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WEIGHTED LACUNARY MAXIMAL FUNCTIONS ON CURVES

JONG-GUK BAK

ABSTRACT. Let $\gamma(t) = (t, t^2, \ldots, t^n) + \mathbf{a}$ be a curve in $\mathbf{R}^n$, where $n \geq 2$ and $\mathbf{a} \in \mathbf{R}^n$. We prove L^p–L^q estimates for the weighted lacunary maximal function, related to this curve, defined by

$$\mathcal{M}_{p,q}f(x) = \sup_{k \in Z} \left| 2^{k(n/p-n/q)} \int_0^1 f(x - 2^k\gamma(t))\, dt \right|, \quad f \in C_0^\infty(\mathbf{R}^n).$$

If $n = 2$ or 3 our results are (nearly) sharp.

Let $n \geq 2$ and fix a vector $\mathbf{a} \in \mathbf{R}^n$. Let $\gamma(t) = (t, t^2, \ldots, t^n) + \mathbf{a}$, for $t \in \mathbf{R}$. Consider the curve $\Gamma = \{\gamma(t) : \ 0 \leq t \leq 1\} \subset \mathbf{R}^n$, and the measure μ supported on Γ given by $d\mu\big(\gamma(t)\big) = dt$. That is, μ acts on functions f by $\langle \mu, f \rangle = \int_0^1 f\big(\gamma(t)\big)\, dt$. For $r > 0$ a dilate μ_r of μ is defined by

$$\langle \mu_r, f \rangle = \int_0^1 f\big(r\gamma(t)\big)\, dt,$$

or equivalently, μ_r may be defined by the equation $\widehat{\mu_r}(\xi) = \hat{\mu}(r\xi)$. Here $\widehat{\ }$ denotes the Fourier transform in $\mathbf{R}^n$. A dilate of a distribution ν is defined similarly.

In analogy with the spherical maximal function introduced by E. M. Stein (see [S3]), one may define the maximal function $\mathcal{N}$ associated to the curve Γ, with $\mathbf{a} = (0, \ldots, 0, 1)$ say, by

$$\mathcal{N}f(x) = \sup_{r>0} |\mu_r * f(x)| = \sup_{r>0} \left| \int_0^1 f\big(x - r\gamma(t)\big)\, dt \right|, \quad f \in C_0^\infty(\mathbf{R}^n).$$

If $n = 2$ this is a variant of the spherical (circular) maximal function and it is known that $\mathcal{N}$ is bounded on L^p if and only if $p > 2$ (see [B], [MSS], [So]). On the other hand if $n \geq 3$ it is at present unknown whether there is some $p < \infty$ for which $\mathcal{N}$ is bounded on $L^p(\mathbf{R}^n)$.

Let us now abbreviate the lacunary dilate μ_{2^k} as μ_k $(k \in \mathbf{Z})$. The corresponding lacunary maximal function may then be defined by

$$\mathcal{M}f(x) = \sup_{k \in \mathbf{Z}} |\mu_k * f(x)| = \sup_k \left| \int_0^1 f\big(x - 2^k\gamma(t)\big)\, dt \right|, \quad f \in C_0^\infty(\mathbf{R}^n).$$

In contrast to $\mathcal{N}$ it is well known that $\mathcal{M}$ is bounded on $L^p(\mathbf{R}^n)$ for $p \in (1, \infty]$ (see [DR], [S3]; also see [C]).

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The purpose of this note is to study the L^p–L^q mapping properties of a weighted version of the lacunary maximal function:

$$\mathcal{M}_{p,q}f(x) = \mathcal{M}_{1/p-1/q}f(x) = \sup_{k\in\mathbf{Z}} |2^{k(n/p-n/q)}\mu_k * f(x)|, \quad f \in C_0^\infty(\mathbf{R}^n).$$

(A weighted maximal function (for the sphere) was first considered by Oberlin [O2]. As was noted there, homogeneity implies that $\mathcal{M}_{p,q}$ can only be bounded from L^r to L^s when $1/r - 1/s = 1/p - 1/q$.)

