phi)\gamma_{1_p}(\Phi)\big),$$

and

$$|\lambda_\psi^{(2)}| \le 3p.$$

In this case, the corresponding eigenvalue λ_ψ of the adjacency matrix δ satisfies the inequality $|\lambda_\psi| \le 6p$.

The parameters in this case correspond to global elliptic packets $\Pi(\iota) \in \Pi_e(\mathbf{G})$ with $\iota \in \Pi_s(\mathbf{H})$.

b) $\rho_2 = \gamma_2 \otimes \sigma_2$ with γ_2 one-dimensional and σ_2 two dimensional. In this case ${}^L A'_\psi = {}^L\big(U(1) \times U(1)\big)$. The representation σ_2 of $\mathrm{SL}(2, \mathbb{C})$ is the identity representation [13, II, Section 3]. We have

$$\begin{aligned}\rho_{\phi_\psi}(\omega) &= \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big) = \gamma_1(\omega) \oplus \gamma_2(\omega)\sigma_2\big(d(|\omega|^{1/2}, |\omega|^{-1/2})\big)\\ &= \gamma_1(\omega) \oplus \gamma_2(\omega) d(|\omega|^{1/2}, |\omega|^{-1/2}) = \gamma_1(\omega) \oplus d\big(\gamma_2(\omega)|\omega|^{1/2}, \gamma_2(\omega)|\omega|^{-1/2}\big).\end{aligned}$$

At the place p, the parameter ϕ_ψ is unramified and the representation $\rho_{\phi_{\psi_p}}$ of $L_{\mathbb{Q}_p}$ corresponds to the semisimple conjugacy class in $\mathrm{GL}(3, \mathbb{C})$ given by

$$\begin{aligned}\rho_{\phi_{\psi_p}}(\Phi) &= \gamma_{1_p}(\Phi) \oplus d\big(\gamma_{2_p}(\Phi)|\Phi|^{1/2}, \gamma_{2_p}(\Phi)|\Phi|^{-1/2}\big)\\ &= d\big(\gamma_{2_p}(\Phi)|\Phi|^{1/2}, \gamma_{1_p}(\Phi), \gamma_{2_p}(\Phi)|\Phi|^{-1/2}\big).\end{aligned}$$

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This semisimple conjugacy class corresponds to the unramified character $\chi = (\chi_1, \chi_2, \chi_3)$ of the subgroup of diagonal matrices of $\mathrm{GL}_3(\mathbb{Q}_p)$ with

$$\chi_1(a) = \big(\gamma_{2_p}(\Phi)|\Phi|^{1/2}\big)^{\mathrm{ord}_p(a)}$$
$$\chi_2(a) = \big(\gamma_{1_p}(\Phi)\big)^{\mathrm{ord}_p(a)}$$
$$\chi_3(a) = \big(\gamma_{2_p}(\Phi)|\Phi|^{-1/2}\big)^{\mathrm{ord}_p(a)}$$

for $a \in \mathbb{Q}_p^*$. By Tate's Theorem we have

$$\lambda_\psi^{(1)} = p^{\frac{1}{2}\cdot 2}\big(\chi_1(p) + \chi_2(p) + \chi_3(p)\big) = p\big(\gamma_{2_p}(\Phi)|\Phi|^{1/2} + \gamma_{1_p}(\Phi) + \gamma_{2_p}(\Phi)|\Phi|^{-1/2}\big),$$

and since $|\gamma_{1_p}(\Phi)| = |\gamma_{2_p}(\Phi)| = 1$, it follows that

$$|\lambda_\psi^{(1)}| \leq p(p^{-1/2} + 1 + p^{1/2}).$$

Similarly

$$\lambda_\psi^{(2)} = p\big(\gamma_{2_p}(\Phi)|\Phi|^{1/2}\gamma_{1_p}(\Phi) + \gamma_{2_p}^2(\Phi) + \gamma_{1_p}(\Phi)\gamma_{2_p}(\Phi)|\Phi|^{-1/2}\big),$$

and

$$|\lambda_\psi^{(2)}| \leq p(p^{-1/2} + 1 + p^{1/2}).$$

In this case, the corresponding eigenvalue λ_ψ of the adjacency matrix δ satisfies the inequality $|\lambda_\psi| \leq 2p(p^{1/2} + 1 + p^{-1/2})$. They are exceptional eigenvalues which make the building fail to be a Ramanujan type building with bound $6p$.
The parameters in this case correspond to the global packets in $\Pi_a(\mathbf{G})$. The representations in the A-packets are non-tempered, which is the reason the A-packets correspond to parameters whose attached representations are non-trivial on the $\mathrm{SL}(2,\mathbb{C})$-part.

