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EXTENDED CHROMATIC POLYNOMIALS

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1. Introduction. Let G be a finite graph with non-empty vertex set $\mathscr{V}(G)$ and edge set $\mathscr{E}(G)$ (see [**2**]). Let λ be a positive integer. Tutte [**5**] defines a λ-colouring of G as a mapping of $\mathscr{V}(G)$ into the set $I_\lambda = \{1, 2, 3, \ldots, \lambda\}$ with the property that two ends of any edge are mapped onto distinct integers. The elements of I_λ are commonly called "colours." If $P(G, \lambda)$ represents the number of λ-colourings of G, it is well known that $P(G, \lambda)$ can be expressed as a polynomial in λ. For this reason $P(G, \lambda)$ is usually referred to as the chromatic polynomial of G.

The chromatic polynomial $P(G, \lambda)$ was first suggested as an approach to the four-colour conjecture. To quote Tutte [**5**]: "... many people are specially interested in the value $\lambda = 4$. There is a long-standing conjecture that $P(M, 4)$ is positive for every triangulation M." Although the four-colour conjecture remains, chromatic polynomials are of interest in themselves and occupy a prominent place in the literature.

Many well known combinatorial problems seem to suggest other chromatic polynomials in much the same way as the four-colour conjecture prompted the definition of $P(G, \lambda)$. For example, consider the class of Ramsey numbers $R(k_1, \ldots, k_\lambda; 2)$. (See [**1**; **4**; **3**, Chapter IV].) For $k = (k_1, \ldots, k_\lambda)$, define $E_k(G, \lambda)$ to be the number of mappings f of $\mathscr{E}(G)$ into I_λ, which have the property that for each $\nu = 1, \ldots, \lambda$, $f^{-1}(\nu)$ does not contain all the edges of any complete subgraph on as many as k_ν vertices. For G the complete graph K_n on n vertices, if $E_k(K_n, \lambda) > 0$, then $n \leqq R(k_1, \ldots, k_\lambda; 2)$. The determination of the largest integer n for which $E_k(K_n, \lambda) > 0$, for example in a case with $\lambda \geqq 4$ and $k_1 \geqq 3, \ldots, k_\lambda \geqq 3$, would be a determination of one of the numbers $R(k_1, \ldots, k_\lambda; 2)$ which have been unknown for a long time. (It is known that $R(4, 4; 2) = 17$, and that $R(3, 3, 3; 2) = 16$.)

Of equal interest are the chromatic polynomials $V_k(G, \lambda)$ and $T_k(G, \lambda)$. Hence $V_k(G, \lambda)$ is the number of mappings of $\mathscr{V}(G)$ into I_λ which are not constant on the vertices of any complete k-subgraph of G; and $T_k(G, \lambda)$ is the number of mappings of the set of triangles of G into I_λ which are not constant on the triangles of any complete k-subgraph of G. In § 3 we specialize $E_k(G, \lambda)$ to the case $k_1 = \ldots = k_\lambda = k$, where now we regard k as a single positive integer (rather than as an ordered set of λ positive integers). Of course V_k and T_k have analogous generalizations, which, like the general E_k, are not studied in the present paper. Also, for the main results of this paper, G will be K_n.

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2. The chromatic polynomials $V_k(G, \lambda)$**.** By a (V, k, λ)-colouring of a graph G we mean a mapping of $\mathscr{V}(G)$ into I_λ which is not constant on the vertices of any complete k-subgraph of G.

Our polynomial $V_2(G, \lambda)$, and the traditional chromatic polynomial $P(G, \lambda)$ as defined by Tutte [**5**], are similar but not identical. Under Tutte's definition, if G has a loop, then $P(G, \lambda)$ is 0. On the other hand, if x denotes the loop of G, then $V_2(G, \lambda) = V_2(G\backslash x, \lambda)$. In fact we have the following lemma.

LEMMA 1. *If G has a loop x, then* $V_k(G, \lambda) = V_k(G\backslash x, \lambda)$.

The proof is immediate from the definition of a (V, k, λ)-colouring.

LEMMA 2. *Let G be the union of components* $H_1, \ldots, H_n$. *Then*

$$V_k(G, \lambda) = \prod_{i=1}^{n} V_k(H_i, \lambda).$$

Proof. Since $H_i \cap H_j = \emptyset$ for $i \neq j$, each combination of (V, k, λ)-colourings of the H_i yields a (V, k, λ)-colouring of G; and each (V, k, λ)-colouring of G is some combination of (V, k, λ)-colourings of the H_i.