It appears that the mapping properties of $\mathcal{M}_{p,q}$ are closely related to those of the convolution operator $Tf = \mu * f$. Let

$$\Delta = \Delta_n = \left\{(1/p, 1/q) \in [0,1]\times[0,1] : 0 \le \frac{1}{p} - \frac{1}{q} \le \frac{2}{n(n+1)}, \frac{1}{q} \ge \frac{n-1}{np}, \frac{1}{q} \ge \frac{n}{(n-1)p} - \frac{1}{n-1}\right\}.$$

Thus Δ is the closed trapezoid (triangle when $n = 2$) with vertices $(0,0)$, $(1,1)$, $D = \big((n^2-n+2)/(n^2+n), (n-1)/(n+1)\big)$, and $D' = \big(2/(n+1), (2n-2)/(n^2+n)\big)$. For T to be bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ it is necessary that $(1/p, 1/q) \in \Delta$ (see *e.g.* [M]). When $n = 2$ or 3 the complete mapping properties of T are known: T is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ if and only if $(1/p, 1/q) \in \Delta$ (see [O1]). But when $n \ge 4$ the only known sufficient condition is that T is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ if $(1/p, 1/q)$ belongs to the closed triangle with vertices $(0,0)$, $(1,1)$, and $E = \big((n^2+n+2)/(2n^2+2n), (n^2+n-2)/(2n^2+2n)\big)$, where E is the midpoint of the line segment DD' (see [M]). Thus when $n \ge 4$ there is a large gap between the known necessary and sufficient conditions.

Note that $\mathcal{M}_{p,q}$ may not be bounded unless $(1/p, 1/q) \in \Delta$, since $\mu * f$ is pointwise dominated by $\mathcal{M}_{p,q}f$. We obtain the following positive result for $\mathcal{M}_{p,q}$ in $\mathbf{R}^3$. It affirms a conjecture of Oberlin. The letter C will denote a constant which may not be the same at each occurrence, but always independent of $\ell \in \mathbf{Z}$ and f (or ξ). Let Δ^o denote the interior of Δ.

THEOREM. *Let $n = 3$. Then*

$$\|\mathcal{M}_{p,q}f\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)} \tag{1}$$

if $(\frac{1}{p}, \frac{1}{q}) \in \Delta^o$, or if $p = q \in (1, \infty]$.

When $p = q \in (1, \infty]$ this is the known result about $\mathcal{M}$ mentioned above. Let us give a brief outline of its proof. The L^2 estimate follows from the decay of $\hat{\mu}$ and a Littlewood-Paley decomposition of f (as in Lemma 1 below). The L^p estimates for $1 < p < 2$ (the other values of p being trivial) are then deduced by applying a "bootstrap" argument (an iterated interpolation argument) similar to the one appearing in [NSW] (see also [DR], [S3]). The proof of the estimates (1) for the points in Δ^o is similar: it may be based on a Littlewood-Paley decomposition of f, and certain uniform oscillatory integral estimates due to Oberlin [O3] and McMichael [M] (see Lemma 2 below), and the convolution properties of μ in $\mathbf{R}^3$ ([O1]; see above), combined with complex interpolation and a bootstrap argument. A similar argument also shows that (1) holds in $\mathbf{R}^n$ ($n \geq 2$) whenever $(1/p, 1/q)$ belongs to the open triangle with vertices $(0,0)$, $(1,1)$ and E, where E is as above.

It may be an interesting problem to determine what happens on the boundary of Δ (see *e.g.* [Ch1, Theorem 4]). It might also be worth pointing out that (1) holds independent of the vector $\mathbf{a}$, in particular when $\mathbf{a} = 0$, although there are related maximal functions whose properties when $\mathbf{a} = (0, \ldots, 0, 1)$ and when $\mathbf{a} = 0$, say, are very different.

To prove the theorem we first need to state two lemmas. Fix a nonnegative function $\phi \in C_0^\infty(\mathbf{R})$ such that ϕ is supported in the interval $(1/2, 2)$ and $\sum_{j \in \mathbf{Z}} \phi(2^j t) \equiv 1$ for $t > 0$. For $j \in \mathbf{Z}$ the Littlewood-Paley operator P_j is defined by $\widehat{P_j f}(\xi) = \phi_j(|\xi|)\hat{f}(\xi) = \phi(2^j|\xi|)\hat{f}(\xi)$, for $f \in C_0^\infty(\mathbf{R}^n)$, say. Thus $f = \sum_{j \in \mathbf{Z}} P_j f$.

The following lemma is standard (see [DR]). It follows by Plancherel's theorem from the hypotheses on the decay of the Fourier transform of ν and the support properties of ϕ_j.

