1
Holonomies and the group of loops

1.1 Introduction

In this chapter we will introduce holonomies and some associated concepts which will be important in the description of gauge theories to be presented in the following chapters. We will describe the group of loops and its infinitesimal generators, which will turn out to be a fundamental tool in describing gauge theories in the loop language.

Connections and the associated concept of parallel transport play a key role in locally invariant field theories like Yang–Mills and general relativity. All the fundamental forces in nature that we know of may be described in terms of such fields. A connection allows us to compare points in neighboring fibers (vectors or group elements depending on the description of the particular theory) in an invariant form. If we know how to parallel transport an object along a curve, we can define the derivative of this object in the direction of the curve. On the other hand, given a notion of covariant derivative, one can immediately introduce a notion of parallel transport along any curve.

For an arbitrary closed curve, the result of a parallel transport in general depends on the choice of the curve. To each closed curve γ in the base manifold with origin at some point o the parallel transport will associate an element H of the Lie group G associated to the fiber bundle. The parallel transported element of the fiber is obtained from the original one by the action of the group element H. The path dependent object $H(\gamma)$ is usually called the *holonomy*. It has been considered in various contexts in physics and given different names. For instance, it is known as the *Wu–Yang phase factor* in particle physics.

Curvature is related to the failure of an element of the fiber to return to its original value when parallel transported along a small closed curve. When evaluated on an infinitesimal closed curve with basepoint o, the

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holonomy has the same information as the curvature at o. Knowledge of the holonomy for any closed curve with a base point o allows one, under very general hypotheses, to reconstruct the connection at any point of the base manifold up to a gauge transformation. An important fact about holonomies is their invariance under the set of gauge transformations which act trivially at the base point. We will later show that this will imply that the physical configurations of any gauge theory can be faithfully and uniquely (up to transformations at the base point) represented by their holonomies. They can therefore be used to encode all the kinematical information about the theory in question.

Since the early 1960s several descriptions of gauge theories in terms of holonomies have been considered. They seem to be particularly well suited to study the non-perturbative features at the quantum level. In recent years interest in the non-local descriptions of gauge theories has been greatly increased by the introduction of a new set of canonical variables that allow one to describe the phase space of general relativity in a manner that resembles an $SU(2)$ Yang–Mills theory. In fact, holonomies may well provide a common geometrical framework for all the fundamental forces in nature

A generalization of the notion of holonomy may be defined intrinsically without any reference to connections. It will turn out that this point of view has more than a purely mathematical interest and is the origin of important results that are relevant to the physical applications. Holonomies can be viewed as homomorphisms from a group structure defined in terms of equivalence classes of closed curves onto a Lie group G. Each equivalence class of closed curves is what we will technically call a loop and the group structure defined by them is called the group of loops.

The group of loops is the basic underlying structure of all the non-local formulations of gauge theories in terms of holonomies. In particular, when quantizing the theory, wavefunctions in the "loop representation" are really functions dependent on the elements of the group of loops*. This is the physical reason why it is important to understand the structure of the group of loops, since it is the "arena" where the quantum loop representation takes place.

In spite of the fact that the group of loops is not a Lie group, it is possible to define infinitesimal generators for it. When they are represented in the space of functions of loops, they give rise to differential operators in loop space. Some of these operators have appeared in various physical contexts and have been given diverse names such as "area derivative",

* In this context the group of loops is usually referred to as "loop space" and we will loosely use this terminology when it does not give rise to ambiguities. Notice that it is not related to the "loop groups" in the main mathematical literature.

"keyboard derivative", "loop derivative". In most of these presentations the group properties of loops were largely ignored and this resulted in various inconsistencies. In the approach we follow in this chapter all these operators arise simply and consistently as representations of the infinitesimal generators of the group of loops.

In many presentations, loop space is formulated with parametrized curves. In this context differential operators are usually written in terms of functional derivatives. The group structure of loops is hidden by these formulations and it is easy to overlook it, again leading to inconsistencies. In this book we will deal with unparametrized loops which allow for a cleaner formulation, only resorting to parametrizations for some particular results.

This chapter is structured in the following way. In section 1.2 we define the group of loops and discuss its topology and its action on open paths. In section 1.3 we introduce the infinitesimal generators of the group and their differential representation. We also introduce differential operators acting on open paths. In section 1.3.3 we introduce the connection derivative, its relation to the loop derivative and to usual notions of gauge theory. In section 1.3.4 we discuss the contact and functional derivatives in loop space and their relations with diffeomorphisms. In section 1.4 we introduce the idea of representations of the group of loops in a Lie group and we retrieve the classical kinematics of gauge theories. We end with a summary of the ideas developed in this chapter.

1.2 The group of loops

We start by considering a set of parametrized curves on a manifold M that are continuous and piecewise smooth. A curve p is a map

$$p : [0, s_1] \cup [s_1, s_2] \cdots [s_{n-1}, 1] \ \rightarrow \ M \tag{1.1}$$

smooth in each closed interval $[s_i, s_{i+1}]$ and continuous in the whole domain. There is a natural composition of parametrized curves. Given two piecewise smooth curves p_1 and p_2 such that the end point of p_1 is the same as the beginning point of p_2, we denote by $p_1 \circ p_2$ the curve:

$$p_1 \circ p_2(s) = \begin{cases} p_1(2s), & \text{for } s \in [0, 1/2] \\ p_2(2(s-1/2)) & \text{for } s \in [1/2, 1]. \end{cases} \tag{1.2}$$

The curve traversed in the opposite orientation ("opposite curve") is given by

$$p^{-1}(s) := p(1-s). \tag{1.3}$$

In what follows, we will mainly be interested in unparametrized curves. We will therefore define an equivalence relation by identifying the curve

p and $p \circ \phi$ for all orientation preserving differentiable reparametrizations $\phi : [0,1] \rightarrow [0,1]$. It is important to note that the composition of unparametrized curves is well defined and independent of the members of the equivalence classes used in their definition.

We will now consider closed curves $l, m, \ldots$, that is, curves which start and end at the same point o. We denote by L_o the set of all these closed curves. The set L_o is a semi-group under the composition law $(l, m) \rightarrow l \circ m$. The identity element ("null curve") is defined to be the constant curve $i(s) = o$ for any s and any parametrization. However, we do not have a group structure, since the opposite curve l^{-1} is not a group inverse in the sense that $l \circ l^{-1} \neq i$.

Holonomies are associated with the parallel transport around closed curves. In the case of a trivial bundle the connection is given by a Lie-algebra-valued one form A_a on M. The parallel transport around a closed curve $l \in L_o$ is a map from the fiber over o to itself given by the path ordered exponential (for the definition of path ordered exponential see reference [1]),

$$H_A(l) = P \exp \int_l A_a(y) dy^a. \tag{1.4}$$

In the general case of a principal fiber bundle $P(M, G)$ with group G over M the holonomy map is defined as follows. We choose a point $\hat{o}$ in the fiber over o and by using the connection A we lift the closed curve l in M to a curve $\hat{l}$ in P such that the beginning point is

$$\hat{l}(0) = \hat{o} \tag{1.5}$$

and the end point is given by

$$\hat{l}(1) = \hat{l}(0) H_A(l), \tag{1.6}$$

which defines $H_A(l)$. The holonomy H_A is an element of the group G and the product denotes the right action of G. The main property of H_A is

$$H_A(l \circ m) = H_A(l) H_A(m). \tag{1.7}$$

A change in the choice of the point on the fiber over o replacing $\hat{o}$ for $\hat{o}' = \hat{o} g$ induces the transformation

$$H'_A(l) = g^{-1} H_A(l) g. \tag{1.8}$$

In order to transform the set L_o into a group, we need to introduce a further equivalence relation. The rationale for this relation is to try to identify all closed curves leading to the the same holonomy for all smooth connections, since curves with the same holonomy carry the same information towards building the physical quantities of the theory. The classes of equivalence under this relation are what we will from now on call *loops*

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and we will denote them with Greek letters, to distinguish them from the individual curves which form the equivalence classes. Several definitions of this equivalence relation have been proposed. Each of them sheds some light on the group structure so we will take a minute to consider them in some detail.

Definition 1
Let

$$H_A : L_o \to G \tag{1.9}$$

be the holonomy map of a connection A defined on a bundle $P(M, G)$. Two curves $l, m \in L_o$ are equivalent [2] [4] $l \sim m$ iff

$$H_A(l) = H_A(m) \tag{1.10}$$

for every bundle $P(M, G)$ and smooth connection A.

