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TWISTED ALEXANDER POLYNOMIAL FOR THE BRAID GROUP

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In this paper, we study the twisted Alexander polynomial, due to Wada [11], for the Jones representations [6] of Artin's braid group.

1. INTRODUCTION

The twisted Alexander polynomial for finitely presentable groups was introduced by Wada in [11]. Let Γ be such a group with a surjective homomorphism to $\mathbb{Z} = \langle t \rangle$. (We shall treat only the case of an infinite cyclic group, although the case of any free Abelian group is considered in [11].) To each linear representation

$$\rho : \Gamma \to GL(n, R)$$

of the group Γ over a unique factorisation domain R, we shall assign a rational expression $\Delta_{\Gamma,\rho}(t)$ in the indeterminate t with coefficients in R, which is called the twisted Alexander polynomial of Γ associated to ρ. This polynomial is well-defined up to a factor of εt^e, where $\varepsilon \in R^\times$ is a unit of R and $e \in \mathbb{Z}$.

The twisted Alexander polynomial is a generalisation of the original Alexander polynomial (see [3]) in the following sense. Namely the Alexander polynomial of Γ with the Abelianisation $\alpha : \Gamma \to \langle t \rangle$ is written as

$$\Delta_\Gamma(t) = (1 - t)\Delta_{\Gamma,\mathbf{1}}(t),$$

where $\mathbf{1}$ is the trivial 1-dimensional representation of Γ.

As a remarkable application, Wada shows in [11] that Kinoshita-Terasaka and Conway's 11-crossing knots are distinguished by the twisted Alexander polynomial. The notion of Alexander polynomials twisted by a representation and its applications have appeared in several papers (see [5, 7, 8, 10]).

In this paper, we consider the twisted Alexander polynomial in the case where Γ is not all the group of a knot. To be more precise, we shall deal with Artin's braid group B_n ($n \geqslant 3$) and its Jones representation [6] as our object. Here it is known that the Jones

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representations of B_n arise from the Hecke algebra of type A_{n-1} and are in one-to-one correspondence with Young diagrams with n-boxes. Therefore it is natural to raise the following problem: *Describe the twisted Alexander polynomial $\Delta_{B_n,\rho}(t)$ for all the Jones representations ρ of the braid group. In other words, can we see $\Delta_{B_n,\rho}(t)$ from Young diagrams?*

The purpose of this paper is to investigate behaviour of $\Delta_{B_n,\rho}(t)$ for several important series of Jones representations. Among them the most interesting object is perhaps the Burau representation of the braid group (see [**2**]), which corresponds to the Young diagram of type $(2,1,\ldots,1)$. In this case, the twisted Alexander polynomial has the following notable property.

THEOREM 1.1. *Let $\beta : B_n \to GL\big(n-1, \mathbb{Z}[s^{\pm 1}]\big)$ be the Burau representation of the braid group B_n and $\alpha : B_n \to \mathbb{Z} = \langle t \rangle$ be the Abelianisation. Then the twisted Alexander polynomial $\Delta_{B_n,\beta}(t)$ is given by*

$$\Delta_{B_n,\beta}(t) = \begin{cases} 1 - st^2 & (n = 3) \\ 1 & (n \geqslant 4). \end{cases}$$

As is well known, the Burau representation is faithful for $n = 3$, not for $n \geqslant 5$ (see [**1**]) and unsolved for $n = 4$ at the time of writing. Thus it would be interesting to study a relation between the faithfulness of the Burau representation and the twisted Alexander polynomial.