THEOREM 1. *If G is any finite graph, then* $V_k(G, \lambda)$ *can be expressed as a polynomial in* λ *with the following properties:*

(i) *The coefficient* a_i *of* λ^i *is an integer for all i.*

(ii) *Coefficient* $a_i \neq 0$ *only if* $c(G) \leqq i \leqq m$, *where* $c(G)$ *is the number of components of G, and m is the cardinality of* $V(G)$.

(iii) *Coefficient* a_m *is* 1.

Proof. Let $a_{n,k}$ be the number of mappings of $V(G)$ onto I_n which are not constant on any complete k-subgraph of G. Clearly, the $a_{n,k}$ are integers for all integers k and n. Also, since necessarily $a_n = 0$ for $n > m$, we have

$$V_k(G, \lambda) = \sum_{n=1}^{m} \binom{\lambda}{n} a_{n,k}$$

where $\binom{\lambda}{n}$ is taken to be 0 if $\lambda < 0$ or $\lambda < n$. Clearly $V_k(G, \lambda)$ is a polynomial in λ with integer coefficients. The term of highest degree, λ^m, of $V_k(G, \lambda)$ is obtained from the above expression when $n = m$. Since $\binom{\lambda}{n}$ contains a factor λ for each value of n, we see that a non-zero constant term in $V_k(G, \lambda)$ is not possible. The lower bound of $c(G)$ for the exponent of λ is now a consequence of Lemma 2. Note that $c(G)$ is by no means a strict lower bound on the exponents of λ in $V_k(G, \lambda)$. For $k > 2$ we can obtain an arbitrarily large connected graph G which contains no complete k-subgraph. For such a graph G, $V_k(G, \lambda) = m$, although $c(G) = 1$.

Clearly $V_2(K_n, \lambda) = P(K_n, \lambda)$, and we obtain directly

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THEOREM 2. *The polynomial* $V_3(K_n, \lambda)$ *is given by*

$$\frac{\lambda!}{(\lambda - n)!} + \sum_{i=1}^{(n-1)/2} \frac{2\lambda! n!}{[\lambda - (n + 2i + 1)/2]! 2^{(n-2i+1)/2}(2i - 1)!(n - 2i + 1)}$$

if n *is odd; and by*

$$\frac{\lambda!}{(\lambda - n)!} + \sum_{i=0}^{(n-2)/2} \frac{2\lambda! n!}{[\lambda - (n + 2i)/2]! 2^{(n-2i)/2}(2i)!(n - 2i)}$$

if n *is even.*

Proof. Let n be an odd integer. Suppose first that $\lambda \geqq n$. Define a t-partition $\mathscr{P}$ of a graph G to be a partition of the set $\mathscr{V}(G)$ such that each part of $\mathscr{P}$ contains at most t elements. At most two vertices of K_n can share the same colour in a $(V, 3, \lambda)$-colouring since any three vertices of K_n are the vertices of a triangle. Hence each $(V, 3, \lambda)$-colouring of K_n is a colouring of some 2-partition $\mathscr{P}$ of K_n in which distinct parts of $\mathscr{P}$ receive different colours. There can be $1, 3, 5, \ldots,$ or n one-element parts to a 2-partition of K_n. If $\mathscr{P}$ has k one-element parts, then it has $(n - k)/2$ two-element parts and $(n + k)/2$ parts in all. There are therefore

$$\sum_{i=0}^{(n-k-2)/2} \frac{(n - 2i)!}{(n - 2i - 2)!(n - k)} = \frac{2n!}{2^{(n-k)/2}k!(n - k)}$$

2-partitions with k one-element parts. Colouring each part of $\mathscr{P}$ differently requires $(n + k)/2$ colours. There are $\binom{\lambda}{(n + k)/2}$ ways of choosing $(n + k)/2$ colours, and $[(n + k)/2]!$ ways to colour the $(n + k)/2$ parts of $\mathscr{P}$. We see now that there are

$$\sum_{\text{odd } k=1}^{n-2} \frac{2\lambda! n!}{[\lambda - (n + k)/2]! 2^{(n-k)/2}k!(n - k)}$$

ways of colouring the 2-partitions of K_n with at least one two-element part. There are clearly $\lambda!/(\lambda - n)!$ ways to colour the 2-partition of K_n with n one-element parts. Hence for odd n,

$$V_3(K_n, \lambda) = \lambda!/(\lambda - n)! + \sum_{\text{odd } k=1}^{n-2} \frac{2\lambda! n!}{[\lambda - (n + k)/2]! 2^{(n-k)/2}k!(n - k)}.$$

We now replace k by $2i - 1$, and let i range from 1 to $(n - 1)/2$, to obtain a conventional summation. Thus the theorem is established for odd n. For even n the proof is similar.

