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RESEARCH ARTICLE

Galois points and Cremona transformations

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Abstract

In this article, we study Galois points of plane curves and the extension of the corresponding Galois group to $\mathrm{Bir}(\mathbb{P}^2)$. We prove that if the Galois group has order at most 3, it always extends to a subgroup of the Jonquières group associated with the point P. Conversely, with a degree of at least 4, we prove that it is false. We provide an example of a Galois extension whose Galois group is extendable to Cremona transformations but not to a group of de Jonquières maps with respect to P. In addition, we also give an example of a Galois extension whose Galois group cannot be extended to Cremona transformations.

1. Introduction

Let k be an algebraically closed field. Let C be an irreducible plane curve in $\mathbb{P}^2$. Giving a point $P \in \mathbb{P}^2$, we consider the projection $\pi_P|_C : C \dashrightarrow \mathbb{P}^1$, which is the restriction of the projection $\pi_P : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with centre P. Let $K_P = \pi_P^*(k(\mathbb{P}^1))$. If $P \in C$ [resp. $P \notin C$], we say that P is an inner [resp. outer] Galois point for C if $k(C)/K_P$ is Galois. In this case, we write $G_P = \mathrm{Gal}(k(C)/K_P)$ and call it the Galois group at P. A de Jonquières map is a birational map φ for which there exist $P \in \mathbb{P}^2$ such that φ preserves the pencil of lines passing through P. The group of a de Jonquières transformations preserving the pencil of lines passing through a given point $P \in \mathbb{P}^2$ is denoted by $\mathrm{Jonq}_P \subset \mathrm{Bir}(\mathbb{P}^2)$, this corresponding to ask that ϕ preserves a pencil of lines through the point P. As in ref. [4], we are interested in the extension of elements of G_P to $\mathrm{Bir}(\mathbb{P}^2)$. There are two interesting questions:

Question 1.1. *If P is Galois, does G_P extends to* $\mathrm{Bir}(\mathbb{P}^2)$*?*

Question 1.2. [8] *If an element extends to* $\mathrm{Bir}(\mathbb{P}^2)$*, does it extend to a de Jonquières map? i.e. to an element $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ with $\pi_P \circ \varphi = \pi_P$?*

Consider a point $P \in \mathbb{P}^2$ with multiplicity m_P on an irreducible plane curve C in $\mathbb{P}^2$ of degree d, we will show later that the extension $[k(C) : K_P]$ has degree $d - m_P$. Our first main result is the following theorem, that considers the case of degree 3.

Theorem A. *Let $P \in \mathbb{P}^2$, let $C \subset \mathbb{P}^2$ be an irreducible curve. If the extension $k(C)/K_P$ is Galois of degree at most 3, then G_P always extends to a subgroup of* $\mathrm{Jonq}_P \subseteq \mathrm{Bir}(\mathbb{P}^2)$.

Theorem A resulted from Theorem 3.2, which provides more information on the Galois extensions of degree at most 3 and the related Galois Groups at a point P. This encourages us to study higher-degree Galois extensions and determine if their Galois groups G_P can always be extended to $\mathrm{Bir}(\mathbb{P}^2)$ as well as to the group of de Jonquières map with respect to P. The following theorem gives a negative answer to Question 1.2.

Theorem B. *Let k be a field of characteristic $char(k) \neq 2$ containing a primitive fourth root of unity, and let C be the irreducible curve defined by the equation $X^4 - 4ZYX^2 - ZY^3 + 2Z^2Y^2 - YZ^3 = 0$, then the point $P = [1:0:0]$ is an outer Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 4. The group G_P extends to $\mathrm{Bir}(\mathbb{P}^2)$ but not to Jonq_P.*

The following result gives a negative answer to Question 1.1 (see also [8, Example 5]); it follows from Lemma 5.2.

Theorem C. *Let k be a field with $char(k) \neq 5$ that contains a primitive 5th root of unity, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u:v] \mapsto [uv^6 - u^7 : u^5(u^2 + v^2) : v^5(u^2 + v^2)]$. We define $C := \overline{\phi(\mathbb{P}^1)}$ which is an irreducible curve of $\mathbb{P}^2$, then the point $P = [1:0:0]$ is an inner Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 5. Moreover, the identity is the only element of the Galois group that extends to $\mathrm{Bir}(\mathbb{P}^2)$.*

Remark 1.3. *After this article was uploaded to the ArXiv, [5] was uploaded. Theorem 1 in [5] corresponds to the case $k = \mathbb{C}$ of Theorem A.*

2. Preliminaries

The concept of Galois points for irreducible plane curve $C \subset \mathbb{P}^2$ was introduced by [6], [8], [1]. In order to study the extension of an element in G_P to $\mathrm{Bir}(\mathbb{P}^2)$, we need the following lemma.