Definition 2
We start by defining loops which are equivalent to the identity. A closed curve l is called a *tree*[5] or *thin* [6] if there exists a homotopy of l to the null curve in which the image of the homotopy is included in the image of l. This kind of curves does not "enclose any area" of M. Two closed curves $l, m \in L_o$ are equivalent $l \sim m$ iff $l \circ m^{-1}$ is thin. Obviously a thin curve is equivalent to the null curve.

Definition 3 [7]
Given the closed curves l and m and three open curves p_1, p_2 and q such that

$$l = p_1 \circ p_2 \tag{1.11}$$
$$m = p_1 \circ q \circ q^{-1} \circ p_2 \tag{1.12}$$

then $l \sim m$.

There is a fourth definition, due to Chen [7], that requires the use of a set of objects (Chen integrals, which we will call "loop multitangents") that we will define in chapter 2, but we will not discuss it here.

It can be shown that definitions 2 and 3 are equivalent. Moreover, it is also immediate to notice that two curves equivalent under definitions 2 or 3 are also equivalent under definition 1. The reciprocal is not obvious. Partial results can be found in reference [7] and a complete proof for piecewise analytic curves has been presented by Ashtekar and Lewandowski [40].

With any of these definitions one can show that the composition between loops is well defined and is again a loop. In other words if $\alpha \equiv [l]$

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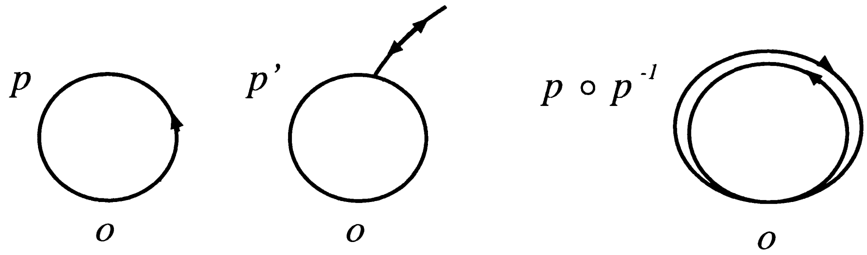

Fig. 1.1. Curves p and p' differ by a tree. The composition of a curve and its inverse is a tree.

and $\beta \equiv [m]$ then $\alpha \circ \beta = [l \circ m]$ where by [] we denote the equivalence classes. From now on we will denote loops with greek letters, to distinguish them from curves[†].

Notice that with the equivalence relation defined, it makes sense to define an inverse of a loop. Since the composition of a curve with its opposite yields a tree (see figure 1.1) it is natural, given a loop α, to define its inverse α^{-1} by $\alpha \circ \alpha^{-1} = \iota$ where ι is the set of closed curves equivalent to the null curve (thin loops or trees). α^{-1} is the set of curves opposite to the elements of α.

We will denote the set of loops basepointed at o by $\mathcal{L}_o$. Under the composition law given by $\circ$ this set is a non-Abelian group, which is called the group of loops.

A well known result [5] is that any homomorphism,

$$\mathcal{L}_o \to G, \tag{1.13}$$

where G is a Lie group, defines a *holonomy* associated with a "generalized" connection. By generalized we mean that the connection will not, in general, be a smooth function (for instance it could be distributional or worse). One can, by imposing extra smoothness conditions [6, 4] on the homomorphism, ensure that a differentiable principal fiber bundle and a connection are defined such that H is the holonomy of this connection. Recall that under a homomorphism, the composition law of the group of loops is mapped onto the composition law of the Lie group G,

$$H(\alpha \circ \beta) = H(\alpha)H(\beta), \tag{1.14}$$

[†] Notice that in this book we will use the word loop in a very precise sense, denoting the holonomic-equivalent classes of curves. Other equivalences can be considered. The idea of a group of loops has appeared in other unrelated contexts [42]. For this reason some authors have proposed calling the holonomic equivalence classes "hoops" to avoid confusion [3].

and that inverses are mapped to each other,

$$H(\alpha^{-1}) = (H(\alpha))^{-1}. \tag{1.15}$$

We will come back to this property in section 1.4 when we discuss the infinitesimal generators and their relations to the physical quantities.

From now on we will routinely use functions of loops, such as the holonomy that we just introduced. Obviously, not any function of curves qualifies as a function of loops. An immediate example of this would be to consider the length of a curve, which takes different values on the different curves that form the equivalence class defining a loop.

It is useful to introduce a notion of continuity in loop space, since we will be frequently using functions defined on this space. We will define two loops α and β to be close, in the sense that α in a neighborhood $U_\epsilon(\beta)$ if there exist at least two parametrized curves $a(s) \in \alpha$ and $b(s) \in \beta$ such that $a(s) \in U_\epsilon(b(s))$ with the usual topology of curves in the manifold‡. With this topology, the group of loops is a topological group.

It is convenient for future use to introduce an equivalence relation for open curves similar to the one we introduced for closed curves. We will call the equivalence classes of open curves "paths". Given two open curves p_o^x and q_o^x from the basepoint to a point x in the manifold, we will define these curves to be equivalent iff $p_o^x {q^{-1}}_o^x$ is a tree§. We will denote paths with Greek letters as we do for loops, but indicating the origin and end points, as in α_o^x. Given two different paths starting and ending at the same points, it is immediate to see that the composition of one with the opposite of the other is a loop. Analogously one can compose loops with paths to produce new paths with the same end points. Furthermore, the notion of topology introduced for loops can immediately be generalized to paths. However, paths cannot be structured into a group, since it is not possible to compose, in general, two paths to form a new path (the end of one of them has to coincide with the beginning of the other in order to do this).

1.3 Infinitesimal generators of the group of loops

We will now consider a representation of the group of loops given by operators acting on continuous functions under the topology introduced in the previous section. We will introduce a set of differential operators

‡ Lewandowski [4], elaborating on a suggestion by Barrett [6] has introduced a topology defined in terms of homotopies of loops. The group of loops endowed with this topology is a topological Haussdorff group.

§ From now on we will interchangeably use the notations ${q^{-1}}_o^x$ and q_x^o to designate the same object, the curve q traversed from x to o. A similar convention will be adopted for paths.

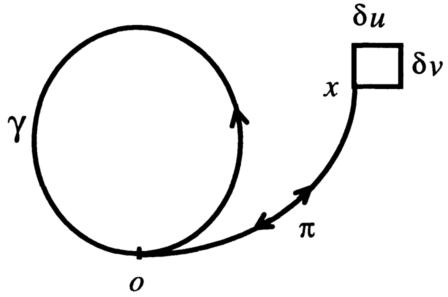

Fig. 1.2. The infinitesimal loop that defines the loop derivative.

acting on these functions that are related to the infinitesimal generators of the group of loops, in terms of which one can construct the elements of the group. In later chapters we will show that these operators are related to physical quantities of gauge theories. Although the explicit introduction of the differential operators will be made in a coordinate chart, we will show that the definitions do not depend on the particular chart chosen. A more intrinsic definition, also making use of the properties of the group of loops has been proposed by Tavares [43].

1.3.1 The loop derivative

Given $\Psi(\gamma)$ a continuous, complex-valued function of $\mathcal{L}_o$ we want to consider its variation when the loop γ is changed by the addition of an infinitesimal loop $\delta\gamma$ basepointed at a point x connected by a path π_o^x to the basepoint of γ, as shown in figure 1.2. That is, we want to evaluate the change in the function when changing its argument from γ to $\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma$. In order to do this we will consider a two-parameter family of infinitesimal loops $\delta\gamma$ that contain in a particular coordinate chart the curve obtained by traversing the vector u^a from x^a to $x^a + \epsilon_1 u^a$, the vector v^a from $x^a + \epsilon_1 u^a$ to $x^a + \epsilon_1 u^a + \epsilon_2 v^a$, the vector $-u^a$ from $x^a + \epsilon_1 u^a + \epsilon_2 v^a$ to $x^a + \epsilon_2 v^a$ and the vector $-v^a$ from $x^a + \epsilon_2 v^a$ back to x^a as shown in figure 1.2. We will denote these kinds of curves with the notation¶ $\delta u \delta v \delta \bar{u} \delta \bar{v}$.

¶ In order not to clutter the notation we will not distinguish between curves and paths here. We also drop the ϵ_i dependence of each path. The path $\delta\bar{u} \equiv (\delta u)^{-1}$.