A more satisfactory formula is given by:

THEOREM 3. *Let* n, k *be integers with* $n \geqq k \geqq 3$, *and let*

$$N = \left[\frac{n}{k - 1}\right].$$

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Then

$$V_k(K_n, \lambda) = \sum_{n=0}^{N} \left\{ \left[\prod_{i=1}^{n} \binom{n-(k-1)(i-1)}{k-1} \right] V_2(\lambda, m) V_{k-1}(K_{n-mk+m}, \lambda - m) \right\}.$$

Proof. First, any $(V, k-1, \lambda)$-colouring of K_n is a (V, k, λ)-colouring of K_n so that $V_k(K_n, \lambda) \geqq V_{k-1}(K_n, \lambda)$. Next any (V, k, λ)-colouring of K_n which is not a $(V, k-1, \lambda)$-colouring must be constant on at least one complete $(k-1)$-subgraph of K_n. We shall call a (V, k, λ)-colouring of K_n which is constant on exactly n $(k-1)$-subgraphs an $E_V(n, k-1, \lambda)$-colouring of K_n. There are

$$\prod_{i=1}^{n} \binom{n-(k-1)(i-1)}{k-1}$$

ways to choose n pair-wise disjoint $(k-1)$-subgraphs of K_n. If each of these n $(k-1)$-subgraphs is to be monochromatic, there are $V_2(\lambda, m)$ ways to colour them so as to avoid a monochromatic k-subgraph. For each such colouring of m K_{k-1}-subgraphs there are $V_{k-1}(K_{n-m(k-1)}, \lambda - m)$ ways to colour the remaining vertices of K_n to obtain an $E_V(m, k-1, \lambda)$-colouring of K_n. Hence there are

$$\left[\prod_{i=1}^{m} \binom{n-(k-1)(i-1)}{k-1} \right] V_2(\lambda, m) V_{k-1}(K_{n-m(k-1)}, \lambda - m)$$

$E_V(m, k-1, \lambda)$-colourings of K_n. Since m can range from 0 to $\left[\frac{n}{k-1}\right]$, the theorem follows if we understand that

$$\prod_{i=1}^{0} \binom{n-(k-1)(i-1)}{k-1} = V_2(\lambda, 0) = 1$$

which is standard (see [**5**]).

Determination of $V_k(G, \lambda)$ provides more information than is at first realized. We have, for example, the following relationships:

$$V_k(G, 1) = a_{1,k}$$
$$V_k(G, 2) = a_{2,k} + 2a_{1,k}$$
$$V_k(G, 3) = a_{3,k} + 3a_{2,k} + 3a_{1,k}$$
$$\vdots$$
$$V_k(G, m) = a_{m,k} + \sum_{i=0}^{m-1} \binom{m}{i} a_{n-i,k}$$
$$\vdots$$
$$V_k(G, \lambda) = \frac{\lambda!}{m!(\lambda - m)!} + \sum_{i=0}^{m-1} \binom{\lambda}{m-i} a_{m-i,k}.$$

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Since $a_m = m!$ for all graphs G, this system of equations provides a nice check of the correctness of a calculated $V_k(G, \lambda)$. Also, knowledge of the a_i is important in itself.

THEOREM 4. *For any graph G, if $V_2(G, \lambda) > 0$, then*

$$V_3\left(G, \left[\frac{\lambda + 1}{2}\right]\right) > 0.$$

Proof. Suppose there exists a map Λ: $V(G) \rightarrow I_\lambda$ which is not constant on any 2-subgraph of G. If λ is even, define Λ': $V(G) \rightarrow I_{\lambda/2}$ by

$$\Lambda'(v) = \begin{cases} \Lambda(v) & \text{if} \quad \Lambda(v) \leqq \frac{\lambda}{2} \\ \Lambda(v) - \frac{\lambda}{2} & \text{if} \quad \Lambda(v) > \frac{\lambda}{2}. \end{cases}$$