On the other hand, the twisted Alexander polynomials of the braid group for Jones representations have a *symmetry* in the following sense. Let Y and Y' be Young diagrams corresponding to a conjugate partition (that is, reflecting the diagram in the 45° line). Further let $^{-} : \Lambda_q[t^{\pm 1}] \to \Lambda_q[t^{\pm 1}]$ denote the involution induced by the transformation $t \mapsto -q^{-1}t^{-1}$, where $\Lambda_q = \mathbb{Z}[q^{\pm 1/2}]$ is the Laurent polynomial ring. Then it can be shown that

THEOREM 1.2. $\Delta_{B_n,Y'}(t) = \overline{\Delta_{B_n,Y}(t)}$ *holds up to a factor of* εt^e $\big(\varepsilon \in {\Lambda_q}^{\times},\ e \in \mathbb{Z}\big)$.

Therefore to give a complete answer to the previous problem, it is sufficient to compute the twisted Alexander polynomials for about 'half' the Young diagrams.

Now we describe the contents of this paper. In the next section, we review the definition of the twisted Alexander polynomial for finitely presentable groups. In Section 3, we briefly recall the Burau representation of the braid group and prove Theorem 1.1. The final section is devoted to a systematic study for the Jones representation. In particular, we explicitly compute the twisted Alexander polynomials for Young diagrams up to 5-boxes.

2. TWISTED ALEXANDER POLYNOMIAL

Suppose that the group Γ has the presentation

$$P(\Gamma) = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle.$$

Let $\rho : \Gamma \to GL(n, R)$ be a linear representation and let $\alpha : \Gamma \to \mathbb{Z} \cong \langle t \rangle$ be a surjective homomorphism. We denote the group ring $R[\mathbb{Z}] \cong R[t^{\pm 1}]$ by Λ_t. Then $\rho \otimes \alpha$ defines a ring homomorphism

$$\mathbb{Z}[\Gamma] \to M(n, \Lambda_t),$$

where $M(n, \Lambda_t)$ denotes the matrix algebra of degree n over Λ_t. Let F_u denote the free group on generators $x_1, \ldots, x_u$ and denote by $\Phi : \mathbb{Z}[F_u] \to M(n, \Lambda_t)$ the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[\Gamma]$ induced by the presentation and the map $\mathbb{Z}[\Gamma] \to M(n, \Lambda_t)$ given by $\rho \otimes \alpha$.

Let us consider the $v \times u$ matrix M whose (i, j) component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(n, \Lambda_t),$$

where $\dfrac{\partial}{\partial x}$ denotes the free differential calculus (see [**4**]). This matrix M is called the Alexander matrix of the presentation $P(\Gamma)$ associated to the representation ρ.

For $1 \leqslant j \leqslant u$, let us denote by M_j the $v \times (u-1)$ matrix obtained from M by removing the jth column. Now we regard M_j as a $vn \times (u-1)n$ matrix with coefficients in Λ_t. For a $(u-1)n$-tuple of indices

$$I = \left(i_1, \ldots, i_{(u-1)n}\right) \quad \left(1 \leqslant i_1 \leqslant \cdots \leqslant i_{(u-1)n} \leqslant vn\right),$$

we denote by M_j^I the $(u-1)n \times (u-1)n$ square matrix consisting of the i_kth rows of the matrix M_j, where $k = 1, \ldots, (u-1)n$.

The following two lemmas are the foundation of the definition of the twisted Alexander polynomial (see [**11**] for the proof).

Lemma 2.1. $\det \Phi(1 - x_j) \neq 0$ *for some* j.

Lemma 2.2. $\det M_j^I \det \Phi(1 - x_k) = \pm \det M_k^I \det \Phi(1 - x_j)$ *for* $1 \leqslant j < k \leqslant u$ *and for any choice of the indices* I.

We denote by $Q_j(t) \in \Lambda_t$ the greatest common divisor of $\det M_j^I$ for all the choices of the indices I. It should be noted that the Laurent polynomial ring Λ_t over a unique factorisation domain R is again a unique factorisation domain. The Laurent polynomial $Q_j(t)$ is well-defined up to a factor of εt^e where $\varepsilon \in R^\times$ is a unit of R and $e \in \mathbb{Z}$. If $v < u - 1$ then we define $Q_j(t)$ to be the zero polynomial.