For a given π and γ a loop differentiable function depends only on the infinitesimal vectors $\epsilon_1 u^a$ and $\epsilon_2 v^a$. We will assume it has the following expansion with respect to them,

$$\begin{aligned}\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = \; & \Psi(\gamma) + \epsilon_1 u^a Q_a(\pi_o^x)\Psi(\gamma) + \epsilon_2 v^a P_a(\pi_o^x)\Psi(\gamma) \\ & + \tfrac{1}{2}\epsilon_1\epsilon_2(u^a v^b + v^a u^b) S_{ab}(\pi_o^x)\Psi(\gamma) \\ & + \tfrac{1}{2}\epsilon_1\epsilon_2(u^a v^b - v^a u^b)\Delta_{ab}(\pi_o^x)\Psi(\gamma). \end{aligned} \tag{1.16}$$

where Q, P, S, Δ are differential operators on the space of functions $\Psi(\gamma)$. If ϵ_1 or ϵ_2 vanishes or if u is collinear with v then $\delta\gamma$ is a tree and all the terms of the right-hand side except the first one must vanish. This means that $Q = P = S = 0$. Since the antisymmetric combination $(u^a v^b - v^a u^b)$ vanishes, Δ need not be zero. That is, a function is loop differentiable if for any path π_o^x and vectors u, v, the effect of an infinitesimal deformation is completely contained in the path dependent antisymmetric operator $\Delta_{ab}(\pi_o^x)$,

$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = (1 + \tfrac{1}{2}\sigma^{ab}(x)\Delta_{ab}(\pi_o^x))\Psi(\gamma), \tag{1.17}$$

where $\sigma^{ab}(x) = 2\epsilon_1\epsilon_2(u^{[a}v^{b]})$ is the element of area of the infinitesimal loop $\delta\gamma$. We will call this operator the loop derivative.

Notice that we have proved that for an arbitrary function of loop space, one does not have contributions from the terms Q, P, S in the expansion (1.16). If one considers functions of curves rather than of loops, these terms will in general be present. As an example, they are present if one considers the function given by the length of the curve. On the other hand, not every function of loop space is differentiable. For instance, we will see when we consider knot invariants — functionals of loops invariant under smooth deformations of the loops — that they are not strictly speaking loop differentiable. The reason for this is that sometimes appending an infinitesimal loop could enable us to change the topology of the knots and therefore to induce finite changes in the values of the functions.

Loop derivatives of various kinds were considered by several authors. The idea was introduced by Mandelstam [8]. Later generalizations can be found in the work of Chen [7], Makeenko and Migdal [10, 12], Polyakov [44], Gambini and Trias [13, 14, 15], Blencowe [16] and Brügmann and Pullin [26]. Other references can be found in Loll [17]. The various definitions are not equivalent, and many of them refer to objects that are in reality different from the loop derivative we are defining here. One of the main differences is that in many treatments the infinitesimal loop, instead of being appended at an arbitrary fixed point of the manifold defined by a path π_o^x as is our case, is appended to a point *that lies on the loop.* Since one is considering functions of arbitrary loops that means that the point where the derivative acts has to be redefined when

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considering its value on a new loop. In other words, the domain of the function that results when applying these kinds of derivatives is not the loop space defined in section 1.2, but the space of loops with a marked point. Makeenko and Migdal [12] noticed this fact and this drove them to call it "keyboard derivative."

Notice that this is not the case for the derivative we defined. The result of the application of the loop derivative to a function of a loop is also a function of a loop. For each arbitrary open path there is a different derivative. For these definitions to work it is crucial to have a basepoint, which provides a fixed point for any loop on which to attach the open path that defines the derivative. These considerations are of crucial importance. For instance, we will soon prove that our derivative satisfies Bianchi identities, a fact that cannot be proven for derivatives that act only on points of the loop. The relevance of the group of loops and the path dependence of the loop derivative were first recognized by Gambini and Trias [13, 15].

At the end of section 1.2 we noted that the elements of the group of loops have a natural action on open paths, giving as a result a deformation of the path. We can immediately find an example of this fact in terms of a differential operator defined by simply extending the definition of the loop derivative (1.17) to give for open paths

$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma_o^y) = (1 + \tfrac{1}{2}\sigma^{ab}(x)\Delta_{ab}(\pi_o^x))\Psi(\gamma_o^y). \tag{1.18}$$

We will take some notational latitude to give the same name to the loop derivative acting on paths and on loops. In all cases the context will uniquely determine to which derivative we are referring. Notice that this extension to open paths is not at all clear for derivatives that depend on a point of the loop as is the case of the "keyboard derivative".

1.3.2 Properties of the loop derivative

• **Tensor character.** By its very definition, (1.17), it is immediate to see that the loop derivative has to behave as a tensor under local coordinate transformations containing the end point of the path π_o^x for loop differentiable functions. One need just require that the whole expression be invariant and notice that the loop derivative is contracted with the tensor σ^{ab}. Therefore by quotient law, it must be a tensor. Notice that the loop derivative is really associated with the surface spanned by du^a and dv^b rather than with the individual infinitesimal vectors, being invariant under vector transformations that preserve the element of area.

• **Commutation relations.** The loop derivatives are non-commutative operators. This, as we will see later, is naturally associated with the fact

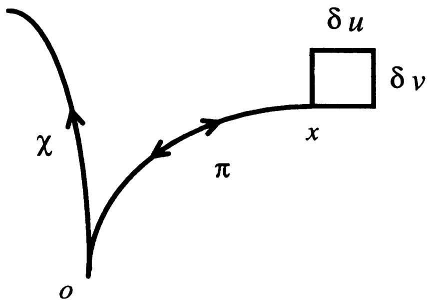

Fig. 1.3. The two paths used to compute the commutation relation

that they correspond to the generators of a non-Abelian group. Their commutation relations can be computed directly from the geometric properties of the group of loops in the following way. Consider two infinitesimal loops $\delta\eta_1$, $\delta\eta_2$ given by

$$\delta\eta_1 = \pi_o^x \circ \delta u \delta v \delta\bar{u}\delta\bar{v} \circ \pi_x^o \quad \text{and} \quad \delta\eta_2 = \chi_o^y \circ \delta q \delta r \delta\bar{q}\delta\bar{r} \circ \chi_y^o \tag{1.19}$$

and with area elements

$$\sigma_1^{ab} = \epsilon_1\epsilon_2(u^a v^b - v^a u^b) \quad \text{and} \quad \sigma_2^{ab} = \epsilon_3\epsilon_4(q^a r^b - r^a q^b). \tag{1.20}$$

Then we can derive the following relation:

$$\begin{aligned} &\Psi(\delta\eta_1 \circ \delta\eta_2 \circ (\delta\eta_1)^{-1} \circ (\delta\eta_2)^{-1} \circ \gamma) = (1 + \tfrac{1}{2}\sigma_1^{ab}\Delta_{ab}(\pi_o^x)) \\ &\times(1 + \tfrac{1}{2}\sigma_2^{cd}\Delta_{cd}(\chi_o^y))(1 - \tfrac{1}{2}\sigma_1^{ef}\Delta_{ef}(\pi_o^x))(1 - \tfrac{1}{2}\sigma_2^{gh}\Delta_{gh}(\chi_o^y))\Psi(\gamma) = \\ &(1 + \tfrac{1}{4}\sigma_1^{ab}\sigma_2^{cd}[\Delta_{ab}(\pi_o^x), \Delta_{cd}(\chi_o^y)])\Psi(\gamma). \end{aligned} \tag{1.21}$$

The first equality follows from the definition of the loop derivative and of the loops $\delta\eta_i$. To prove the second, one expands keeping only terms of first order in each ϵ_i and neglecting those of order ϵ_i^2.

We will now define an open path by composing the two paths we have been using

$$\chi'^y_o = \delta\eta_1 \circ \chi_o^y. \tag{1.22}$$

This allows us to rewrite the loop composed by the first three loops in the argument of Ψ in the left-hand side of equation (1.21) as,

$$\delta\eta_1 \circ \delta\eta_2 \circ (\delta\eta_1)^{-1} = \chi'^y_o \circ \delta q \delta r \delta\bar{q}\delta\bar{r} \circ \chi'^o_y. \tag{1.23}$$

Therefore,

$$\Psi(\delta\eta_1 \circ \delta\eta_2 \circ (\delta\eta_1)^{-1} \circ (\delta\eta_2)^{-1} \circ \gamma) = \\ (1 + \tfrac{1}{2}\sigma_2^{ab}\Delta_{ab}(\chi'^{y}_{o}))(1 - \tfrac{1}{2}\sigma_2^{cd}\Delta_{cd}(\chi^{y}_{o}))\Psi(\gamma). \quad (1.24)$$

And again expanding in ϵs and keeping only the first order in each ϵ_i we get

$$(1 + \tfrac{1}{2}\sigma_2^{ab}\Delta_{ab}(\chi'^{y}_{o}))(1 - \tfrac{1}{2}\sigma_2^{cd}\Delta_{cd}(\chi^{y}_{o}))\Psi(\gamma) = \\ (1 + \tfrac{1}{4}\sigma_1^{ab}\sigma_2^{cd}\Delta_{ab}(\pi^{x}_{o})[\Delta_{cd}(\chi^{y}_{o})])\Psi(\gamma), \quad (1.25)$$

where in the last expression $\Delta_{ab}(\pi^x_o)[\Delta_{cd}(\chi^y_o)]$ represents the action of the first loop derivative only on the path dependence of the second derivative.