Lemma 2.1. *The field extension $k(\mathbb{P}^1) \hookrightarrow k(C)$ induced by π_P has degree $d - m_P$, where m_P is the multiplicity of C at P, and d is the degree of C.*

Proof. Let $F(X, Y, Z) = 0$ be the defining equation of C of degree d. We may fix $P = [1:0:0] \in \mathbb{P}^2$ and choose that C is not the line $Z = 0$. Since P has multiplicity m_P, then the equation of C is $F(X, Y, Z) = F_{m_p}(Y, Z)X^{d-m_P} + \ldots\ldots + F_{d-1}(Y, Z)X + F_d(Y, Z)$, where $F_i(Y, Z)$ is a homogeneous polynomial of Y and Z of degree i ($m_P \le i \le d$) and $F_{m_P}(Y, Z) \neq 0$. Since $F(X, Y, Z)$ is irreducible in $k[X, Y, Z]$ and not a multiple of Z, then $f = F(X, Y, 1) \in k[X, Y]$ is also irreducible in $k[X, Y]$. We can see f as an irreducible polynomial in $\tilde{k}[X]$ with $deg_X(f) = d - m_P$ where $\tilde{k} = k(Y)$. Hence, the extension $k(C)/K_P$ is isomorphic to $(k(Y)[X]/(f))/(k(Y))$, and thus, it has the same degree equal to the degree of the irreducible polynomial $f \in \tilde{k}[X]$, so $[k(C) : \tilde{k}] = deg_X(f) = d - m_P$. □

It is well known [2, Ch. 1, Theorem 4.4] that for any two varieties X and Y, there is a bijection between the set of dominant rational maps $\varphi : X \dashrightarrow Y$, and the set of field homomorphisms $\varphi^\star : k(Y) \to k(X)$. In particular, we obtain:

Lemma 2.2. *For each variety X, we have a group isomorphism $\mathrm{Bir}(X) \xrightarrow{\simeq} \mathrm{Aut}_k(k(X))$ which sends φ to φ^*.*

Lemma 2.3. *For any field k, we have $\mathrm{Aut}_k(k(x)) = \{x \mapsto (ax + b)/(cx + d); a, b, c, d \in k,\ ad - bc \neq 0\}$ and $\mathrm{Aut}_k(k(x)) \cong \mathrm{Bir}(\mathbb{A}^1) \cong \mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1)$.*

Definition 2.4. *Let $P \in \mathbb{P}^2$, we write $\mathrm{Jonq}_P = \{\varphi \in \mathrm{Bir}(\mathbb{P}^2) | \exists\, \alpha \in \mathrm{Aut}(\mathbb{P}^1); \pi_P \circ \varphi = \alpha \circ \pi_P\}$ and call it the Jonquières group of P.*

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Lemma 2.5. *Let $P=[1:0:0]$, by taking an affine chart, a de Jonquières map with respect to P is a special case of a Cremona transformation, of the form*

$$\iota^{-1}\circ \mathrm{Jonq}_P\circ\iota=\{(x,y)\mapsto(\frac{ax+b}{cx+d},\frac{r_1(x)y+r_2(x)}{r_3(x)y+r_4(x)})\}$$

and $\iota:\mathbb{A}^2\hookrightarrow\mathbb{P}^2$, $(x,y)\longmapsto[x:y:1]$, where $a,b,c,d\in k$ with $ad-bc\neq 0$ and $r_1(x),r_4(x),r_2(x),r_3(x)\in k(x)$ with $r_1(x)r_4(x)-r_2(x)r_3(x)\neq 0$.

Proof. We have the following commutative diagram

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, hook, "\iota"] \arrow[d, dashed, "\pi_x"'] & \mathbb{P}^2 \arrow[d, dashed, "\pi_P"] \\ \mathbb{A}^1 \arrow[r, hook, "\psi"] & \mathbb{P}^1, \end{tikzcd}$$

where $\iota:(x,y)\mapsto[x:y:1]$ and $\psi:x\mapsto[x:1]$, which gives the equality

$$\iota^{-1}\circ\mathrm{Jonq}_P\circ\iota=\{f\in\mathrm{Bir}(\mathbb{A}^2)|\exists\alpha\in\mathrm{Bir}(\mathbb{A}^1);\alpha\circ\pi_x=\pi_x\circ f\}.$$

Let $f\in\iota^{-1}\circ\mathrm{Jonq}_P\circ\iota$ given by $(x,y)\mapsto(f_1(x,y),f_2(x,y))$, then $\pi_x\circ f:(x,y)\mapsto f_1(x,y)$. Since $\alpha\circ\pi_x=\alpha(x)$, it follows that $f_1(x,y)$ depends only on x and is of the form $f_1(x,y)=(ax+b)/(cx+d)$ where $a,b,c,d\in k$ and $ad-bc\neq 0$ by Lemma 2.3. From Lemma 2.2, f^* is subjective, so $k(x,y)=k((ax+b)/(cx+d),f_2(x,y))$. To describe the second component $f_2(x,y)$, let us define the birational map $\tau:(x,y)\mapsto((dx-b)/(-cx+a),y)$, hence $\tau\circ f:(x,y)\mapsto(x,f_2(x,y))$ is a birational map since both f and τ are birationals. By Lemma 2.2, $k(x)(f_2(x,y))=k(x)(y)$. Apply Lemma 2.3 over the field $k(x)$, then $f_2(x,y)=(r_1(x)y+r_2(x))/(r_3(x)y+r_4(x))$. □

