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On the Generalized Cyclic Eilenberg-Zilber Theorem

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Abstract. We use the homological perturbation lemma to give an algebraic proof of the cyclic Eilenberg-Zilber theorem for cylindrical modules.

1 Introduction

The original Eilenberg-Zilber theorem (see [7] for a recent account) states that if X and Y are simplicial abelian groups then the total complex of the bicomplex $X \otimes Y$ is chain homotopy equivalent to the diagonal complex of $X \otimes Y$. This result was then generalized by Dold and Puppe to bisimplicial abelian groups [4, 7]: if X is a bisimplicial abelian group then the total complex and the diagonal complex of X are chain homotopy equivalent. This extension is important because in many examples (*e.g.* the bisimplicial group associated to a group action through its translation category), X is not decomposable as the tensor product of two simplicial groups.

Thanks to the work of Connes [3], one knows that a general setup for defining and studying cyclic homology is through cyclic modules. In order to define products and coproducts in cyclic homology and to prove Künneth type formulas, several authors, including Kassel [10], Hood-Jones [8], and Loday [11], have proved Eilenberg-Zilber type theorems for tensor products of cyclic modules associated to algebras. A good reference that compares various methods used by these authors is A. Bauval's article [1]. In ([11], p. 130) one can find an Eilenberg-Zilber theorem for tensor products of two cyclic modules. The most general result in this direction is stated by Getzler and Jones in [6]. The proof is however topological in nature, and is based on the method of acyclic models. In view of the importance of this result for cyclic homology theory, for example in deriving a spectral sequence for the cyclic homology of the crossed product algebra for the action of a group, as in [6], or for the action of a Hopf algebra, we felt it desirable to give a purely algebraic proof of this fact.

In attempting to extend the proof in [11] to this more general setup, we realized that the cyclic shuffle map of [11] has no immediate extension to cylindrical (or even bicyclic) modules, while the shuffle and Alexander-Whitney maps have more or less obvious extensions (see the Remark in Section 3 for more on this). It seems plausible that one should use a different definition for cyclic shuffles. Instead, we use the homological perturbation lemma to obtain an algebraic proof for the cyclic Eilenberg-Zilber theorem for cylindrical modules. Our approach is motivated by A. Bauval's work [1], where a perturbation lemma is used to give an alternative proof

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of the cyclic Eilenberg-Zilber theorem of [11]. In using the perturbation lemma, one has to overcome two difficulties: first, showing that the first term in the perturbation formula is identical with the boundary operator B_t (Proposition 3.3), and secondly, proving that all higher order terms in the perturbation formula actually vanish (Theorem 3.1). We find it remarkable that these results continue to be true in our cylindrical module context. By making use of explicit formulas for the contracting homotopy in the generalized Eilenberg-Zilber theorem for bisimplicial modules, one can, in principle, find an explicit formula for a generalized cyclic shuffle map.

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2 Preliminaries

Let k be a commutative unital ring. Recall that [5] a Λ_∞-module is a simplicial k-module $M = (M_n)_{n\geq 0}$ endowed, for each $n \geq 0$, with automorphisms $\tau_n\colon M_n \to M_n$, such that the following relations hold

$$\begin{aligned}
\delta_i\tau_n &= \tau_{n-1}\delta_{i-1}, \quad 1 \leq i \leq n,\\
\delta_0\tau_n &= \delta_n,\\
\sigma_i\tau_n &= \tau_{n+1}\sigma_{i-1}, \quad 1 \leq i \leq n,\\
\sigma_0\tau_n &= \tau_{n+1}^2\sigma_n.
\end{aligned}$$

Here δ_i and σ_i are the faces and degeneracies of M. In case $\tau_n^{n+1} = 1$ for all $n \geq 0$, we say that M is a cyclic k-module. We denote the categories of Λ_∞ (resp. cyclic) k-modules by $k\Lambda_\infty$ (resp. $k\Lambda$).