All this implies

$$[\Delta_{ab}(\pi^x_o), \Delta_{cd}(\chi^y_o)] = \Delta_{cd}(\chi^y_o)[\Delta_{ab}(\pi^x_o)], \quad (1.26)$$

from which it is immediate to show that

$$\Delta_{ab}(\pi^x_o)[\Delta_{cd}(\chi^y_o)] = -\Delta_{cd}(\chi^y_o)[\Delta_{ab}(\pi^x_o)]. \quad (1.27)$$

These expressions highlight the path dependence of the loop derivative, in the sense that they express the variation of the derivative when the path is varied. We will see at the end of this subsection how these expressions can be naturally interpreted as a group commutator when we prove that the loop derivative is a generator of the group of loops.

This commutation relation can be viewed in a different light by considering its integral expression. In order to do this, we will introduce a loop dependent operator $U(\alpha)$ on the space of functions of loops which has the effect of introducing a finite deformation in the argument of the function,

$$U(\alpha)\Psi(\gamma) \equiv \Psi(\alpha \circ \gamma). \quad (1.28)$$

The operator has a naturally defined inverse,

$$U(\alpha)^{-1} = U(\alpha^{-1}), \quad (1.29)$$

and has a natural composition law,

$$U(\alpha)U(\beta)\Psi(\gamma) = U(\alpha \circ \beta)\Psi(\gamma). \quad (1.30)$$

We now consider the action of the loop derivative evaluated along a deformed path, shown in figure 1.4, on a function of loop, and applying the definition of loop derivative (1.17) we get

$$(1 + \tfrac{1}{2}\sigma^{ab}\Delta_{ab}(\alpha \circ \pi^x_o))\Psi(\gamma) = \Psi(\alpha \circ \pi^x_o \circ \delta\gamma \circ \pi^o_x \circ \alpha^{-1} \circ \gamma), \quad (1.31)$$

where $\delta\gamma$ is the infinitesimal loop associated with the area element σ^{ab}. We then use the definition of the operator U (1.28) to get

$$\Psi(\alpha \circ \pi^x_o \circ \delta\gamma \circ \pi^o_x \circ \alpha^{-1} \circ \gamma) = U(\alpha)(1 + \tfrac{1}{2}\sigma^{ab}\Delta_{ab}(\pi^x_o))U(\alpha)^{-1}\Psi(\gamma), \quad (1.32)$$

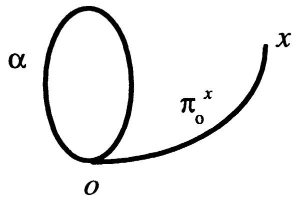

Fig. 1.4. The deformed path used to derive the integral expression of the commutation relations

from which we can read off the identity,

$$\Delta_{ab}(\alpha \circ \pi_o^x) = U(\alpha)\Delta_{ab}(\pi_o^x)U(\alpha)^{-1}, \tag{1.33}$$

which expresses the transformation property of the loop derivative under finite deformations of its path dependence. We will see at the end of this chapter that this expression is the reflection in the language of loops of the gauge covariance of the field tensor in a gauge theory.

• **Bianchi identities.** There is a second set of relations that again can be directly obtained from the geometric properties of the group of loops. One can readily see that they are a reflection of the usual Bianchi identities of Yang–Mills theories. In order to describe them we need to introduce a new differential operator, which we will call the end point derivative or Mandelstam covariant derivative [8], that acts on functions of open paths.

Given a function of an open path $\Psi(\pi_o^x)$, a local coordinate chart at the point x and a vector in that chart u^a, we define the Mandelstam derivative by considering the change in the function when the path is extended from x to $x + \epsilon u$ by the infinitesimal path δu shown in figure 1.5 as

$$\Psi(\pi_o^x \circ \delta u) = (1 + \epsilon u^a D_a)\Psi(\pi_o^x). \tag{1.34}$$

We denote the new path as $\pi_o^{x+\epsilon u}$. If one performs a coordinate transformation, noting that ϵu^a is a vector and applying the quotient law, it is immediate to see that D_a transforms as a one-form.

Having introduced this operator, we are now ready to derive the Bianchi identity. As usual, the fundamental idea is that "the boundary of a boundary vanishes". In the group of loops language, this can be expressed by considering a thin loop ι. A representative curve of this loop has the shape of a box with sides δu, δv and δw, connected to the origin by the path π_o^x as shown in figure 1.6.

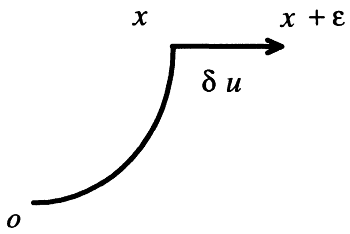

Fig. 1.5. The extended path defining the Mandelstam derivative

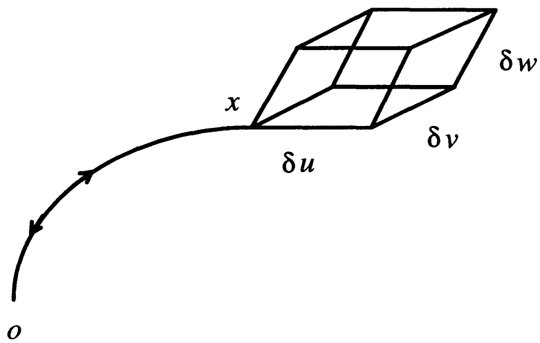

Fig. 1.6. The loop used to derive the Bianchi identity for the loop derivative.

The curve δu represents paths that go from a generic point x to $x+\epsilon_1 u$ and similarly for δv and δw with increments $\epsilon_2 v$ and $\epsilon_3 w$ respectively. Explicitly,

$$\begin{aligned}\iota = {} & \pi_o^x \circ \delta u\delta v\delta w\delta\bar{v}\delta\bar{w}\delta\bar{u} \circ \pi_x^o \circ \pi_o^x \circ \delta u\delta w\delta\bar{u}\delta\bar{w} \circ \pi_x^o \\ & \circ\pi_o^x \circ \delta w\delta u\delta v\delta\bar{u}\delta\bar{v}\delta\bar{w} \circ \pi_x^o \circ \pi_o^x \circ \delta w\delta v\delta\bar{w}\delta\bar{v} \circ \pi_x^o \\ & \circ\pi_o^x \circ \delta v\delta w\delta u\delta\bar{w}\delta\bar{u}\delta\bar{v} \circ \pi_x^o \circ \pi_o^x \circ \delta v\delta u\delta\bar{v}\delta\bar{u} \circ \pi_x^o . \end{aligned} \tag{1.35}$$

Now, since ι is a tree and therefore indistinguishable from the identity loop, we have

$$\Psi(\gamma) = \Psi(\iota\circ\gamma) \quad \forall\gamma . \tag{1.36}$$

Noting that the tree is built by six infinitesimal loops (the "faces" of the box shown in figure 1.6), each connected to the origin via the path π and the "sides" of the box, we can rewrite this identity in terms of loop derivatives as

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$$\Psi(\gamma) = (1 + \epsilon_2\epsilon_3 v^a w^b \Delta_{ab}(\pi_o^{x+\epsilon_1 u}))(1 + \epsilon_1\epsilon_3 u^c w^d \Delta_{cd}(\pi_o^x))$$
$$\times (1 + \epsilon_1\epsilon_2 u^e v^f \Delta_{ef}(\pi_o^{x+\epsilon_3 w}))(1 + \epsilon_3\epsilon_2 w^g v^h \Delta_{gh}(\pi_o^x))$$
$$\times (1 + \epsilon_3\epsilon_1 w^i u^j \Delta_{ij}(\pi_o^{x+\epsilon_2 v}))(1 + \epsilon_2\epsilon_1 v^k u^l \Delta_{kl}(\pi_o^x))\Psi(\gamma). \quad (1.37)$$

Collecting the terms of first order in each ϵ_i, applying the definition of the Mandelstam derivative and noting that u, v, w are arbitrary we get the final form of the Bianchi identities for the loop derivatives,

$$D_a\Delta_{bc}(\pi_o^x) + D_b\Delta_{ca}(\pi_o^x) + D_c\Delta_{ab}(\pi_o^x) = 0. \quad (1.38)$$

We will soon see applications of the above derived identities and their crucial role in the formulation of gauge theories in terms of loops.