From the above two lemmas, we can define the twisted Alexander polynomial of Γ associated to the representation ρ to be the rational expression

$$\Delta_{\Gamma,\rho}(t) = \frac{Q_j(t)}{\det \Phi(1 - x_j)}$$

provided $\det \Phi(1 - x_j) \neq 0$.

REMARK 2.1. Up to a factor of εt^e ($\varepsilon \in R^\times$, $e \in \mathbb{Z}$), this is in fact an invariant of the group Γ, the associated homomorphism α and the representation ρ (see [**11**, Theorem 1]). Further the twisted Alexander polynomial does not depend on the choice of the basis for the representation space.

3. Burau representation of the braid group

Let B_n denote the braid group of n strings. The group B_n is generated by the $n-1$ elements $\sigma_1, \dots, \sigma_{n-1}$ which satisfy the following two kinds of relations;

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \big(|i-j| \geqslant 2\big),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (i = 1, \dots, n-2).$$

The Burau representation of B_n is one of the so-called Magnus representations which are defined for a subgroup of the automorphism group of the free group F_n. By its definition this representation maps each element of B_n to an $n \times n$ matrix, but we easily notice that it reduces to an $(n-1) \times (n-1)$ representation. The (reduced) Burau representation

$$\beta : B_n \to GL\big(n-1, \mathbb{Z}[s^{\pm 1}]\big),$$

is explicitly defined by

$$\sigma_1 \mapsto \begin{pmatrix} -s & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad \sigma_{n-1} \mapsto I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ s & -s \end{pmatrix},$$
$$\sigma_i \mapsto I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ s & -s & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2} \quad (1 < i < n-1),$$

where I_n denotes the $n \times n$ identity matrix (see [**2**] for more details).

Now let us prove Theorem 1.1. The result for the case $n = 3$ is described in [**11**, Section 4], so that hereafter we shall consider only $n \geqslant 4$.

Let $[i, j]$ denote a relation between the generators σ_i and σ_j. Further we adopt the numbering of relations of B_n as in the following table (for instance, this table is sufficient to compute the matrix M_5 in the case of the group B_6).

	$\partial r_i/\partial\sigma_1$	$\partial r_i/\partial\sigma_2$	$\partial r_i/\partial\sigma_3$	$\partial r_i/\partial\sigma_4$
$r_1=[1,2]$	$1-\sigma_2+\sigma_1\sigma_2$	$-1+\sigma_1-\sigma_2\sigma_1$		
$r_2=[1,3]$	$1-\sigma_3$		σ_1-1	
$r_3=[2,3]$		$1-\sigma_3+\sigma_2\sigma_3$	$-1+\sigma_2-\sigma_3\sigma_2$	
$r_4=[1,4]$	$1-\sigma_4$			σ_1-1
$r_5=[2,4]$		$1-\sigma_4$		σ_2-1
$r_6=[3,4]$			$1-\sigma_4+\sigma_3\sigma_4$	$-1+\sigma_3-\sigma_4\sigma_3$
$r_7=[1,5]$	$1-\sigma_5$			
$r_8=[2,5]$		$1-\sigma_5$		
$r_9=[3,5]$			$1-\sigma_5$	
$r_{10}=[4,5]$				$1-\sigma_5+\sigma_4\sigma_5$

In the above table, the vacant entries are all zero. As pointed out by Wada in [**11**], it is convenient to use relations instead of relators for the computation of the Alexander matrix.

First let us calculate a denominator in the definition of the twisted Alexander polynomial.

Lemma 3.1. $\det\Phi(1-\sigma_{n-1})=(1-t)^{n-2}(1+st)$.