For example, to each unital k-algebra A and an algebra automorphism $g \in \mathrm{Aut}(A)$ one can associate a Λ_∞-module $A_g^\natural$ with $A_{g,n}^\natural = A^{\otimes(n+1)}$, and with faces, degeneracies and τ_n defined by

$$\begin{aligned}
\delta_i(a_0\otimes\cdots\otimes a_n) &= a_0\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\cdots\otimes a_n, \quad 0\leq i\leq n-1,\\
\delta_n(a_0\otimes\cdots\otimes a_n) &= (ga_n)a_0\otimes a_1\otimes\cdots\otimes a_{n-1},\\
\sigma_i(a_0\otimes\cdots\otimes a_n) &= a_0\otimes\cdots\otimes a_i\otimes 1\cdots\otimes a_n, \quad 0\leq i\leq n,\\
\tau_n(a_0\otimes\cdots\otimes a_n) &= ga_n\otimes a_0\cdots\otimes a_{n-1}.
\end{aligned}$$

The cyclic homology groups of a cyclic k-module M can be defined, among other ways, via the bicomplex $\mathcal{B}(M)$ defined by

$$\begin{aligned}
\mathcal{B}_{m,n}(M) &= M_{n-m} \quad \text{if } n\geq m\geq 0,\\
\mathcal{B}_{m,n}(M) &= 0 \quad \text{otherwise.}
\end{aligned}$$

The vertical boundary operator is defined by

$$b = \sum_{i=0}^{m}(-1)^i\delta_i\colon M_m \to M_{m-1},$$

and the horizontal boundary operator is given by

$$B = N\sigma_{-1}(1-t)\colon M_m \to M_{m+1},$$

where

$$t = (-1)^m \tau_m, \quad \sigma_{-1} = \tau_m \sigma_m \colon M_m \to M_m, \quad \text{and} \quad N = 1 + t + \cdots + t^m.$$

One can check that $b^2 = B^2 = bB + Bb = 0$. The cyclic homology of M, denoted by $\mathrm{HC}_*(M)$, is defined to be the total homology of the first quadrant bicomplex $\mathcal{B}(M)$.

If M is only a Λ_∞-module, we can still define the operator B as above and $B^2 = 0$ [6], however $bB + Bb$ need not be zero. As in [6], let $T = 1 - bB - Bb$.

Recall that a mixed complex (C, b, B) is a chain complex (C, b) with a map of degree $+1$, $B\colon C_* \to C_{*+1}$, satisfying $b^2 = B^2 = bB + Bb = 0$ [11]. To any mixed complex C one associates a bicomplex BC in the first quadrant, defined by $\mathrm{BC}_{n,m} = C_{m-n}$ if $m \geq n \geq 0$ and 0 otherwise, with horizontal boundary B and vertical boundary b. By definition, the cyclic homology of C is $\mathrm{HC}_*(C) = H_*\big(\mathrm{Tot}(\mathrm{BC})\big)$, and its Hochschild homology is, $\mathrm{HH}_*(C) = H_*(C, b)$. As for cyclic homology of algebras, here also, we have a short exact sequence of complexes,

$$0 \to (C, b) \to \mathrm{Tot}(\mathrm{BC}) \xrightarrow{S} \mathrm{Tot}(\mathrm{BC})[2] \to 0,$$

where S is the quotient map obtained by factoring by the first column.

An S-morphism of mixed complexes $f\colon (C, b, B) \to (C', b', B')$ is a morphism of complexes $f\colon\ \mathrm{Tot}(\mathrm{BC}) \to \mathrm{Tot}(\mathrm{BC}')$, such that f commutes with S. One can write an S-morphism as a matrix of maps

$$f = \begin{pmatrix} f^0 & f^1 & f^2 & \cdots \\ f^{-1} & f^0 & f^1 & \cdots \\ f^{-2} & f^{-1} & f^0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $f^i\colon C_{*-2i} \to C'_*$, and $[B, f^i] + [b, f^{i+1}] = 0$.

The benefit of S-morphisms may be seen in many cases when we do not have a cyclic map between cyclic modules but we can have a S-morphism. Every S-morphism induces a map $f_*\colon\ \mathrm{HC}_*(C) \to \mathrm{HC}_*(C')$ rendering the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & \mathrm{Tot}(\mathrm{BC}) & \longrightarrow & \mathrm{Tot}(\mathrm{BC})[2] & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f^0} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f[2]} & & \\
0 & \longrightarrow & C' & \longrightarrow & \mathrm{Tot}(\mathrm{BC}') & \longrightarrow & \mathrm{Tot}(\mathrm{BC}')[2] & \longrightarrow & 0
\end{array}$$

We have the following proposition which follows easily from the five lemma:

Proposition 2.1 **([11])** *Let $f\colon (C, b, B) \to (C', b', B')$ be an S-morphism of mixed complexes. Then $f^0_*\colon\ \mathrm{HH}_*(C) \to \mathrm{HH}_*(C')$ is an isomorphism if and only if $f_*\colon\ \mathrm{HC}_*(C) \to \mathrm{HC}_*(C')$ is an isomorphism.*

Let Λ_∞(resp. Λ) be the ∞-cyclic (resp. cyclic) categories of Feigin-Tsygan [5] (resp. Connes [3]). We do not need their actual definitions for this paper.