• **The Ricci identity.** Consider the action of four Mandelstam covariant derivatives along the vectors u, v on a function of an open path $\Psi(\pi_o^x)$. Keeping the terms of first order in $\epsilon_1\epsilon_2$ in the left-hand side of the next expression we get

$$(1 + \epsilon_1 u^a D_a)(1 + \epsilon_2 v^b D_b)(1 - \epsilon_1 u^c D_c)(1 - \epsilon_2 v^d D_d)\Psi(\pi_o^x) =$$
$$(1 + \epsilon_1\epsilon_2 u^a v^b [D_a, D_b])\Psi(\pi_o^x). \quad (1.39)$$

The action of the four covariant derivatives is equivalent to appending an infinitesimal loop at the end of the path π_o^x and therefore can be written in terms of the loop derivative,

$$[D_a, D_b]\Psi(\pi_o^x) = \Delta_{ab}(\pi_o^x)\Psi(\pi_o^x). \quad (1.40)$$

This expression is the loop analogue of the usual expression of the commutator of covariant derivatives in terms of the curvature and we will again see its implications for gauge theories at the end of this chapter.

• **The loop derivative as a generator of the group of loops.** Let us now show that we can, by superposition of loop derivatives, generate any finite loop homotopic to the identity. We need to introduce a parametrization for this proof. Let $\gamma(s)$ be a parametrized curve belonging to the equivalence class defining the finite loop γ with $s \in [0, 1]$. Consider a one-parameter family of parametrized loops $\eta(s, t)$ interpolating smoothly between $\gamma(s)$ and the identity loop, such that $\eta(s, 0)$ is in the equivalence class of the identity loop and $\eta(s, 1) = \gamma(s)$. Consider the curves $\eta(1, s)$ $(= \gamma(s))$ and $\eta(1 - \epsilon, s)$. The two curves are drawn in figure 1.7 and differ by an infinitesimal element of area. The whole purpose of our proof will be to cover the infinitesimal area separating the two mentioned curves with a "checkerboard" of infinitesimal closed curves

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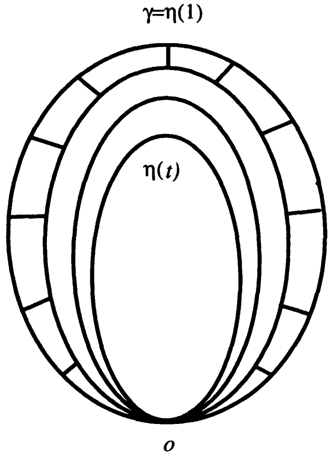

Fig. 1.7. The construction of a finite loop from the loop derivative. The curves $\delta\eta_i$ are determined by two elements in the family $\eta(t)$.

such that along each of them one can define a loop derivative. One can therefore express the curve $\gamma(s)$ as

$$\gamma(s) = \lim_{n\to\infty} \eta(s, 1-\epsilon) \circ \delta\eta_1 \circ \cdots \circ \delta\eta_n, \tag{1.41}$$

where the $\delta\eta_i$ are shown in figure 1.7. Analytically, in terms of differential operators on functions of loops we can write||

$$\begin{aligned}\Psi(\eta(1)) &= \Psi(\eta(1-\epsilon)) \\ &\quad + \epsilon \oint_0^1 ds \dot{\eta}^a(1-\epsilon, s)\eta'^b(1-\epsilon, s)\Delta_{ab}(\eta(1-\epsilon)_o^s)\Psi(\eta(1-\epsilon)),\end{aligned} \tag{1.42}$$

where $\dot{\eta}(t,s) \equiv d\eta(t,s)/ds$ and $\eta'(t,s) \equiv d\eta(t,s)/dt$ It is immediate to proceed from $\eta(1-\epsilon, s)$ inwards just by repeating the same construction, and so continuing until the final curve is the identity. The end result is

$$\Psi(\eta(1)) = \mathrm{T}\exp\left(\int_0^1 dt \oint_0^1 ds \dot{\eta}^a(t,s)\eta'^b(t,s)\Delta_{ab}(\eta(t)_o^s)\Psi(\eta(0))\right) \tag{1.43}$$

where the outer integral is ordered in t (T-ordered). This result is the loop version of the non-Abelian Stokes theorem of gauge theories [18] and it shows that the loop derivative is a generator of loop space, i.e., it allows us to generate any finite loop homotopic to the identity.

Notice that the expression for the finite element of the group involves a superposition of an infinite number of generators associated with different

|| We drop the s dependence of η where it is not relevant.

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paths. This is not the usual situation that one encounters in Lie groups, where it is enough to exponentiate one generator to obtain any element of the group. This is another indication of the non-Lie character of the group of loops. It is a direct consequence of the impossibility of defining a non-integer number of powers of $\delta\eta$.

Finally, identifying the loop derivative as a generator of the group of loops allows us to rewrite in a revealing form the commutation relations of the loop derivatives. Applying the definition of the loop derivative, equation (1.26) can be cast in the following way

$$[\Delta_{ab}(\pi_o^x), \Delta_{cd}(\chi_o^y)] = \lim_{\epsilon_i \to 0} \frac{1}{\sigma^{ab}} (\Delta_{cd}(\delta\eta_1 \circ \chi_o^y) - \Delta_{cd}(\chi_o^y)), \qquad (1.44)$$

which is the usual expression of the commutator in terms of a linear combination of elements of the algebra. So we see that the group of loops formally obeys commutation relations similar to those of a Lie group.

1.3.3 Connection derivative

In section 1.3.1 we introduced the loop derivative. We saw in the previous section that this operator has several properties resembling those of the curvature or field tensor of a gauge theory. We will now introduce a differential operator with properties similar to those of the connection or vector potential of a gauge theory [14, 15, 4]. This operator appears naturally as an intermediate step in the construction of gauge theories from the group of loops. Although one could formulate a gauge theory completely in terms of the path dependent loop derivative alone, the treatment that we will follow will lead us to a more familiar formulation of gauge theories.

Let us consider a covering of the manifold with overlapping coordinate patches. We attach to each coordinate patch $\mathcal{P}^i$ a path $\pi_o^{y_0^i}$ going from the origin of the loop to a point y_0^i in $\mathcal{P}^i$. We also introduce a continuous function with support on the points of the chart $\mathcal{P}^i$ such that it associates to each point x on the patch a path $\pi_{y_0^i}^x$. Given a vector u at x, the connection derivative of a continuous function of a loop $\Psi(\gamma)$ will be obtained by considering the deformation of the loop given by the path $\pi_o^{y_0^i} \circ \pi_{y_0^i}^x \circ \delta u \circ \pi_{x+\epsilon u}^{y_0^i} \circ \pi_{y_0^i}^o$ shown in figure 1.8. The path δu goes from x to $x + \epsilon u$. We will say that the connection derivative δ_a exists and is well defined if the loop dependent function of the deformed loop admits an expansion in terms of ϵu^a given by

$$\Psi(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \gamma) = (1 + \epsilon u^a \delta_a(x))\Psi(\gamma), \qquad (1.45)$$

where we have written π_o^x to denote the path $\pi_o^{y_0^i} \circ \pi_{y_0^i}^x$ and similarly for its inverse.

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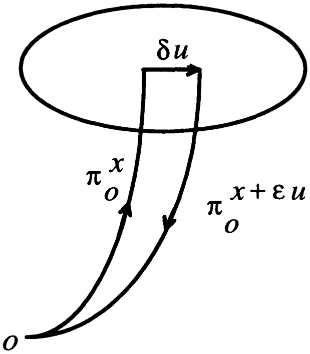

Fig. 1.8. The path that defines the connection derivative

The definition of the connection derivative can be immediately extended to act on functions of open paths, in a similar way to that used for the loop derivative. We will take some notational latitude to give the same name to the connection derivative acting on functions of paths or loops. In all cases the context will uniquely determine to which derivative we are referring.