LEMMA 1. *Suppose that ν is a distribution on $\mathbf{R}^n$ such that for some number $\delta > 0$ $|\hat{\nu}(\xi)| \leq C|\xi|^{-\delta}$, and $|\hat{\nu}(\xi)| \leq C|\xi|^{\delta}$ for $\xi \in \mathbf{R}^n$. Then*

$$\Big\| \Big(\sum_{k \in \mathbf{Z}} |\nu_k * P_{k+\ell} f|^2 \Big)^{1/2} \Big\|_2 \leq C 2^{-\delta|\ell|} \|f\|_2.$$

It follows from the last inequality that

$$\Big\| \Big(\sum_k |\nu_k * f|^2 \Big)^{1/2} \Big\|_2 = \Big\| \Big(\sum_k \Big| \nu_k * \Big(\sum_\ell P_{k+\ell} f \Big) \Big|^2 \Big)^{1/2} \Big\|_2 \leq C \sum_\ell 2^{-\delta|\ell|} \|f\|_2 \leq C\|f\|_2.$$

Certain special cases of the next lemma were proved by Oberlin [O3]. The general version stated below is due to McMichael [M]. Let $\mathcal{P}_N$ be the space of real-valued polynomials on $\mathbf{R}$ of degree at most N.

LEMMA 2. *Given a positive integer N, there exists a constant C_N such that if $\alpha_1, \ldots, \alpha_N$ are nonnegative real numbers with $\sum_{j=1}^N j\alpha_j = 1$, then*

$$\Big| \int_a^b e^{ip(t)} \Big(\prod_{j=1}^N |p^{(j)}(t)|^{\alpha_j} \Big)^{1+is} dt \Big| \leq C_N (1 + |s|)^\sigma$$

if $p \in \mathcal{P}_N$, $a < b$, and $s \in \mathbf{R}$, where $\sigma = \sum_{j=1}^N \alpha_j$.

PROOF OF THEOREM. Following Oberlin and McMichael [M] we define an analytic family of operators by

$$T_z f(x) = \frac{1}{\Gamma\big((z+1)/2\big)^2} \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty f\big(x - \gamma(t) - u\gamma''(t) - v\gamma'''(t)\big) |u|^z |v|^z \, du \, dv \, dt$$

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(initially by this equation for $\operatorname{Re} z > -1$, then for all complex z by analytic continuation). Then $T_z f(x) = \mu^z * f(x)$, where

$$\widehat{\mu^z}(\xi) = C_z \int_0^1 e^{i\gamma(t)\cdot\xi} |\gamma''(t)\cdot\xi|^{-1-z} \cdot |\gamma'''(t)\cdot\xi|^{-1-z}\, dt$$

(see [GS, p. 359]). If $\operatorname{Re} z = -6/5$, it follows from Lemma 2 with $p(t) = \gamma(t)\cdot\xi$, $N = 3$, $\alpha_1 = 0$, and $\alpha_2 = \alpha_3 = 1/5$, that

$$|\widehat{\mu^z}(\xi)| \le C_z \quad \forall \xi \in \mathbf{R}^3,$$

where the constant C_z has at most exponential growth in $|\operatorname{Im} z|$.

Now let G_α be the Bessel kernel of (complex) order α, *i.e.*,

$$\widehat{G_\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha/2},$$

and take $\nu = G_\alpha * \mu^z$, with $\operatorname{Re}\alpha = \varepsilon \in (0, 2/5)$. Then $\hat\nu(\xi) = \widehat{G_\alpha}(\xi)\widehat{\mu^z}(\xi)$. So $|\hat\nu(\xi)| \le C_z(1+|\xi|)^{-\varepsilon}$ if $\operatorname{Re} z = -6/5$. Notice also that $|\hat\nu(\xi)| \le C|\xi|^{2/5}$ if $\operatorname{Re} z = -6/5$. Therefore by Lemma 1

$$\Big\| \sup_k |(G_{\varepsilon+is} * \mu^z)_k * P_{k+\ell} f| \Big\|_2 \le C_z 2^{-\varepsilon|\ell|} \|f\|_2, \quad \text{if } \operatorname{Re} z = -6/5. \tag{2}$$