Definition 2.1 (**[6]**) By a cylindrical k-module we mean a contravariant functor $X\colon \Lambda_\infty \times \Lambda_\infty \to k$-mod, such that for all p, q, $\tau^{q+1}t^{p+1} = \mathrm{id}\colon X_{p,q} \to X_{p,q}$. More explicitly, we have a bigraded sequence of k-modules $X_{n,m}$, $n, m \geq 0$, with horizontal and vertical face, degeneracy and cyclic operators

$$\begin{aligned}
d_i\colon X_{n,m} &\to X_{n-1,m}, \quad n \geq i \geq 0,\\
s_i\colon X_{n,m} &\to X_{n+1,m}, \quad n \geq i \geq 0,\\
t\colon X_{n,m} &\to X_{n,m},\\
\delta_i\colon X_{n,m} &\to X_{n,m-1}, \quad m \geq i \geq 0,\\
\sigma_i\colon X_{n,m} &\to X_{n,m+1}, \quad m \geq i \geq 0,\\
\tau\colon X_{n,m} &\to X_{n,m},
\end{aligned}$$

such that every vertical operator commutes with every horizontal operator and vertical and horizontal operators satisfy the usual Λ_∞-module relations. Moreover, for all p and q the crucial relation $\tau^{q+1}t^{p+1} = \mathrm{id}\colon X_{p,q} \to X_{p,q}$ holds. In this paper we denote the horizontal operators by d_i, s_i, t and the vertical operators by δ_i, σ_i, τ.

Example 2.1 Homotopy colimits of diagrams of simplicial sets are defined as the diagonal of a bisimplicial set ([7], p. 199). We show that if the indexing category is a cyclic groupoid and the functor has certain extra properties, we can turn this bisimplicial set into a cylindrical module. Let I be a groupoid, *i.e.*, a small category in which every morphism is an isomorphism. Recall from [2] that a cyclic structure ε on I is a choice of morphisms $\varepsilon_i \in \mathrm{Hom}(i,i)$ for all $i \in \mathrm{Obj}(I)$, such that for all $f\colon i \to j$, $f\varepsilon_i = \varepsilon_j f$. Let (I,ε) be a cyclic groupoid. We call a functor $Z\colon I \to k\Lambda_\infty$ a *cyclic functor* if $Z(\varepsilon_i)|Z(i)_n = t_n^{n+1}|Z(i)_n$. To each cyclic functor Z we associate a cylindrical k-module $\mathrm{BE}_I\, Z$, such that,

$$\mathrm{BE}_I\, Z_{m,n} = \bigoplus_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} i_m} Z(i_m)_n.$$

We define the following cylindrical structure on $\mathrm{BE}_I\, Z$:

$$\big(\delta_j(x)\big)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} i_m} = \begin{cases} (x)_{i_1 \xrightarrow{g_2} i_2 \xrightarrow{g_3} \cdots \xrightarrow{g_m} i_m} & \text{if } j = 0\\ (x)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{j-1}} i_{j-1} \xrightarrow{g_{j+1}\circ g_j} i_{j+1} \xrightarrow{g_{j+2}} \cdots \xrightarrow{g_m} i_m} & \text{if } 1 \leq j \leq m-1\\ \big(g_m^{-1}(x)\big)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{m-1}} i_{m-1}} & \text{if } j = m, \end{cases}$$

$$\big(\sigma_j(x)\big)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} i_m} = (x)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_j} i_j \xrightarrow{\mathrm{id}} i_j \xrightarrow{g_{j+1}} \cdots \xrightarrow{g_m} i_m},$$

$$\big(\tau(x)\big)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} i_m} = \big(g_m^{-1}(x)\big)_{i_m \xrightarrow{h} i_0 \xrightarrow{g_1} \cdots \xrightarrow{g_{m-1}} i_{m-1}},$$

where $h = (g_m \circ g_{m-1} \circ \cdots \circ g_1)^{-1}$. The horizontal structure is induced by the cyclic structure of $Z(i_m)$. One can check that $\mathrm{BE}_I Z$ is a cylindrical module.