Notice that the deformation introduced in order to define the connection derivative could have been generated by application of successive loop derivatives, as seen in the non-Abelian Stokes theorem. This implies that any function that is loop differentiable should be connection differentiable, and that there is a natural relation between the two derivatives. We will now prove the converse relation between the connection and loop derivatives. It will be quite reminiscent of the well known relation between the connection and the curvature (or vector potential and field) in a gauge theory. In order to do this, let us start by considering the following identity in loop space,

$$\begin{aligned}\delta\gamma &\equiv \pi_o^x \circ \delta u \delta v \delta \bar{u} \delta \bar{v} \circ \pi_x^o \\ &= \pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \pi_o^{x+\epsilon u} \circ \delta v \\ &\quad \circ \pi_{x+\epsilon_1 u+\epsilon_2 v}^o \circ \pi_o^{x+\epsilon_1 u+\epsilon_2 v} \circ \delta \bar{u} \circ \pi_{x+\epsilon_2 v}^o \pi_o^{x+\epsilon_2 v} \circ \delta \bar{v} \circ \pi_x^o, \quad (1.46)\end{aligned}$$

corresponding to the path shown in figure 1.9. Notice that the first definition is written in such a way that it has the structure of the paths we have used to define loop derivatives, whereas the second has the form of the paths used to define connection derivatives. Specifically, it implies the following identity between differential operators,

$$\begin{aligned}(1+\epsilon_1\epsilon_2 u^a v^b \Delta_{ab}(\pi_o^x))\Psi(\gamma) &= (1+\epsilon_1 u^a \delta_a(x))(1+\epsilon_2 v^b \delta_b(x+\epsilon_1 u)) \\ &\times (1-\epsilon_1 u^c \delta_c(x+\epsilon_1 u+\epsilon_2 v))(1-\epsilon_2 v^d \delta_d(x+\epsilon_2 v))\Psi(\gamma). \quad (1.47)\end{aligned}$$

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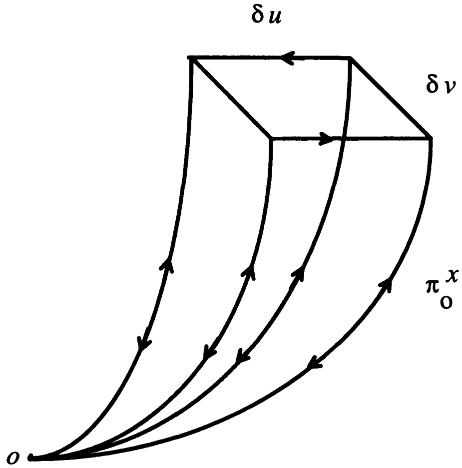

Fig. 1.9. The path that defines the relation between connection derivatives and loop derivatives.

We now expand to first order in $\epsilon_1\epsilon_2$ and get

$$\Delta_{ab}(\pi_o^x) = \partial_a\delta_b(x) - \partial_b\delta_a(x) + [\delta_a(x), \delta_b(x)]. \tag{1.48}$$

Notice that we have obtained the loop derivative for the path π_o^x given by $\pi_o^{y_0^i} \circ \pi_{y_0^i}^x$. This path is uniquely prescribed by the function defining the connection derivative. The loop derivative defined by (1.48) automatically satisfies the Bianchi identities due to the fact that it is obtained by a construction in loop space that is totally similar to that used to derive the identities themselves.

The idea of introducing the connection derivative was to provide us in the language of loops with a notion of connection or vector potential similar to that of gauge theories. However, the connection in a gauge theory is gauge dependent. How does this dependence manifest itself in the language of loops? It does so through the choice of prescription of path used to compute the connection derivative. We will now study this in some detail.

Let us consider a connection derivative at a point x and two prescriptions for choosing the path from the origin to x, given by the continuous functions $\pi_o^x = f(x)$ and $\chi_o^x = g(x)$, as shown in figure 1.10.

We again consider two equivalent paths in the group of loops,

$$\chi_o^x \circ \delta u \circ \chi_{x+\epsilon u}^o = \chi_o^x \circ \pi_x^o \circ \pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \pi_o^{x+\epsilon u} \circ \chi_{x+\epsilon u}^o. \tag{1.49}$$

We also introduce a point dependent operator $U(x)$ constructed from the loop dependent deformation operator $U(\gamma)$ defined in (1.28) and the loop

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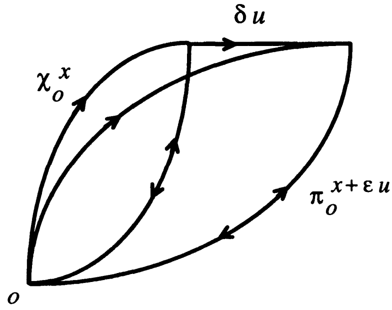

Fig. 1.10. The path dependence of the connection derivative.

associated with the point x, $\chi_o^x \circ \pi_x^o$, by $U(x) \equiv U(\chi_o^x \circ \pi_x^o)$. This gives the identity between operators:

$$(1 + \epsilon u^a \delta_a^{(\chi)}(x))\Psi(\gamma) = U(x)(1 + \epsilon u^a \delta_a^{(\pi)}(x))U^{-1}(x + \epsilon u)\Psi(\gamma). \quad (1.50)$$

From which we can immediately compute the change in the connection derivative due to a change in the prescription of the path,

$$\delta_a^{(\chi)}(x) = U(x)\delta_a^{(\pi)}(x)U(x)^{-1} + U(x)\partial_a U(x)^{-1}, \quad (1.51)$$

and we see that it is totally analogous to the transformation law for a gauge connection under changes of gauge.

The usual relation between connections and holonomies in a local chart in a gauge theory can also be written in this language. This relation is just an expression of the fact that the infinitesimal generators associated with connections allow us to construct finite loops. As shown in figure 1.11 one uses infinitesimal increments generated by the connection derivative to build a loop. The expression of this fact in loop space is

$$\gamma = \lim_{n\to\infty} \delta\gamma_1 \circ \delta\gamma_2 \circ \cdots \circ \delta\gamma_n, \quad (1.52)$$

where $\delta\gamma_i = \pi_o^{x_i}\delta u_i \pi^0_{x_i+\epsilon u_i}$, and this relation connects the deformation operator with the connection derivative,

$$U(\gamma) = \lim_{k\to\infty} (1 + (x_2 - x_1)^a \delta_a(x_1)) \times(1 + (x_3 - x_2)^a \delta_a(x_2)) \cdots (1 + (x_1 - x_k)^a \delta_a(x)). \quad (1.53)$$

We can rewrite the above expression as the following path ordered exponential,

$$U(\gamma) = \mathrm{P}\exp\left(\int_\gamma dy^a \delta_a(y)\right). \quad (1.54)$$

This again is reminiscent of the familiar expression for gauge theories, which yields the holonomy in terms of the path ordered exponential of a

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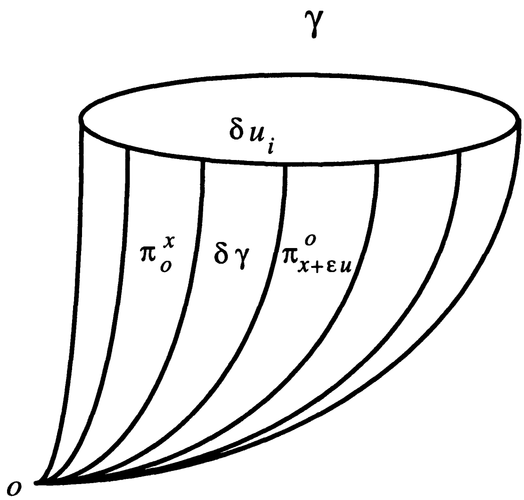

Fig. 1.11. How to generate a finite loop using the infinitesimal generators.

connection.

1.3.4 Contact and functional derivatives

The differential operators that we have introduced up to now are characteristic of the group structure of the group of loops and find no direct analogues in the space of parametrized curves. It is worthwhile analyzing whether there is any relation between these operators and the usual functional derivative $\delta/\delta p(s)$ acting on functionals of parametrized curves $\Psi[p(s)]$. The operator in loop space that allows us to make this connection is the contact derivative $C_a(x)$, which plays an important role in diffeomorphism invariant theories, as we will see later.

We define the contact derivative of a function of loops in terms of the expression

$$C_a(x)\Psi(\gamma) = \oint_\gamma dy^b \delta(x-y)\Delta_{ab}(\gamma_o^y)\Psi(\gamma), \tag{1.55}$$

where γ_o^y is the portion of the loop γ going from the basepoint to y. This operator was first introduced in the chiral formulation of Yang–Mills theories in the loop representation by Gambini and Trias [13]. It can be considered as the projection of the loop derivative on the tangent to the loop γ,

$$X^a(x,\gamma) = \oint_\gamma dy^a \delta(x-y). \tag{1.56}$$

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This expression involving the tangent to the loop will play a role in the ideas of chapter 2.