Proof: We can easily compute

$$\det\Phi(1-\sigma_{n-1})=\begin{vmatrix}1-t & & & \\ & \ddots & & \\ & & 1-t & 0\\ & & -st & 1+st\end{vmatrix}=(1-t)^{n-2}(1+st),$$

so that the proof is complete. ∎

Lemma 3.2. *$\det M^I_{n-1}$ is divided by $(1-t)^{n-2}(1+st)$ for all the choices of the indices I.*

Proof: We denote the matrix M_{n-1} by the column vectors $\mathbf{m}_j$ $(1\leqslant j\leqslant n-2)$:

$$M_{n-1}=\left(\Phi\left(\frac{\partial r_i}{\partial\sigma_j}\right)\right)=(\mathbf{m}_1,\dots,\mathbf{m}_{n-2}),$$

where

$$\mathbf{m}_j={}^t\left(\Phi\left(\frac{\partial r_1}{\partial\sigma_j}\right),\cdots,\Phi\left(\frac{\partial r_l}{\partial\sigma_j}\right)\right)$$

and l is the number of relations of B_n (namely, $l=(n-1)(n-2)/2$). We then notice the fact that the jth column vector in each $\mathbf{m}_j$ contains a common divisor $1-t$. In fact, we

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see from the previous table that non-trivial blocks in $\mathbf{m}_j$ are the following $n-2$ matrices.

$$n-2\left\{\begin{array}{ll}\Phi(\sigma_1-1) & -1+t \text{ is the unique non-trivial component}\\ \cdots & \text{in the } j\text{th column}\\ \Phi(\sigma_{j-2}-1) & \\ \Phi(-1+\sigma_{j-1}-\sigma_j\sigma_{j-1}) & \cdots (*)\\ \Phi(1-\sigma_{j+1}+\sigma_j\sigma_{j+1}) & \cdots (**)\\ \Phi(1-\sigma_{j+2}) & \\ \cdots & 1-t \text{ is the unique non-trivial component}\\ \Phi(1-\sigma_{n-1}) & \text{in the } j\text{th column}\end{array}\right.$$

Clearly the unique non-trivial component in the jth column of the first $j-2$ matrices is $-1+t$. Similaly $1-t$ is the unique non-trivial component in the jth column of the last $n-j-2$ matrices. Further direct calculation shows that the jth column in the matrix $(*)$ is

$${}^t\Big(\underbrace{0,\ldots,0}_{j-2},t-t^2,-1+t,\underbrace{0,\ldots,0}_{n-j-1}\Big)$$

and the jth column in the matrix $(**)$ is

$${}^t\Big(\underbrace{0,\ldots,0}_{j-1},1-t,-st+st^2,\underbrace{0,\ldots,0}_{n-j-2}\Big),$$

so that we can take a term $1-t$ from the jth column in each $\mathbf{m}_j$ $(1\leqslant j\leqslant n-2)$ as a divisor. Hence we have $(1-t)^{n-2}$ as a common divisor of the matrix M_{n-1}.

Next if we add the $(n-1), 2(n-1), \ldots, (n-3)(n-1)$th columns and $-t$ times the $\big\{(n-2)(n-1)-1\big\}$th column to the $(n-2)(n-1)$th column in the matrix M_{n-1}, then the $(n-2)(n-1)$th column vector becomes

$${}^t\Big(\underbrace{0,\ldots,0}_{m},\overbrace{\underbrace{0,\ldots,0}_{n-2},1+st,\ldots,\underbrace{0,\ldots,0}_{n-2},1+st}^{(n-1)(n-3)},\underbrace{0,\ldots,0}_{n-3},-t-st^2,1+st\Big),$$

where $m=\big((n-1)(n-2)(n-3)\big)\big/2$. Therefore we can take a term $1+st$ as a common divisor from this column. For later use we denote the resulting matrix by $\overline{M}_{n-1}$. Accordingly we can conclude that $(1-t)^{n-2}(1+st)\mid\det M^I_{n-1}$ holds for any index I. $\square$

In order to show that the matrix $\overline{M}_{n-1}$ does not have any other common divisor with respect to the column vectors (that is, $(1-t)^{n-2}(1+st)$ is the greatest common polynomial), we need the following lemma.