We apply the above construction to the following situation. Let G be a (discrete) group acting by unital automorphisms on an unital k-algebra A. Let $I = G$ be the category with G as its set of objects and $\mathrm{Hom}(g_1, g_2) = \{h \in G \mid hg_1h^{-1} = g_2\}$. Define a cyclic structure ε on I by $\varepsilon_g = g$, for all $g \in G$. Obviously (I, ε) is a cyclic groupoid. Define a functor $Z\colon I \to k\Lambda_\infty$ by $Z(g) = A_g^\natural$ and $Z(h)\colon Z(g_1) \to Z(g_2)$ the map induced by h. It is a cyclic functor. It follows that $\mathrm{BE}_I Z$ is a cylindrical module. The cylindrical module $X = \mathrm{BE}_I Z$ can be identified as follows. We have

$$X_{m,n} = \bigoplus_{G^{m+1}} A^{\otimes(n+1)} \cong kG^{\otimes(m+1)} \otimes A^{\otimes(n+1)},$$

where kG is the group algebra of the group G over k. The isomorphism is defined by

$$\phi_{m,n}\colon \mathrm{BE}_I Z_{m,n} \to kG^{\otimes(m+1)} \otimes A^{\otimes(n+1)},$$
$$\phi_{m,n}\big((a_0, a_1, \dots, a_n)_{i_0 \xrightarrow{g_1} i_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} i_m}\big) = (i_m^{-1} g_m \cdots g_1, g_1^{-1}, \dots, g_m^{-1} | a_0, a_1, \dots, a_n).$$

Under this isomorphism the vertical and horizontal cyclic maps are given by:

$$\begin{aligned}
\tau(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_0, \dots, g_m | g^{-1} \cdot a_n, a_0, \dots, a_{n-1}), \\
\delta_i(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_0, \dots, g_m | a_0, \dots, a_i a_{i+1}, \dots, a_n), && 0 \le i < n \\
\delta_n(g_0, \dots, g_m | a_0, \dots, a_n) &= \big(g_0, \dots, g_m | (g^{-1} \cdot a_n) a_0, \dots, a_{n-1}\big), \\
\sigma_i(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_0, \dots, g_m | a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n), && 0 \le i \le n, \\
t(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_m, g_0, \dots, g_{m-1} | g_m \cdot a_0, \dots, g_m \cdot a_n), \\
d_i(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_0, \dots, g_i g_{i+1}, \dots, g_m | a_0, \dots, a_n), && 0 \le i < m, \\
d_m(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_m g_0, g_1, \dots, g_{m-1} | g_m \cdot a_0, \dots, g_m \cdot a_n), \\
s_i(g_0, \dots, g_m | a_0, \dots, a_n) &= (g_0, \dots, g_i, 1, g_{i+1}, \dots, g_m | a_0, \dots, a_n), && 0 \le i \le m,
\end{aligned}$$

2 where $g = g_0 g_1 \cdots g_m$.

This shows that, in this particular case, our cylindrical module $\mathrm{BE}_I Z$ reduces to the cylindrical module associated in [6] to the action of a group on an algebra.

3 Proof of the Main Theorem

Let X be a cylindrical k-module, and let $\mathrm{Tot}(X)$ denote its total complex, defined by $\mathrm{Tot}(X)_n = \bigoplus_{p+q=n} X_{p,q}$. Consider the horizontal, and vertical b-differentials $b^h = \sum_{j=0}^n (-1)^j d_j$ and $b^v = \sum_{i=0}^m (-1)^i \delta_i$, and let $b = b^h + b^v$. Similarly we define B^h, B^v, and $B = T^v B^h + B^v$, where $T^v = (1 - b^v B^v - B^v b^v)$. The following lemma is proved in [6].

Lemma 3.1 *$\big(\mathrm{Tot}(X), b, B\big)$ is a mixed complex.*

We can also define the diagonal $d(X)$ of a cylindrical module X. It is a cyclic module with $d(X)_n = X_{n,n}$, and the cyclic structure: $d_i\delta_i$ as the i-th face, $s_i\sigma_i$ as the i-th degeneracy and $t\tau$ as the cyclic map. Note that the cylindrical condition $\tau^{q+1}t^{p+1} = \mathrm{id}\colon X_{p,q} \to X_{p,q}$ is needed in order to show that $(\tau t)^{n+1} = \mathrm{id}\colon X_{n,n} \to X_{n,n}$. Associated to this cyclic module we have a mixed complex $\big(d(X), b_d, B_d\big)$.

The following definition is from [4, 13]. It extends the standard shuffle map, originally defined on the tensor product of simplicial modules, to bisimplicial modules.