Now suppose $\Lambda'(v_i) = \alpha (i = 1, 2, 3)$ where v_1, v_2, and v_3 are the vertices of a 3-subgraph of G, and $1 \leqq \alpha \leqq \lambda/2$. Then either $\Lambda(v_i) = \Lambda(v_j) = \alpha$, or $\Lambda(v_i) = \Lambda(v_j) = \alpha + \lambda/2$, for some $i, j \in I_3$. In either case we reach a contradiction to the definition of Λ, since v_i and v_j are vertices of a 2-subgraph of G for all $i, j \in I_3$. Hence Λ' is a $(V, 3, \lambda/2)$-colouring of G. If λ is odd, define Λ'': $V(G) \rightarrow I_{(\lambda+1)/2}$ by

$$\Lambda'' = \begin{cases} \Lambda(v) & \text{if} \quad \Lambda(v) \leqq \frac{\lambda + 1}{2} \\ \Lambda(v) - \frac{\lambda + 1}{2} & \text{if} \quad \Lambda(v) > \frac{\lambda + 1}{2}. \end{cases}$$

Using an argument similar to the one employed above we see that Λ'' is a $(V, 3, (\lambda + 1)/2)$-colouring of G.

Hence the existence of a $(V, 2, \lambda)$-colouring of G implies the existence of a $(V, 2, [(\lambda + 1)/2])$-colouring of G, and the theorem follows.

COROLLARY 1. *For any loopless planar graph G, $V_3(G, 3) > 0$.*

Proof. If G is planar and loopless, then $V_2(G, 5) > 0$.

COROLLARY 2. *The truth of the four-colour conjecture implies $V_3(G, 2) > 0$ for every planar graph G.*

3. The chromatic polynomials $E_k(G, \lambda)$.

By an (E, k, λ)-colouring of a graph G we mean a mapping of $E(G)$ into I_λ that is not constant on the edges of any k-subgraph of G.

LEMMA 3. *Let G be the union of subgraphs $H_1, H_2, H_3, \ldots, H_n$ such that $H_i \cap H_j = \emptyset$ or a singleton vertex, for each (i, j) with $i \neq j$. Then*

$$E_k(G, \lambda) = \prod_{i=1}^{n} E_k(H_i, \lambda).$$

Proof. The proof is similar to that of Lemma 2.

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THEOREM 5. *If G is any finite graph with* $|E(G)| > 0$, *then* $E_k(G, \lambda)$ *can be expressed as a polynomial in* λ *with the following properties:*

(i) *The coefficient* a_i *of* λ^i *is an integer for all i.*

(ii) $a_i \neq 0$ *only if* $C(G) \leqq i \leqq p$, *where* $C(G)$ *is the number of non-trivial components of G, and* $p = |E(G)|$.

(iii) $A_p = 1$.

Proof. The proof is similar to that of Theorem 1, if we replace $a_{n,k}$ by $b_{n,k}$, where $b_{n,k}$ is the number of mappings of $E(G)$ onto I_n which are non-constant on the edges of any k-subgraph of G.

LEMMA 4. *The edge-polynomial* $E_3(K_3, \lambda) = \lambda^3 - \lambda$.

Proof. There are λ^3 colourings of the edges of K_3 with λ colours. However, λ of these are not $(E, 3, \lambda)$-colourings. Hence $E_3(K_3, \lambda) = \lambda^3 - \lambda$.

Rather than use the subtractive approach, as in the proof of Lemma 4, we can count the number of $(E, 3, \lambda)$-colourings directly. Label the edges of K_3 as x_1, x_2, and x_3. The number of $(E, 3, \lambda)$-colourings with x_1 and x_2 of the same colour is $\lambda(\lambda - 1)$. The number of $(E, 3, \lambda)$-colourings with x_1 and x_2 of different colours is $\lambda^2(\lambda - 1)$. Hence as before,

$$E_3(K_3, \lambda) = \lambda(\lambda - 1) + \lambda^2(\lambda - 1) = \lambda^3 - \lambda.$$

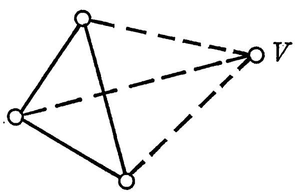
LEMMA 5. *The edge-polynomial*

$$E_3(K_4, \lambda) = \lambda^6 - 4\lambda^4 + 6\lambda^2 - 3\lambda.$$

Proof. There are λ^6 colourings of the edges of K_4 with λ colours. Of the λ^6 colourings, λ are constant on K_4. Also $4\lambda[\lambda(\lambda - 1)(\lambda - 2) + 3(\lambda - 1)^2 + (\lambda - 1)]$ colourings are constant on a single triangle of K_4. Here 4λ corresponds to the four triangles of K_4 and the λ ways in which one can have all of its edges coloured the same; $\lambda(\lambda - 1)(\lambda - 2)$, $3(\lambda - 1)^2$, and $(\lambda - 1)$ indicate the number of ways of colouring the remaining three edges of K_4 all differently, two alike, and all alike, respectively. Finally, there are $6\lambda(\lambda - 1)$ colourings which are constant on exactly two triangles of K_4—there are six ways to leave out an edge, λ ways to colour the two triangles, and $\lambda - 1$ ways to colour the remaining edge. Hence