Lemma 3.3. *There exist the indices I, I', I'' such that*

(i) $\det\overline{M}^I_{n-1}=(1+st)^{n-3}(1-st^2)^{n-2}(1-t+t^2)^{(n-2)(n-3)}$ $(n\geqslant 4)$,

(ii) $\det\overline{M}^{I'}_{n-1}=(1+st)^{n-3}(1-t)^{(n-2)(n-3)}$ $(n\geqslant 5)$ *and*

(iii) $\det\overline{M}^{I''}_{n-1}$ *does not contain* $1+st$ *as a divisor for* $n\geqslant 4$.

PROOF:

(i) Let us consider the index I corresponding to the $n-2$ row-blocks $[1,2]$, $[2,3]$, $[3,4]$, ..., $[n-2,n-1]$ in the matrix M_{n-1}. Since

$$\det \Phi(1-\sigma_{j+1}+\sigma_j\sigma_{j+1}) = (1-t)(1+st)\Big(1-st^2\Big)\Big(1-t+t^2\Big)^{n-3},$$

we have

$$\begin{aligned}\det M^I_{n-1} &= \prod_{j=1}^{n-2} \det \Phi(1-\sigma_{j+1}+\sigma_j\sigma_{j+1}) \\ &= \Big\{(1-t)(1+st)\Big(1-st^2\Big)\Big\}^{n-2}\Big(1-t+t^2\Big)^{(n-2)(n-3)}.\end{aligned}$$

Hence we obtain

$$\det \overline{M}^I_{n-1} = (1+st)^{n-3}\Big(1-st^2\Big)^{n-2}\Big(1-t+t^2\Big)^{(n-2)(n-3)}$$

for $n \geqslant 4$.

(ii) For $n \geqslant 5$, if we choose an index I' such that the diagonal blocks of a square matrix $M^{I'}_{n-1}$ consist of $n-3$ matrices $\Phi(1-\sigma_{n-1})$ and one $\Phi(\sigma_1-1)$, we have

$$\begin{aligned}\det M^{I'}_{n-1} &= \Big\{(1-t)^{n-2}(1+st)\Big\}^{n-3}(-1+t)^{n-2}(-1-st) \\ &= \pm(1-t)^{(n-2)^2}(1+st)^{n-2},\end{aligned}$$

where we have used Lemma 3.1.

(iii) In the matrix $\overline{M}_{n-1}$ for $n \geqslant 4$, if we suitably replace the last row in each $[1,n-1],[2,n-1],\ldots,[n-2,n-1]$ row-blocks by another row, we can easily choose an index I'' so that $\det \overline{M}^{I''}_{n-1} \neq 0$ and does not contain the term $1+st$ as a divisor.

This completes the proof of the lemma. $\square$

PROPOSITION 3.4. $Q_{n-1}(t) = (1-t)^{n-2}(1+st)$.

PROOF: In the case $n=4$, from (i) of the above lemma and the existence of an index J such that

$$\det \overline{M}^J_3 = (1-t)\Big(1+st-st^2\Big)\Big(1+st-st^2+st^3-s^2t^3\Big),$$

we can deduce the result. For $n \geqslant 5$, it is clear from Lemmas 3.2 and 3.3. $\square$

By virtue of Lemma 3.1 and Proposition 3.4, Theorem 1.1 immediately follows. This completes the proof.

REMARK 3.1. As is well known, the braid group B_n is generated by two elements (for example, σ_1 and $\xi = \sigma_1\ldots\sigma_{n-1}$ is a generating system). However we can not say indiscriminately that this presentation is more useful than Artin's one for computing the twisted Alexander polynomial.