Definition 3.1 Let X be a bisimplicial module. Define $\nabla_{n,m}\colon X_{n,m} \to X_{n+m,n+m}$ by

$$\nabla_{n,m} = \sum_{\eta\in \mathrm{Sh}_{m,n}} (-1)^{\eta} s_{\bar\eta(n+m)} \cdots s_{\bar\eta(m+1)}\sigma_{\bar\eta(m)}\cdots\sigma_{\bar\eta(1)},$$

where $\mathrm{Sh}_{m,n} \subset S_{n+m}$, is the set of shuffles in the symmetric group of order $n+m$, defined by $\eta \in \mathrm{Sh}_{m,n}$ if and only if $\eta(1) < \eta(2) < \cdots < \eta(m)$, and $\eta(m+1) < \cdots < \eta(m+n)$. Here $\bar\eta(j) = \eta(j) - 1$, $1 \le j \le n+m$. We define the shuffle map

$$\mathrm{Sh}\colon \bigoplus_{p+q=n} X_{p,q} \to X_{n,n},$$

by

$$\mathrm{Sh} = \sum_{p+q=n} \nabla_{p,q}.$$

Proposition 3.1 *The shuffle map $\mathrm{Sh}\colon\ \mathrm{Tot}(X) \to d(X)$ is a map of b-complexes of degree 0.*

Proof We must show that $b_d \circ \mathrm{Sh} = \mathrm{Sh} \circ b^h + \mathrm{Sh} \circ b^v$. All elements in the left hand side are of form

$$d_i\delta_i s_{\bar\mu(m+n)}\cdots s_{\bar\mu(m+1)}\sigma_{\bar\mu(m)}\cdots\sigma_{\bar\mu(1)}.$$

It would be better to divide these elements into five parts:

1. $i = 0$, or $i = m+n$.
2. $1 \le i \le m+n$, and $i \in \{\mu(1),\dots,\mu(m)\}$, $i+1 \in \{\mu(m+1),\dots,\mu(m+n)\}$.
3. $1 \le i \le m+n$, and $i+1 \in \{\mu(1),\dots,\mu(m)\}$, $i \in \{\mu(m+1),\dots,\mu(m+n)\}$.
4. $1 \le i \le m+n$, and $i, i+1 \in \{\mu(1),\dots,\mu(m)\}$.
5. $1 \le i \le m+n$, and $i, i+1 \in \{\mu(m+1),\dots,\mu(m+n)\}$.

For part 1, let $i = 0$ (we leave to the reader the rest of this case). We have

$$d_0\delta_0 s_{\bar\mu(m+n)}\cdots s_{\bar\mu(m+1)}\sigma_{\bar\mu(m)}\cdots\sigma_{\bar\mu(1)} = s_{\bar{\bar\mu}(m+n)}\cdots s_{\bar{\bar\mu}(m+1)}\sigma_{\bar{\bar\mu}(m)}\cdots\sigma_{\bar{\bar\mu}(2)}d_0.$$

It is obvious that if we define $\rho(i) = \mu(i+1) - 1$, then ρ is also a shuffle and the result is in $\mathrm{Sh} \circ b^v$. For case 2, let $\mu(k) = i$, and $\mu(j) = i+1$, where $m+1 \le i \le m+n$, and

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$1 \le j \le m$. Now let $\alpha = \mu \circ (i, i+1)$. Then it is easy to check that α is also a shuffle and we have $i+1 \in \{\alpha(1), \dots, \alpha(m)\}$ and $i \in \{\alpha(m+1), \dots, \alpha(m+n)\}$. On the other hand we have

$$d_i\delta_i s_{\bar\alpha(m+n)}\cdots s_{\bar\alpha(m+1)}\sigma_{\bar\alpha(m)}\cdots\sigma_{\bar\alpha(1)} = d_i\delta_i s_{\bar\mu(m+n)}\cdots s_{\bar\mu(m+1)}\sigma_{\bar\mu(m)}\cdots\sigma_{\bar\mu(1)},$$

and sign $\mu = -$ sign α. So elements of case 2 cancel the elements of case 3. Now let us do case 4. We assume $\mu(s) = i$, $\mu(s+1) = i+1$, where $1 \le s \le m$. We have

$$\begin{aligned} &d_i\delta_i s_{\bar\mu(m+n)}\cdots s_{\bar\mu(s+2)} s_i s_{i-1} s_{\bar\mu(s-1)}\cdots s_{\bar\mu(m+1)}\sigma_{\bar\mu(m)}\cdots\sigma_{\bar\mu(1)} \\ &\qquad = s_{\bar{\bar\mu}(m+n)}\cdots s_{\bar{\bar\mu}(s+2)} s_{i-1} s_{\bar\mu(s-1)}\cdots s_{\bar\mu(m+1)}\sigma_{\bar\theta(m)}\cdots\sigma_{\bar\theta(1)}, \end{aligned}$$

where

$$\theta(j) = \begin{cases} \mu(j) & \text{if } \mu(j) < i \\ \bar\mu(j) & \text{if } \mu(j) > i+1. \end{cases}$$