$$\begin{aligned} E_3(K_4, \lambda) &= \lambda^6 - \lambda - 4\lambda[\lambda(\lambda - 1)(\lambda - 2) + 3(\lambda - 1)^2 + (\lambda - 1)] - 6\lambda(\lambda - 1) \\ &= \lambda^6 + 4\lambda^4 + 6\lambda^2 - 3\lambda. \end{aligned}$$

Again we offer an alternate method of counting. Choose a vertex v of K_4.



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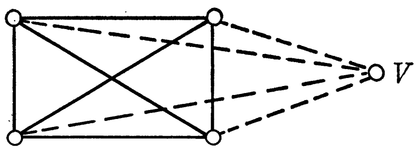
There are $\lambda E_3(K_3, \lambda - 1)(E, 3, \lambda)$-colourings of K_4 with all three edges at v coloured alike; $3\lambda(\lambda - 1)\left[\frac{\lambda - 1}{\lambda} E_3(K_3, \lambda)\right]$ $(E, 3, \lambda)$-colourings of K_4 with exactly two of the three edges at v coloured alike; and $\lambda(\lambda - 1)(\lambda - 2)$ $E_3(K_3, \lambda)(E, 3, \lambda)$-colourings of K_4 with each of the three edges at v coloured differently. Hence,

$$E_3(K_4, \lambda) = \lambda E_3(K_3, \lambda - 1) + 3\lambda(\lambda - 1)\left[\frac{\lambda - 1}{\lambda} E_3(K_3, \lambda)\right] + \lambda(\lambda - 1)(\lambda - 2)E_3(K_3, \lambda)$$
$$= \lambda^6 - 4\lambda^4 + 6\lambda^2 - 3\lambda.$$

LEMMA 6. *The edge-polynomial*

$$E_3(K_5, \lambda) = \lambda^{10} - 10\lambda^8 + 45\lambda^6 - 15\lambda^5 - 100\lambda^4 + 105\lambda^3 - 20\lambda^2 - 6\lambda.$$

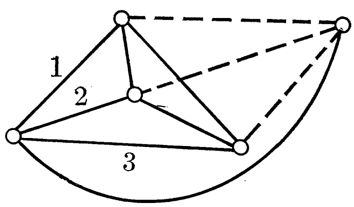
Proof. Choose any vertex v of K_5.



There are $\lambda E_3(K_4, \lambda - 1)(E, 3, \lambda)$-colourings of K_5 with all four edges at v coloured alike. When exactly three of the four edges at v are coloured alike, there are

$$4\lambda(\lambda - 1)[E_3(K_3, \lambda - 1) + (\lambda - 1)E_3(K_3, \lambda - 2) + 3(\lambda - 1)E_3(K_3, \lambda - 1) + 3(\lambda - 1)(\lambda - 2)E_3(K_3, \lambda - 1) + \lambda(\lambda - 1)(\lambda - 2)E_3(K_3, \lambda - 1)]$$

possible $(E, 3, \lambda)$-colourings. Here the term $4\lambda(\lambda - 1)$ includes the four choices of three edges at v, the λ ways to colour them, and the $\lambda - 1$ ways to colour the remaining edge at v.



For each choice of colour c for the three like edges at v, the terms in brackets correspond respectively to the cases: (1) edges 1, 2, and 3 coloured c, (2) edges 1, 2, 3 coloured alike with some colour other than c, (3) two of 1, 2, and 3 coloured c, (4) two of 1, 2, and 3 coloured alike with some colour other than c, and (5) each of 1, 2, and 3 coloured differently. There are

$$6\lambda(\lambda - 1)(\lambda - 2)\left[\frac{(\lambda - 1)}{\lambda} E_3(K_4, \lambda)\right]$$

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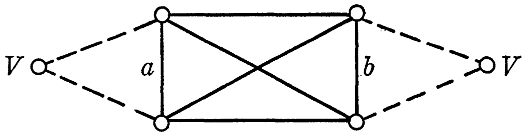
$(E, 3, \lambda)$-colourings of K_5 with exactly two edges at v coloured alike. Suppose next that there are two edges of one colour and two edges of another colour, incident with v.