We have $\|G_{\varepsilon+is}\|_1 \le C\big|\Gamma\big((\varepsilon+is)/2\big)\big|^{-1}$ (see [S1, p. 132]). And we can see that $\mu^{i\tau}$ is bounded (as a function of ξ) if $\tau \in \mathbf{R}$, by making the change of variables $(t, u, v) \longrightarrow y = (y_1, y_2, y_3)$ given by $y = \gamma(t) + u\gamma''(t) + v\gamma'''(t) = (t, t^2 + 2u, 1 + t^3 + 6ut + 6v)$ in the integral for $T_{i\tau} f(x) = \mu^{i\tau} * f(x)$, and noting that the Jacobian is a constant. Thus

$$\|(G_{\varepsilon+is} * \mu^{i\tau}) * f\|_\infty \le \|G_{\varepsilon+is} * \mu^{i\tau}\|_\infty \|f\|_1 \le \|G_{\varepsilon+is}\|_1 \|\mu^{i\tau}\|_\infty \|f\|_1 \le C_{\varepsilon,s} C_\tau \|f\|_1,$$

where the constant $C_{\varepsilon,s}C_\tau$ has at most exponential growth in s and τ. Hence by homogeneity we have

$$\Big\| \sup_k |2^{3k}(G_{\varepsilon+is} * \mu^z)_k * P_{k+\ell} f| \Big\|_\infty \le C_{\varepsilon,s} C_\tau \|f\|_1, \quad \text{if } \operatorname{Re} z = 0. \tag{3}$$

To interpolate (2) and (3) we consider an analytic family of vector-valued linear operators defined by

$$S_z(f) = \{2^{k(3+5z/2)}(G_{\varepsilon+is} * \mu^z)_k * P_{k+\ell} f\}_{k\in\mathbf{Z}}$$

(with $\varepsilon + is$ and ℓ fixed). Observe that (2) may be restated as boundedness of S_z from L^2 to $L^2(\ell^\infty)$ (a mixed-norm space):

$$\Big\| \|S_z(f)\|_{\ell^\infty(\mathbf{Z})} \Big\|_{L^2(\mathbf{R}^3)} \le C 2^{-\varepsilon|\ell|} \|f\|_2, \quad \text{if } \operatorname{Re} z = -6/5;$$

and (3) as

$$\Big\| \|S_z(f)\|_{\ell^\infty(\mathbf{Z})} \Big\|_{L^\infty(\mathbf{R}^3)} \le C \|f\|_1, \quad \text{if } \operatorname{Re} z = 0.$$

Therefore by complex interpolation in the mixed-norm setting (see [BP], [O2]) we obtain

$$\Big\|\sup_k |2^{k/2}(G_{\varepsilon+is} * \mu)_k * P_{k+\ell}f|\Big\|_{12/5} \leq C2^{-(5/6)\varepsilon|\ell|}\|f\|_{12/7}, \tag{4}$$

since $\mu^{-1} = \mu$.

Now fix a number $\delta \in (0, 1/3)$. By Theorem 2 in [S2, p. 324] we have $|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-1/3}$. So $|(G_{-\delta+is} * \mu)^\wedge(\xi)| = |(G_{-\delta+is})^\wedge(\xi)| \cdot |\hat{\mu}(\xi)| \leq C$. Hence by Plancherel's theorem

$$\Big\|\sup_k |(G_{-\delta+is} * \mu)_k * P_{k+\ell}f|\Big\|_2 \leq \Big\|\Big(\sum_k |(G_{-\delta+is} * \mu)_k * P_{k+\ell}f|^2\Big)^{1/2}\Big\|_2 \leq C\|f\|_2. \tag{5}$$

We now apply complex interpolation again to the analytic family

$$S^\alpha(f) = \{2^{k(\alpha+\delta)/2(\varepsilon+\delta)}(G_\alpha * \mu)_k * P_{k+\ell}f\}_{k\in\mathbf{Z}}.$$

Since $G_0 * \mu = \mu$, (4) and (5) thus yield

$$\Big\|\sup_k |2^{k(3/p_0-3/q_0)}\mu_k * P_{k+\ell}f|\Big\|_{q_0} \leq C2^{-\varepsilon(p_0)|\ell|}\|f\|_{p_0} \tag{6}$$

for some $\varepsilon(p_0) > 0$ if $2 > p_0 > 12/7$ and $q_0 = p_0'$ (the conjugate exponent of p_0). (By choosing $\varepsilon > 0$ small enough in (4) we may get (6) for points $(1/p_0, 1/p_0') \in \Delta^o$ arbitrarily close to the point $(7/12, 5/12)$.) Since $f = \sum_{\ell\in\mathbf{Z}} P_{k+\ell}f$, an immediate consequence of (6) is that

$$\|\mathcal{M}_{p_0,q_0}f\|_{q_0} \leq C\|f\|_{p_0}. \tag{6'}$$

This proves (1) for points $(1/p, 1/q)$ in Δ^o lying on the line of duality $1/p + 1/q = 1$.