It is easy to show that the permutation

$$\begin{pmatrix} 1 & 2\cdots m & m+1\cdots s-1 & s & s+1\cdots m+n-1 \\ \theta(1) & \theta(2)\cdots\theta(m) & \mu(m+1)\cdots\mu(s-1) & i-1 & \bar\mu(m+1)\cdots\bar\mu(m+n) \end{pmatrix}$$

is a $(m-1, n)$-shuffle. Similarly one can do case 5 and then by counting the proof is finished. ■

In a similar way to the shuffle map, the Alexander-Whitney map also extends to bisimplicial modules [13]. Define $A_{p,q}\colon X_{n,n} \to X_{p,q}$, where $p+q = n$, by

$$A_{p,q} = (-1)^{p+q}\delta_{p+1}\cdots\delta_n \underbrace{d_0\cdots d_0}_{p \text{ times}},$$

and let

$$A = \sum_{p+q=n} A_{p,q}\colon d(X)_n \to \mathrm{Tot}(X)_n.$$

Both maps Sh and A induce maps on the normalized complexes, denoted by $\bar A\colon \bar d(X) \to \overline{\mathrm{Tot}}(X)$ and $\overline{\mathrm{Sh}}\colon \overline{\mathrm{Tot}}(X) \to \bar d(X)$.

Remark If $X = M \otimes N$ is the tensor product of two cyclic modules, one can define the cyclic shuffles [11]

$$\mathrm{Sh}'\colon M_p \otimes N_q \to M_{p+q+2} \otimes N_{p+q+2}.$$

For example, if $M = A^\natural$ and $N = B^\natural$ are cyclic modules of associative unital algebras, we have $\mathrm{Sh}'_{p,q}(x,y) = \sigma_{-1}(x) \perp \sigma_{-1}(y)$, where

$$\perp\colon A^\natural_p \otimes B^\natural_q \to (A \otimes B)^\natural_{p+q},$$

is defined by

$$(a_0, a_1, \dots, a_p) \perp (b_0, b_1, \dots, b_q) = \sum_{\mu} \mu^{-1} \cdot (a_0 \otimes b_0, a_1 \otimes 1, \dots a_p \otimes 1, 1 \otimes b_1, \dots 1 \otimes b_q).$$

The summation is over all cyclic (p, q)-shuffles in S_{p+q}. The point is that one has to use μ^{-1}, as opposed to μ as appears in [11] (p. 127), in the above formula. This was also noticed by Bauval [1] and Loday (private communication). When A and B are commutative algebras, the definition of cyclic shuffle, and the fact that $(\mathrm{Sh}, \mathrm{Sh}')$: $\overline{\mathrm{Tot}}(X) \to \bar{d}(X)$ defines an S-map which is a quasi-isomorphism, are essentially due to G. Rinehart (*cf.* [11]).

Now, one approach to prove the cyclic Eilenberg-Zilber theorem for cylindrical modules would be to extend this cyclic shuffle map to the case where X is not decomposable to a tensor product. However, no natural extension exists, exactly because one has to use μ^{-1} in the above formula for the cyclic shuffles. For example, with $p = 2$ and $q = 0$, $\mathrm{Sh}'_{2,0}(a_0, a_1, a_2|b_0)$ contains a term of the form $(1, a_0, a_2, 1, a_1|1, 1, 1, b_0, 1)$, corresponding to the cyclic shuffle $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{smallmatrix}\right)$. It is clear that a term like (a_0, a_2, a_1) can not be produced from (a_0, a_1, a_2) via the cyclic or simplicial maps.