Pick a pair of non-adjacent edges a and b in K_4 as in the figure above. (There are three such pairs.) The number of $(E, 3, \lambda)$-colourings A for which a and b are coloured alike is given by

$$A = \lambda(\lambda - 1)[\lambda^3 + \lambda^2 - 3\lambda + 1].$$

The number of $(E, 3, \lambda)$-colourings D for which a and b are coloured differently is given by

$$D = \lambda(\lambda - 1)[\lambda^4 - 4\lambda^2 + 2].$$

Hence the number of $(E, 3, \lambda)$-colourings with two edges of one colour and two edges of another colour at v is

$$3\lambda(\lambda - 1)\left\{\left(1 - \frac{2}{\lambda}\right)A + \left[1 - \frac{2}{\lambda} + \frac{1}{\lambda(\lambda - 1)}\right]D\right\}.$$

Finally, there are $\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)E_3(K_4, \lambda)(E, 3, \lambda)$-colourings of K_5 with all of the edges at v coloured differently.

Combining the terms derived above, we obtain

$$\begin{aligned} E_3(K_5, \lambda) &= \lambda E_3(K_4, \lambda - 1) + 4\lambda(\lambda - 1)[E_3(K_3, \lambda - 1) \\ &\quad + (\lambda - 1)E_3(K_3, \lambda - 2) + 3(\lambda - 1)E_3(K_3, \lambda - 1) \\ &\quad + 3(\lambda - 1)(\lambda - 2)E_3(K_3, \lambda - 1) \\ &\quad + \lambda(\lambda - 1)(\lambda - 2)E_3(K_3, \lambda - 1)] + 3\lambda(\lambda - 1)\left\{\left(1 - \frac{2}{\lambda}\right)A \right. \\ &\quad \left. + \left[1 - \frac{2}{\lambda} + \frac{1}{\lambda(\lambda - 1)}\right]D\right\} + 6(\lambda - 1)(\lambda - 2)E_3(K_4, \lambda) \\ &\quad + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)E_3(K_4, \lambda) \\ &= \lambda^{10} - 10\lambda^8 + 45\lambda^6 - 15\lambda^5 - 100\lambda^4 + 105\lambda^3 - 20\lambda^2 - 6\lambda. \end{aligned}$$

For $E_k(G, \lambda)$ we have the following system of equations, analogous to the system mentioned above for $V_k(G, \lambda)$:

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$$E_k(G, 1) = b_{1,k}$$

$$E_k(G, 2) = b_{2,k} + 2b_{1,k}$$

$$E_k(G, 3) = b_{3,k} + 3b_{2,k} + 3b_{1,k}$$

$$\vdots$$

$$E_k(G, p) = \sum_{i=0}^{p-1} \binom{p}{i} b_{p-i,k}$$

$$\vdots$$

$$E_k(G, \lambda) = \sum_{i=0}^{p-1} \binom{\lambda}{p-i} b_{p-i,k},$$

where $p = |E(G)|$, and where for small λ it is understood that the factorial expressions in the denominators are evaluated as they were earlier.

Remark. Using the above system of equations, one can easily get the computer to calculate the $b_{i,k}$, $i = 1, 2, 3, \ldots, p$, provided that $E_k(G, \lambda)$ is known. For the given $E_3(K_4, \lambda)$ we obtain $b_{6,3} = 6!$, and for the given $E_3(K_5, \lambda)$ we obtain $b_{10,3} - 10!$. Since for K_4 and K_5 we find, by direct counts, the values of $b_{2,3}$ and $b_{3,3}$ which also agree, this is strong presumptive evidence for the correctness of our polynomials $E_3(K_4, \lambda)$ and $E_3(K_5, \lambda)$. Direct counting and the computer yield the following values:

K_4		K_5	
$b_{2,3} =$	8	$b_{2,3} =$	12
$b_{3,3} =$	396	$b_{3,3} =$	17,100
$b_{4,3} =$	1,464	$b_{4,3} =$	474,480
$b_{5,3} =$	1,800	$b_{5,3} =$	3,922,200
$b_{6,3} =$	6!	$b_{6,3} =$	14,552,640
		$b_{7,3} =$	28,224,000
		$b_{8,3} =$	29,836,800
		$b_{9,3} =$	16,329,600
		$b_{10,3} =$	10!

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