An important property of the contact derivative is that it is the generator of diffeomorphisms on functions of loops. Given the infinitesimal diffeomorphism

$$x \longrightarrow x'^a = x^a + \epsilon u^a, \tag{1.57}$$

the expression for $\Psi(\gamma'_\epsilon)$, where γ'_ϵ is the loop obtained by "dragging along" γ with the diffeomorphism (1.57), is given by

$$\begin{aligned}\Psi(\gamma'_\epsilon) &= \left(1 + \epsilon \int d^3x u^a(x)\mathcal{C}_a(x)\right)\Psi(\gamma) \\ &= \left(1 + \epsilon \oint_\gamma dy^b u^a(y)\Delta_{ab}(\gamma_o^y)\right)\Psi(\gamma).\end{aligned} \tag{1.58}$$

To prove that $\mathcal{C}_a$ actually is a generator of diffeomorphisms, we will show that it satisfies the corresponding algebra,

$$\left[\int d^3xN^a(x)\mathcal{C}_a(x), \int d^3yM^a(y)\mathcal{C}_a(y)\right] = \int d^3x\mathcal{L}_{\vec{M}}N^a(x)\mathcal{C}_a(x), \tag{1.59}$$

where N^a, M^a are arbitrary vector fields on the three-manifold.

We will start with an auxiliary calculation that will prove useful in what follows. We will evaluate the action of the loop derivative on a contact derivative. To this end we construct the following expression, which holds due to the very definition of loop derivative,

$$\begin{aligned}&(1 + \tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\int_\gamma dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)\Psi(\gamma) := \\ &\int_{\delta\gamma_z\circ\gamma} dy^a\delta(y-x)\Delta_{ab}((\delta\gamma_z\circ\gamma)_o^y)\Psi(\delta\gamma_z\circ\gamma).\end{aligned} \tag{1.60}$$

In this expression, $\delta\gamma_z$ is the infinitesimal loop added to γ through a path from the origin up to the point z. That is, we evaluate the action of an infinitesimal deformation of area σ^{cd} acting on the contact derivative.

We now expand the right-hand side of (1.60), partitioning the domain of integration into the portions after and before the action of the deformation and use the definition of the loop derivative to expand $\Psi(\delta\gamma\circ\gamma)$,

$$\begin{aligned}&\int_o^z dy^a\Delta_{ab}(\gamma_o^y)\delta(y-x)(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) \\ &+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z) + v^a\delta(z+u-x)\Delta_{ab}(\gamma_o^{z+u}) \\ &-u^a\delta(z+u+v-x)\Delta_{ab}(\gamma_o^{z+u+v}) - v^a\delta(z+u-x)\Delta_{ab}(\gamma_o^{z+u+v+\bar{u}})\} \\ &\times(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma) \\ &+\int_z^1 dy^a\delta(y-x)\Delta_{ab}((\delta\gamma_z\circ\gamma)_o^y)(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma).\end{aligned} \tag{1.61}$$

The last term in this expression can be rewritten as

$$\oint_\gamma dy^a \delta(x-y)\Theta(y-z)(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))[\Delta_{ab}(\gamma_o^y)](1+\tfrac{1}{2}\sigma^{ef}\Delta_{ef}(\gamma_o^z))\Psi(\gamma) \tag{1.62}$$

where $\Theta(y-z)$ is a Heaviside function that orders points along the loop, i.e., it is 1 if z precedes y and zero otherwise. We will be able to combine the zeroth order contribution of this term with the first term in (1.61). It should be noticed that the first loop derivative does not act on everything to its right but only on the path inside the second loop derivative γ_o^y, a fact that as before we denote by enclosing it in brackets. We now consider the expansion of the terms containing the infinitesimally shifted loop. They can be expressed with the use of the Mandelstam derivative,

$$\Delta_{ab}(\gamma_o^{z+u}) = (1+u^c D_c)\Delta_{ab}(\gamma_o^z), \tag{1.63}$$

$$\Delta_{ab}(\gamma_o^{z+u+v}) = (1+v^d D_d)(1+u^c D_c)\Delta_{ab}(\gamma_o^z), \tag{1.64}$$

$$\delta(z+u-x) = (1+u^a\partial_a)\delta(z-x). \tag{1.65}$$

We now expand (1.61) again

$$\begin{aligned}
&\oint_\gamma dy^a \delta(y-x)\Delta_{ab}(\gamma_o^y)(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma)\\
&+\{u^a\delta(z-x)\Delta_{ab}(\gamma_o^z)+v^a(1+u^e\partial_e)\delta(z-x)(1+u^d D_d)\Delta_{ab}(\gamma_o^z)\\
&-u^a(1+v^c D_c)\delta(z-x)(1+v^d D_d)\Delta_{ab}(\gamma_o^z)-v^a\delta(z-x)\Delta_{ab}(\gamma_o^z)\}\\
&\times(1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma)+\tfrac{1}{2}\oint_\gamma dy^a\delta(y-x)\Theta(y-z)\sigma^{cd}\\
&\times\Delta_{cd}(\gamma_o^z)[\Delta_{ab}(\gamma_o^y)](1+\tfrac{1}{2}\sigma^{cd}\Delta_{cd}(\gamma_o^z))\Psi(\gamma).
\end{aligned} \tag{1.66}$$

Of the terms in braces, it can be readily seen that only contributions proportional to $u^a v^b$ are present, neglecting terms of higher order. The other terms combine to give the original expression. We can finally read off the contribution of the loop derivative,

$$\begin{aligned}
&\Delta_{cd}(\gamma_z)\oint_\gamma dy^a\delta(x-y)\Delta_{ab}(\gamma_o^y)\Psi(\gamma)=\\
&2(\partial_{[c}\delta(z-x)\Delta_{d]b}(\gamma_o^z)+\delta(z-x)D_{[c}\Delta_{d]b}(\gamma_o^z))\Psi(\gamma)\\
&+\oint_\gamma dy^a\Theta(y-z)\delta(y-x)\Delta_{cd}(\gamma_o^z)[\Delta_{ab}(\gamma_o^y)]\Psi(\gamma)\\
&+\oint_\gamma dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)\Delta_{cd}(\gamma_o^z)\Psi(\gamma).
\end{aligned} \tag{1.67}$$

With this calculation in hand, it is straightforward to compute the successive action of two diffeomorphisms,

$$C(\vec{N})\,C(\vec{M})\,\Psi(\gamma)=\int d^3wN^d(w)\oint_\gamma dz^c\delta(w-z)\Delta_{cd}(\gamma_o^z)$$
$$\times\int d^3xM^b(x)\oint_\gamma dy^a\delta(y-x)\Delta_{ab}(\gamma_o^y)\Psi(\gamma). \quad (1.68)$$

Expanding this expression we get six terms

$$\int d^3w\int d^3xN^d(w)M^b(x)$$
$$\times\{\oint_\gamma dz^c\delta(w-z)\partial_c\delta(z-x)\Delta_{db}(\gamma_o^z)\Psi(\gamma)$$
$$-\oint_\gamma dz^c\delta(w-z)\partial_d\delta(z-x)\Delta_{cb}(\gamma_o^z)\Psi(\gamma)$$
$$+\oint_\gamma dz^c\delta(w-z)\delta(z-x)D_c\Delta_{db}(\gamma_o^z)\Psi(\gamma)$$
$$-\oint_\gamma dz^c\delta(w-z)\delta(z-x)D_d\Delta_{cb}(\gamma_o^z)\Psi(\gamma)$$
$$+\oint_\gamma dz^c\oint_\gamma dy^a\delta(w-z)\delta(y-x)\Theta(y-z)\Delta_{cd}(\gamma_o^z)[\Delta_{ab}(\gamma_o^y)]\Psi(\gamma)$$
$$+\oint_\gamma dz^c\oint_\gamma dy^a\delta(w-z)\delta(y-x)\Delta_{ab}(\gamma_o^y)\Delta_{cd}(\gamma_o^z)\Psi(\gamma)\}. \quad (1.69)$$

We should now subtract the same terms with the replacement $\vec{N}\leftrightarrow\vec{M}$. Since the calculation is tedious but straightforward we describe in words how the terms combine. The fifth and sixth terms, when combined with the similar terms coming from the substitution $\vec{N}\leftrightarrow\vec{M}$ cancel taking into account the commutation relations for the loop derivatives (1.26). The first and third terms combined with the first of the substitution $\vec{N}\leftrightarrow\vec{M}$ form a total derivative. The fourth term, combined with the third and fourth of the substitution $\vec{N}\leftrightarrow\vec{M}$ cancel due to the Bianchi identities of the loop derivatives. Finally, the second terms combine to produce exactly $C(\mathcal{L}_{\vec{N}}\vec{M})$, which is the correct result of the calculation.