Proposition 3.2 *$\overline{\mathrm{Sh}}$ and $\bar{A}$ define a deformation retraction of $\bar{d}(X)$ to $\overline{\mathrm{Tot}}(X)$, i.e. there is a homotopy $h\colon \bar{d}(X) \to \bar{d}(X)$ such that $\bar{A} \circ \overline{\mathrm{Sh}} = 1$ and $\overline{\mathrm{Sh}} \circ \bar{A} = 1 + bh + hb$.*

Proof The existence of h is part of the generalized Eilenberg-Zilber theorem [7]. We just prove that $\bar{A} \circ \overline{\mathrm{Sh}} = 1$. Let us calculate the action of $\bar{A} \circ \overline{\mathrm{Sh}}$ on a typical element $x \in X_{p,q}$. For every $\overline{A_{p',q'}}$ with $(p', q') \neq (p, q)$, we have $\overline{A_{p',q'}} \circ \overline{\mathrm{Sh}}(x) = 0$, so we should only check the identity $\overline{A_{p,q}} \circ \overline{\mathrm{Sh}}(x) = x$. For $\mu \in \mathrm{Sh}_{p,q}$ for simplicity let $\mu \cdot x = (-1)^{\mu} s_{\bar{\mu}(n)} \cdots s_{\bar{\mu}(p+1)} \sigma_{\bar{\mu}(p)} \cdots \sigma_{\bar{\mu}(1)}(x)$, and $\mu_{p,q} = (q + 1, \dots, n, 1, \dots, q)$. Then it is not difficult to verify that $\overline{A_{p,q}}(\mu.x) = 0$ for every $\mu \neq \mu_{p,q}$, and $\overline{A_{p,q}}(\mu_{p,q}.x) = x$. ■

The first applications of perturbation theory to cyclic homology go back to the work of C. Kassel in [9]. Our proof of the generalized Eilenberg-Zilber theorem, Theorem 3.1 below, is also based on homological perturbation theory. We recall the necessary definitions and results from [1, 9]. A chain complex (L, b) is called a *deformation retract* of a chain complex (M, b) if there are chain maps

$$L \xrightarrow{g} M \xrightarrow{f} L$$

and a chain homotopy $h\colon M \to M$ such that

$$fg = \mathrm{id}_L \quad \text{and} \quad gf = \mathrm{id}_M + bh + hb.$$

The retraction is called *special* if in addition

$$hg = fh = h^2 = 0.$$

It is easy to see that any retraction data as above can be replaced by a special retraction [9]. Now we perturb the differential of the "bigger" complex M to $b + B$ so that $(b+B)^2 = 0$. It is natural to ask whether the differential of L can be perturbed to a new differential $b+B_\infty$ so that $(L, b+B_\infty)$ is a deformation retraction of $(M, b+B)$. The homological perturbation lemma asserts that, under suitable conditions, this is possible. More precisely, assume the retraction is special, L and M have bounded below increasing filtrations, g and f preserve the filtration and h decreases the filtration. Then it is easy to check that the following formulas are well defined and define a special deformation retract of $(M, b + B)$ to $(L, b + B_\infty)$:

$$\begin{aligned} h_\infty &= h \sum_{m \geq 0} (Bh)^m, \\ g_\infty &= (1 + h_\infty B) g, \\ f_\infty &= f(1 + Bh_\infty), \\ B_\infty &= f(1 + Bh_\infty) Bg. \end{aligned}$$

To apply the perturbation lemma to our problem, we need to know that the perturbed differential $b+B_\infty$ coincides with the existing differential. This means we have to show that the first term in the series for B_∞ coincides with B_t and all other terms vanish. The next proposition verifies the first part of the claim. It is a generalization of Lemma IV.1 in [1].

Proposition 3.3 *Let X be a cylindrical module and let $B_t = T^v B^h + B^v$ and $B_d = B^h B^v$ be the total and diagonal B-differentials on $\overline{\mathrm{Tot}}(X)$ and $\bar{d}(X)$, respectively. Then $\bar{A} B_d \overline{\mathrm{Sh}} = B_t$.*

Proof Let $x \in X_{p,q}$. As in the proof of Proposition 3.2, we have

$$\overline{\mathrm{Sh}}(x) = \sum_{\mu \in \mathrm{Sh}_{p,q}} \mu \cdot x.$$

Let $n = p + q$. In

$$\bar{A} B_d \overline{\mathrm{Sh}}(x) = \sum_{r+s=n+1} \bar{A}_{r,s} B_d \overline{\mathrm{Sh}}(x),$$

because we are working with normalized chains, all parts are zero except for $r = p+1$, $s = q$ or $r = p$, $s = q + 1$. We denote the first part by S_1 and the second part by S_2. We show that $S_1 = T^v B^h(x)$ and $S_2 = B^v(x)$.