Case (3) $\rho = \rho_1 \oplus \rho_2 \oplus \rho_3$ where, for $i = 1, 2, 3$, $\rho_i = \gamma_i \otimes \sigma_i$ are irreducible one-dimensional representations of $A_\mathbb{Q} \times \mathrm{SL}(2,\mathbb{C})$ with γ_i character of $A_\mathbb{Q}$ and σ_i the trivial one-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$. We only consider the case when ρ_1, ρ_2, ρ_3 are not equivalent to each other since otherwise the group S_ψ is infinite and the corresponding global packet is not discrete. In this case $S_\psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Since $S_\psi = \{d(s_1, s_2, s_3) : s_i = \pm 1\}$ it follows that the centralizer in $\hat{G}$ of S_ψ consists of elements of the form $g = d(a_{11}, a_{22}, a_{33})$. Thus the centralizer in $\hat{G}$ of S_ψ is isomorphic to $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C}) = \hat{C}$ and the endoscopic support of the global packet Π_ψ corresponding to ψ is ${}^L A_\psi = {}^L C$.

In this case,

$$\rho_{\phi_\psi}(\omega) = \rho\big(\omega \times d(|\omega|^{1/2}, |\omega|^{-1/2})\big) = \gamma_1(\omega) \oplus \gamma_2(\omega) \oplus \gamma_3(\omega).$$

At the place p, the parameter ϕ_ψ is unramified and the representation $\rho_{\phi_{\psi_p}}$ of $L_{\mathbb{Q}_p}$ corresponds to the semisimple conjugacy class in $\hat{G} = \mathrm{GL}(3,\mathbb{C})$ given by

$$\rho_{\phi_{\psi_p}}(\Phi) = \gamma_{1_p}(\Phi) \oplus \gamma_{2_p}(\Phi) \oplus \gamma_{3_p}(\Phi) = d\big(\gamma_{1_p}(\Phi), \gamma_{2_p}(\Phi), \gamma_{3_p}(\Phi)\big).$$

This semisimple conjugacy class corresponds to an unramified character $\chi = (\chi_1, \chi_2, \chi_3)$ of the subgroup of diagonal matrices of $\mathrm{GL}_3(\mathbb{Q}_p)$ with

$$\chi_i(a) = \big(\gamma_{i_p}(\Phi)\big)^{\mathrm{ord}_p(a)}, \quad i = 1, 2, 3,$$

for $a \in \mathbb{Q}_p^*$. By Tate's Theorem we have

$$\lambda_\psi^{(1)} = p^{\frac{1}{2}\cdot 2}\big(\chi_1(p) + \chi_2(p) + \chi_3(p)\big) = p\big(\gamma_{1_p}(\Phi) + \gamma_{2_p}(\Phi) + \gamma_{3_p}(\Phi)\big),$$

and since $|\gamma_{i_p}(\Phi)| = 1$ for $i = 1, 2, 3$, it follows that

$$|\lambda_\psi^{(1)}| \le 3p.$$

Similarly

$$\lambda_\psi^{(2)} = p\big(\gamma_{1_p}(\Phi)\gamma_{2_p}(\Phi) + \gamma_{1_p}(\Phi)\gamma_{3_p}(\Phi) + \gamma_{2_p}(\Phi)\gamma_{3_p}(\Phi)\big),$$

and

$$|\lambda_\psi^{(2)}| \le 3p.$$

In this case, the corresponding eigenvalue λ_ψ of the adjacency matrix δ satisfies the inequality $|\lambda_\psi| \le 6p$.

The parameters in this case correspond to global elliptic packets $\Pi(\iota) \in \Pi_e(\mathbf{G})$ with $\iota \in \Pi_e(\mathbf{H})$ such that $\iota = \iota(\theta)$ for a character θ of $\mathbf{C} = U(1) \times U(1) \times U(1)$ which is not semi-regular.

The three cases above exhaust all possibilities for eigenvalues of the adjacency matrix δ of the underlying graph of each connected component of the building quotient $\mathbb{B}$. This proves part a) of the main theorem.

Part b) follows from Theorem 14.6.3 in [19] which shows that if G' is a compact form of $U(3)$ arising from a division algebra, then $\Pi(\mathbf{G}') = \Pi_s(\mathbf{G}')$. Therefore, in this case the A-packets do not occur and there are no exceptional eigenvalues for the connected components of the building $\mathbb{B}$. This concludes the proof of the theorem. ∎

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