We end by pointing out the relation between the contact derivative and the usual functional derivative. This can be immediately recognized by noticing that we can write $\int d^3xu^a(x)\mathcal{C}_a(x)$ in terms of parametrized curves by

$$\int d^3xu^a(x)\mathcal{C}_a(x)=\oint_0^1 dsu^a(p(s))\frac{\delta}{\delta p^a(s)}, \quad (1.70)$$

where $p(s)$ is one of the parametrized curves in the equivalence class of the loop γ. The way to see this is to notice that if γ is the equivalence class of curves $[p^a(s)]$, then $\Psi(\gamma') = \Psi[p^a(s) + \epsilon u^a(p(s))]$. Then,

$$\mathcal{C}_a(x) = \oint ds\delta(x - p(s))\frac{\delta}{\delta p(s)^a}, \tag{1.71}$$

which relates the two derivatives. It should be noticed that these two derivatives only agree when acting on functions of loops. The expression on the right of (1.70) can act on functions of parametrized curves, on which the contact derivative is not defined.

1.4 Representations of the group of loops

In previous sections we derived several relations between generators of the group of loops. These relations were independent of any particular representation. We will now study their form in the context of a particular representation in terms of a given gauge group. We will see that from them emerges the kinematical structure of gauge theories.

Gauge theories arise as representations (homomorphisms $\mathcal{H}$) of the group of loops onto some gauge group G,

$$\mathcal{H} : \mathcal{L}_0 \to G, \tag{1.72}$$

i.e.,

$$\gamma \longrightarrow H(\gamma), \tag{1.73}$$

such that $H(\gamma_1)H(\gamma_2) = H(\gamma_1 \circ \gamma_2)$.

Let us assume we are considering a specific Lie group, for instance $SU(N)$, with $N^2 - 1$ generators X^i such that $\mathrm{Tr}X^i = 0$ and

$$[X^i, X^j] = C^{ij}_k X^k, \tag{1.74}$$

where C^{ij}_k are the structure constants of the group in question. We will assume that the representation is loop differentiable**. This will enable us to obtain the usual local objects associated with the gauge theory (curvature and connection) from the loop language.

Let us compute the action of the connection derivative in this representation. We use the same prescriptions as in the previous section

$$(1 + \epsilon u^a\delta_a(x))H(\gamma) = H(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \gamma) = H(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o)H(\gamma). \tag{1.75}$$

Since the loop $\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o$ is close to the identity loop (with the topology of loop space) and since H is a continuous, differentiable representation,

$$H(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o) = 1 + i\epsilon u^a A_a(x), \tag{1.76}$$

** If it is not loop differentiable, instead of dealing with holonomies we will have "generalized" holonomies, which are not derived from a smooth connection.

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where $A_a(x)$ is an element of the algebra of the group, in our example of $SU(N)$. That is, $A_a(x) = A_a^i X^i$. Therefore, we see that through the action of the connection derivative,

$$\delta_a(x)H(\gamma) = iA_a(x)H(\gamma). \tag{1.77}$$

Following similar steps one obtains the action of the loop derivative,

$$\Delta_{ab}(\pi_o^x)H(\gamma) = iF_{ab}(x)H(\gamma), \tag{1.78}$$

where F_{ab} is an algebra-valued antisymmetric tensor field.

Remember that π_o^x is only fixed by the prescription. Changing the prescription for π_o^x is the way to change the gauge. Suppose we change the prescription by $\pi_o^x \to \pi'^x_o = \pi'^x_o \circ \pi_x^o \circ \pi_o^x$. We then simply use equation (1.33) that in terms of the field reads

$$F'_{ab}(x) = H(x)F_{ab}(x)H(x)^{-1}, \tag{1.79}$$

where $H(x)$ is a shorthand for $H(\pi'^x_o \circ \pi_x^o)$.

From equation (1.48) we immediately get the usual relation defining the curvature in terms of the potential,

$$F_{ab}(x) = \partial_a A_b(x) - \partial_b A_a(x) + i[A_a, A_b]. \tag{1.80}$$

Gauge transformations in terms of the connection are immediate from equation (1.51),

$$A_a(x)' = H(x)A_a(x)H(x)^{-1} - iH(x)\partial_a H(x)^{-1}. \tag{1.81}$$

Let us act with the deformation operator $U(\eta)$ introduced in section 1.3.3 on the representation $H(\gamma)$,

$$U(\eta)H(\gamma) = H(\eta \circ \gamma) = H(\eta)H(\gamma). \tag{1.82}$$

Now, applying formula (1.54),

$$U(\eta)H(\gamma) = \mathrm{P}\exp\left(\oint_\eta dy^a \delta_a(y)\right) H(\gamma), \tag{1.83}$$

and substituting equation (1.77) and comparing terms we get

$$H(\eta) = \mathrm{P}\exp\left(i\oint_\eta dy^a A_a(y)\right), \tag{1.84}$$

which shows that the representation we are considering is given in terms of the usual expression for the holonomy of the connection A_a.

Up to now, open paths have not played any relevant physical role. We will now show that open paths are naturally related to the inclusion of material fields coupled to gauge theories. We will consider matter fields that transform under the fundamental representation of the gauge group considered in our example, $SU(N)$.

We will describe the matter field at the point x through a path dependent object $\Psi(\pi_o^x)$. The natural extension of the representation introduced at the beginning of this section to the case of open paths is to consider the composition of an open path and a loop, defined by

$$\Psi(\gamma \circ \pi_o^x) \equiv H(\gamma)\Psi(\pi_o^x). \tag{1.85}$$

As in other cases, the role of the path choice will be to fix a gauge choice. A local description in a fixed gauge is obtained by fixing a family of paths, each of which are associated with each point in the manifold. The functions Ψ will now become functions of points labeled by the fixed prescription π used to determine the paths $\Psi^{(\pi)}(x)$. The prescription is given through a continuous function from the points of the manifold into the paths $f(x) = \pi_o^x$. Notice that if we change the prescription for the path $\pi_o^x \longrightarrow {\pi'}_o^x = {\pi'}_o^x \circ \pi_x^o \circ \pi_o^x$ we get

$$\Psi^{(\pi')}(x) = H({\pi'}_o^x \circ \pi_x^o)\Psi^{(\pi)}(x). \tag{1.86}$$

The Mandelstam derivative D_a behaves as the usual covariant derivative of a gauge theory. Consider its action on a function of an open path,

$$\begin{aligned}(1+\epsilon u^a D_a)\Psi(\pi_o^x) = \Psi(\pi_o^x \circ \delta u) &= \Psi(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \pi_o^{x+\epsilon u}) \\ &= (1+\epsilon u^a \delta_a(x))(1+\epsilon u^b \partial_b)\Psi(\pi_o^x),\end{aligned} \tag{1.87}$$

expanding in ϵ and keeping terms of first order, we get

$$D_a\Psi(\pi_o^x) = \partial_a\Psi^{(\pi)}(x) + \delta_a(x)\Psi(\pi_o^x). \tag{1.88}$$

Using the relation between the function of a deformed path and the holonomy (1.85), we get

$$D_a\Psi(\pi_o^x) = \partial_a\Psi^{(\pi)}(x) + iA_a(x)\Psi^{(\pi)}(x). \tag{1.89}$$

The usual form of the Ricci identity,

$$[D_a, D_a] = iF_{ab}, \tag{1.90}$$

can be obtained directly from the above expression or by considering the representation of equation (1.40).

1.5 Conclusions

We have seen how gauge theories arise as representations of the group of loops. All the usual kinematical concepts of gauge theories are reflections of properties of the group of loops.

It is important to realize that the identities and properties that we proved in this chapter for the loop and connection derivative *do not* depend on any choice of gauge group to represent the group of loops. In this sense one can think of the corresponding generators of the group as associated with an "abstract" curvature and connection. It is only when one considers a particular representation of the group of loops in terms of a gauge group that these quantities adopt the usual meaning of connections and curvatures in gauge theories.

In the next chapter we will introduce more techniques that will put us in a better position to deal with loops in the context of quantum theories in the loop representation.