For $0 \leq i \leq n$ and $\mu \in \mathrm{Sh}_{p,q}$, let

$$S_1(i, \mu) = \bar{A}_{p+1,q} \tau_{n+1} \sigma_n t_{n+1} s_n \tau_n^i t_n^i (\mu . x).$$

The reader can easily check that $S_1(i,\mu) = 0$ for all $0 \le i \le q-1$ and all $\mu \in \mathrm{Sh}_{p,q}$. For the rest of the elements in S_1, we have $S_1(i,\mu) = 0$ for all $q \le i \le n$ and all $\mu \ne \mu_{p,q,i}$, where

$$\mu_{p,q,i} = (1,2,\dots,n-i,n+q-i+1,\dots,n,n-i+2,\dots,n+q-i).$$

Now, we have $S_1(i,\mu_{p,q,i}) = (-1)^{(i-q)p} t_{p+1} s_p t_p^{i-q} \tau_q^{q+1}$. We have shown that $S_1 = T^v B^h(x)$.

Next, we compute S_2. For $0 \le i \le n$ and $\mu \in \mathrm{Sh}_{p,q}$ let

$$S_2(i,\mu) = \bar{A}_{p,q+1} \tau_{n+1} \sigma_n t_{n+1} s_n \tau_n^i t_n^i(\mu.x).$$

For $q+1 \le i \le p+q$ and all $\mu \in \mathrm{Sh}_{p,q}$ we have $S_2(i,\mu) = 0$, and if we denote

$$\alpha_{i,p,q} = (q-i+1,\dots,n-i,1,2,\dots,q-i,n-i+1,\dots,n),$$

then $S_2(i,\mu) = 0$ for all $0 \le i \le q$ and all $\mu \ne \alpha_{p,q,i}$. Finally, we have $S_2(i,\alpha_{p,q,i}) = (-1)^{iq} \tau_{q+1} \sigma_q \tau_q^i$ for $0 \le i \le q$. We have shown that $S_2 = B^v(x)$. The proposition is proved. ∎

Now we are in a position to combine the perturbation lemma with the above proposition to prove:

Theorem 3.1 *Let X be a cylindrical module. Then there exists an S-map of mixed complexes, $f\colon \mathrm{Tot}(X) \to d(X)$, such that $f_0 = \mathrm{Sh}$ is the shuffle map and f is a quasi-isomorphism.*

Proof It suffices to prove the statement for the normalized complexes. By Proposition 3.2, $\big(\overline{\mathrm{Tot}}(X), b\big)$ is a deformation retract of $\big(\bar{d}(X), b\big)$. So, applying the perturbation lemma, we have to show that all the extra terms in the perturbation series vanish. Now the normalized homotopy operator is induced from the original homotopy operator $h\colon X_{n,n} \to X_{n+1,n+1}$. Dold and Puppe show that the operator h is universal, in the sense that it is a linear combination (with integral coefficients) of simplicial morphisms of X (p. 213, Satz 2.9 in [4]). One knows that any order preserving map $\Phi\colon [n] \to [m]$ between finite ordinals can be uniquely decomposed as,

$$\Phi = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_s},$$

such that $i_1 \le i_2 \le \cdots \le i_r$ and $j_1 < j_2 < \cdots < j_s$ and δ_{i_k} are cofaces and σ_{j_k} are codegeneracies (*cf. e.g.* Loday [11], p. 401). On dualizing this and combining it with the Dold-Puppe result, it follows that the homotopy operator $h\colon X_{n,n} \to X_{n+1,n+1}$ is a linear combination of operators of the form $\sigma_i^v \sigma_i^h \circ g$, where σ_i^v, σ_i^h are vertical and horizontal degeneracy operators and g is another operator whose specific form is not important for the sake of this argument. Now we look at the perturbation formula. We are claiming that the induced operator on the normalized chains

$$\bar{A} \circ B_d \circ h_\infty \circ B_d \circ \overline{\mathrm{Sh}}\colon \overline{\mathrm{Tot}}(X) \to \overline{\mathrm{Tot}}(X),$$

where $h_\infty = h\sum_{m\ge 0}(B_d h)^m$ is zero. This follows from the above observation since

$$\mathrm{Image}(B_d \circ h_\infty \circ B_d \circ \mathrm{Sh}) \subset \mathrm{Image}(B_d \circ h) \subset \text{degenerate chains}.$$ ∎

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