4. Jones representation of the braid group

Let $H(q,n)$ denote the Hecke algebra of type A_{n-1}. Namely, it is an algebra over $\mathbb{C}(q)$ generated by $g_1, \dots, g_{n-1}$ with the following three kinds of relations;

$$\begin{aligned} {g_i}^2 &= (q-1)g_i + q, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\ g_i g_j &= g_j g_i \quad \big(|i-j| \geqslant 2\big), \end{aligned}$$

where q is a parameter. Thus the correspondence $\sigma_i \mapsto g_i$ defines a group homomorphism from the braid group B_n to $H(q,n)$. Jones observed in [**6**] that for q close to 1, the simple $H(q,n)$ modules (or *quadratic* irreducible representations of B_n) are in one-to-one correspondence with Young diagrams with n-boxes. Moreover their decomposition rules are the same as for the symmetric group $\mathfrak{S}_n$. We call them the Jones representation of the braid group.

Now let us record the picture of Young diagrams up to 5-boxes. Then there is an ambiguity in assigning diagrams to representations by the row-column symmetry. To fix this problem, we use $(2) = \square\square$ to represent the trivial representation of $\mathfrak{S}_2$.

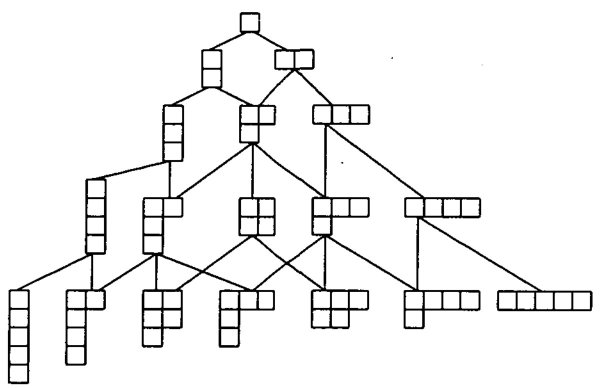

Figure 4.1

The lines connecting different rows represent the restrictions of representations. That is, they describe the irreducible decomposition of a given irreducible representation of B_n when it is restricted to B_{n-1}.

Example 4.1. The trivial representation $\rho = (n)$ of B_n is given by $\rho(\sigma_i) = q$ for $1 \leqslant i \leqslant n-1$ (see [**6**, Note 4.7]). Then we have

$$\Delta_{B_n,\rho}(t) = \begin{cases} \dfrac{1 - qt + q^2t^2}{1-qt} & (n = 3, 4) \\ \dfrac{1}{1-qt} & (n \geqslant 5). \end{cases}$$

For the parity representation $\rho = (1, \dots, 1)$ defined by $\rho(\sigma_i) = -1$, the same calculation shows that

$$\Delta_{B_n,\rho}(t) = \begin{cases} \dfrac{1+t+t^2}{1+t} & (n = 3, 4) \\ \dfrac{1}{1+t} & (n \geqslant 5). \end{cases}$$

Now as the first interesting property for Jones representations, we show the symmetry of the twisted Alexander polynomials. Let π_Y be a representation of the Hecke algebra $H(q, n)$ corresponding to a Young diagram Y and ρ_Y denote one for the braid group B_n. Namely they satisfy

$$\rho_Y(\sigma_i) = \pi_Y(g_i) \quad (1 \leqslant i \leqslant n-1)$$

for each generator of B_n and $H(q, n)$. Further let Y' be the Young diagram reflecting Y in the 45° line. Then

$$\rho_{Y'}(\sigma_i) = \pi_{Y'}(g_i) = -q\pi_Y(g_i)^{-1},$$

because the symmetry of Y and Y' corresponds to the automorphism $g_i \mapsto -qg_i^{-1}$ of $H(q, n)$ (see [**6**, Note 4.6]). We therefore obtain

$$\begin{aligned} \Phi'(\sigma_i) &= (\rho_{Y'} \otimes \alpha)(\sigma_i) \\ &= -qt\pi_Y(g_i)^{-1} \\ &= \left(-q^{-1}t^{-1}\pi_Y(g_i)\right)^{-1}. \end{aligned}$$