We now extend (1) to points that lie off the line of duality. Fix $\beta = 1/p_0 - 1/q_0 \in (0, 1/6)$ and let L denote the (open) line segment $L = L_\beta = \{(1/p, 1/q) \in \Delta^o : 1/p - 1/q = \beta\}$. Since μ is a positive measure, if $\{f_k\}$ is a sequence of functions, (6′) implies that

$$\Big\|\sup_k |2^{k(3/p_0-3/q_0)}\mu_k * f_k|\Big\|_{q_0} \leq \Big\|\mathcal{M}_{p_0,q_0}\big(\sup_j |f_j|\big)\Big\|_{q_0} \leq C\Big\|\sup_j |f_j|\Big\|_{p_0}. \tag{6''}$$

(See [NSW] and [Ch2] for related positivity arguments.) Let $(1/a, 1/b)$ denote the right endpoint of L. (At the left endpoint the argument is simpler and a bootstrap argument is not necessary, since $a \geq 2$.) It is known from [O1] that

$$\|\mu * f\|_b \leq C\|f\|_a,$$

which implies by homogeneity that for $k \in \mathbf{Z}$ and the same constant C

$$\|2^{k(3/a-3/b)}\mu_k * f\|_b \leq C\|f\|_a.$$

Since $1 \leq a \leq b$ it is easy to see that

$$\Big\|\Big(\sum_k |2^{k(3/a-3/b)}\mu_k * f_k|^b\Big)^{1/b}\Big\|_b \leq C\Big\|\Big(\sum_k |f_k|^a\Big)^{1/a}\Big\|_a \leq C\Big\|\sum_k |f_k|\Big\|_a. \tag{7}$$

By interpolating (6″) and (7) in the mixed-norm setting we get

$$\Big\|\Big(\sum_k |2^{3\beta k}\mu_k * f_k|^{2b}\Big)^{1/2b}\Big\|_{q_1} \le C\Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_{p_1}, \tag{8}$$

with $1/p_1 - 1/q_1 = \beta$ and $1/p_1 = (1/p_0 + 1/a)/2$. (Thus $(1/p_1, 1/q_1)$ is the midpoint of the line segment joining $(1/p_0, 1/q_0)$ and $(1/a, 1/b)$.) Taking $f_k = P_{k+\ell}f$ in (8) we obtain

$$\Big\|\sup_k |2^{3\beta k}\mu_k * P_{k+\ell}f|\Big\|_{q_1} \le C\Big\|\Big(\sum_k |P_{k+\ell}f|^2\Big)^{1/2}\Big\|_{p_1} \le C\|f\|_{p_1}, \tag{8'}$$

where the last inequality follows from a Littlewood-Paley inequality (see *e.g.* [So, p. 21]). Interpolating (6) and (8′) yields

$$\Big\|\sup_k |2^{3\beta k}\mu_k * P_{k+\ell}f|\Big\|_q \le C2^{-\varepsilon(\beta,p)|\ell|}\|f\|_p, \tag{9}$$

for all $(1/p, 1/q)$ on L lying strictly between $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. Hence we have for the same values of p and q

$$\|\mathcal{M}_{p,q}f\|_q \le C\|f\|_p, \tag{9'}$$

and by the positivity of μ (as before)

$$\Big\|\sup_k |2^{3\beta k}\mu_k * f_k|\Big\|_q \le C\Big\|\sup_j |f_j|\Big\|_p. \tag{9''}$$

We interpolate again with (9″) (in place of (6″) in the interpolation step above) and (7) to get (1) on the entire open line segment with endpoints $(1/p_0, 1/q_0)$ and $(1/p_2, 1/q_2)$, where the latter is the midpoint of the line segment joining $(1/p_1, 1/q_1)$ and $(1/a, 1/b)$. By repeating this process we obtain (1) for any point $(1/p, 1/q)$ on L. ■

It should also be clear from this proof that in the statement of the theorem (1) may be replaced by the following slightly stronger estimate:

$$\Big\|\Big(\sum_k |2^{k(3/p-3/q)}\mu_k * f|^q\Big)^{1/q}\Big\|_q \le C\|f\|_p. \tag{1'}$$

To see this observe that, for instance, the $\sup_k$ on the left hand side of (2) may be replaced by an ℓ^2 norm, so that (4) actually holds with the $\sup_k$ replaced by an $\ell^{12/5}$ norm.

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Department of Mathematics
Florida State University
Tallahassee, Florida 32306
U.S.A.
e-mail: bak@math.fsu.edu

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