The above equality implies that the twisted Alexander polynomial $\Delta_{B_n,\rho_{Y'}}(t)$ coincides with one defined via the ring homomorphism $\sigma_i \mapsto -q^{-1}t^{-1}\pi_Y(g_i)$ up to a factor of εt^e, where ε is a unit of $\Lambda_q = \mathbb{Z}[q^{\pm 1/2}]$ and $e \in \mathbb{Z}$. Moreover the ring homomorphism is nothing but the transformation $t \mapsto -q^{-1}t^{-1}$ for $\Phi(\sigma_i) = t\pi_Y(g_i)$. Accordingly we can conclude that

$$\Delta_{B_n,\rho_{Y'}}(t) = \overline{\Delta_{B_n,\rho_Y}(t)}$$

holds for any symmetric Young diagrams Y and Y'. Here $^-\colon \Lambda_q[t^{\pm 1}] \to \Lambda_q[t^{\pm 1}]$ denotes the involution induced by the above transformation. This completes the proof of Theorem 1.2.

As noted in the introduction (see also [**6**, Note 5.7]), the Burau representation corresponds to the Young diagram Y_β below.

FIGURE 4.2

Therefore as a corollary of Theorems 1.1 and 1.2, we have

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COROLLARY 4.1. $\Delta_{B_n,Y_\beta'}(t) = 1$ *for* $n \geqslant 4$.

As the second fundamental property for Jones representations, we can describe the *decomposition rule* for the twisted Alexander polynomials.

LEMMA 4.2. *Let* $\widetilde{Y}$ *be a Young diagram with* $(n+1)$*-boxes and* $Y_1, \dots, Y_l$ *be Young diagrams with* n*-boxes obtained from* $\widetilde{Y}$ *by removing one box. Then*

$$\Delta_{B_n,\widetilde{Y}}(t) = \prod_{k=1}^{l} \Delta_{B_n,Y_k}(t).$$

PROOF: Since the restriction of $\rho_{\widetilde{Y}}$ to B_n is a direct sum of $\rho_{Y_1}, \dots, \rho_{Y_l}$, the sub-Alexander matrix $\widetilde{M}|_{B_n}$ has a direct sum decomposition as matrices. Hence the claim is clear from the definition of the twisted Alexander polynomial. ∎

Using Lemma 4.2, we can formulate a general form of the twisted Alexander polynomial for Jones representations. Without loss of generality, we can assume $\det \Phi(1-\sigma_1) \neq 0$ for a representation $\rho_{\widetilde{Y}}$ of B_{n+1}. We then have

$$\Delta_{B_{n+1},\widetilde{Y}}(t) = \frac{\widetilde{Q_1}(t)}{\det \Phi(1-\sigma_1)}.$$

Let $p(t)$ be the greatest common divisor of $\det N^I$ for all the choices of the indices I, where N denotes the sub-Alexander matrix of $\widetilde{M}$ defined by

$$N = {}^t\Big(\Phi(\sigma_1 - 1), \dots, \Phi(\sigma_{n-2} - 1), \Phi(-1 + \sigma_{n-1} - \sigma_n\sigma_{n-1})\Big).$$

Here we have decomposed the Alexander matrix $\widetilde{M}$ for $\rho_{\widetilde{Y}}$ as follows:

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ * & N \end{pmatrix}.$$

Further it should be remarked that we adopt the same numbering of relations of B_{n+1} as before (see Section 3). Clearly $p(t)$ is a divisor of $\widetilde{Q_1}(t)$, so that we can write $\widetilde{Q_1}(t) = p(t)q(t)$ for some Laurent polynomial $q(t)$. On the other hand, the twisted Alexander polynomimal $\Delta_{B_n,\widetilde{Y}}(t)$ is calculated by

$$\Delta_{B_n,\widetilde{Y}}(t) = \frac{Q_1(t)}{\det \Phi(1-\sigma_1)},$$

where $Q_1(t)$ is the greatest common divisor of $\det M_1^I$ with respect to the sub-Alexander matrix M of $\widetilde{M}$ corresponding to B_n. We then easily notice that $q(t)$ is a divisor of $Q_1(t)$. Hence we can write $Q_1(t) = q(t)r(t)$. Namely the Laurent polynomial $r(t)$ is essential divisor of $Q_1(t)$ which is not contained in $\widetilde{Q_1}(t)$. Accordingly we obtain

PROPOSITION 4.3. *Under the same assumption of the above lemma, the twisted Alexander polynomial $\Delta_{B_{n+1},\widetilde{Y}}(t)$ is described in the form*

$$\Delta_{B_{n+1},\widetilde{Y}}(t) = A(t) \prod_{k=1}^{l} \Delta_{B_n,Y_k}(t),$$

where $A(t) = p(t)/r(t)$, $p(t)$ and $r(t)$ are the Laurent polynomials defined above.

Finally we compute the twisted Alexander polynomials for Young diagrams up to 5-boxes. For 3 and 4 braids, Figure 4.1 shows that almost all the Hecke algebra representations are essentially Burau representations. We therefore have already finished the computations by virtue of Theorem 1.1, Example 4.1 and Corollary 4.1. Indeed, the 2-dimensional irreducible representation of B_4 corresponding to the Young diagram factors through , so that their twisted Alexander polynomials clearly coincide.

EXAMPLE 4.2. We calculate for $Y =$. Here we use the algorithm proposed by Lascoux and Schutzenberger in [9], which systematically constructs an irreducible representation of the Hecke algebra from a given Young diagram. In this case, the images of generators of B_5 are given by

$$\sigma_1 \mapsto \begin{pmatrix} -1 & \sqrt{q} & 0 & 0 & \sqrt{q} \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & -1 & \sqrt{q} & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ \sqrt{q} & -1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & \sqrt{q} & -1 & \sqrt{q} \\ 0 & 0 & 0 & 0 & q \end{pmatrix},$$

$$\sigma_3 \mapsto \begin{pmatrix} -1 & \sqrt{q} & \sqrt{q} & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & \sqrt{q} & -1 \end{pmatrix}, \quad \sigma_4 \mapsto \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ \sqrt{q} & 0 & -1 & 0 & 0 \\ 0 & \sqrt{q} & 0 & -1 & \sqrt{q} \\ 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

From direct computations, we obtain $Q_1(t) = (1+t)^2(1-qt)^3(1-qt^2)$ and $\widetilde{Q_1}(t) = (1+t)(1-qt)^2$. Thereby the twisted Alexander polynomial is

$$\Delta_{B_5,Y}(t) = \frac{1}{(1+t)(1-qt)}.$$

Further we easily see from Theorem 1.2 that for the conjugate Young diagram Y', $\Delta_{B_5,Y'}(t)$ coincides with the above rational expression.

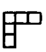
EXAMPLE 4.3. We next examine the representation $Y =$. In this case, matrix representations are as follows:

$$\sigma_1 \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & \sqrt{q} & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \sqrt{q} & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} -1 & \sqrt{q} & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & \sqrt{q} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q} & -1 & \sqrt{q} \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix},$$

$$\sigma_3 \mapsto \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ \sqrt{q} & -1 & 0 & \sqrt{q} & 0 & 0 \\ 0 & 0 & -1 & 0 & \sqrt{q} & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & \sqrt{q} & -1 \end{pmatrix}, \sigma_4 \mapsto \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & \sqrt{q} & 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{q} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

A similar observation to that in the example above shows $Q_1(t) = (1+t)^3(1-qt)^3$ and $\widetilde{Q_1}(t) = 1$. We then conclude

$$\Delta_{B_5,Y}(t) = \frac{1}{(1+t)^3(1-qt)^3}.$$

This is of course invariant under the transformation $t \mapsto -q^{-1}t^{-1}$, because the Young diagram  is self-conjugate.

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