Lemma 2.6. *Let $P,Q\in\mathbb{P}^2$ and $C,D\subset\mathbb{P}^2$ be two irreducible curves, if $\phi\in\mathrm{Bir}(\mathbb{P}^2)$ and $\phi|_C:C\dashrightarrow D$ is birational map, and there exists $\theta\in\mathrm{Aut}(\mathbb{P}^1)$ such that $\pi_Q\circ\phi=\theta\circ\pi_P$, then P is a Galois point of C if and only if Q is a Galois point of D. Moreover, if P is Galois, an element of G_P extends an element of $\mathrm{Bir}(\mathbb{P}^2)$ (respectively Jonq_P) if and only if its image in G_Q extends an element of $\mathrm{Bir}(\mathbb{P}^2)$ (respectively Jonq_Q)*

$$\begin{tikzcd} C \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\pi_P"'] & D \arrow[d, dashed, "\pi_Q"] \\ \mathbb{P}^1 \arrow[r, "\theta"] & \mathbb{P}^1. \end{tikzcd}$$

Proof. Since $\phi|_C$ is birational map from C to D, then $\phi^*|_C:k(D)\to k(C)$ is an isomorphism. Moreover, as $\pi_Q\circ\phi=\theta\circ\pi_P$, we have a commutative diagram

$$\begin{tikzcd} k(D) \arrow[r, "\phi^*|_C"] & k(C) \\ k(\mathbb{P}^1) \arrow[u, hook, "\pi_Q^*"] \arrow[r, "\theta^*"] & k(\mathbb{P}^1). \arrow[u, hook, "\pi_P^*"'] \end{tikzcd}$$

Therefore, $k(D)/\pi_Q^*(k(\mathbb{P}^1))$ is Galois if and only if $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is Galois. In addition, ϕ conjugates Jonq_P to Jonq_Q and sends any element of $\mathrm{Bir}(\mathbb{P}^2)$ that preserves C onto element of $\mathrm{Bir}(\mathbb{P}^2)$ that preserves D. □

Example 2.7. *Let $P=Q\in\{[1:0:0],[0:1:0],[0:0:1]\}$, $\phi:[X:Y:Z]\mapsto[YZ:XZ:XY]$ and $\theta:[Y:Z]\mapsto[Z:Y]$, let $C\subset\mathbb{P}^2$ be an irreducible curve not equal to $x=0$, $y=0$ or $z=0$ and let $D=\phi(C)$, so we have the following diagram*

$$\begin{tikzcd} C \arrow[r, dashed, "\phi"] \arrow[d, dashed, "\pi_P"'] & D \arrow[d, dashed, "\pi_P"] \\ \mathbb{P}^1 \arrow[r, "\theta"] & \mathbb{P}^1. \end{tikzcd}$$

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Thus, if P is a Galois point for C, then P becomes a Galois point for D, this is a particular case of Lemma 2.6 corresponding to [3, *Corollary* 3].

3. Extensions of degree at most three

Lemma 3.1. *Let k be a field and let $L = k[x]/(x^3 + a_2x^2 + a_1x + a_0)$ where $f = x^3 + a_2x^2 + a_1x + a_0$ is a separable irreducible polynomial in $k[x]$, then the field extension L/K is Galois if and only if there exists an element $\sigma \in \mathrm{Gal}(L/K)$ of order 3 such that,*

$$\sigma : x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta} \text{ where } \alpha, \beta, \gamma, \delta \in k \text{ with } \alpha\delta - \beta\gamma \neq 0.$$

Proof. As f is a separable irreducible polynomial of degree 3, the extension L/K is separable of degree 3. It is then Galois if and only if there exists $\sigma \in \mathrm{Aut}(L/K)$ of order 3, so it remains to prove that we can choose σ with the right form. If $\sigma \in \mathrm{Aut}(L/K)$ where, $\sigma : x \mapsto \nu_2x^2 + \nu_1x + \nu_0$ and $\nu_i \in k$ for $i = 0, 1, 2$, so the question here is can we find $\{\alpha, \beta, \gamma, \delta\} \subset K$ with $\alpha\delta - \beta\gamma \neq 0$ such that the following equality holds?

$$\nu_2x^2 + \nu_1x + \nu_0 = \frac{\alpha x + \beta}{\gamma x + \delta}. \tag{1}$$

We can find a solution $\{\alpha = a_2\nu_1\nu_2 - a_1\nu_2^2 + \nu_0\nu_2 - \nu_1^2,\ \beta = a\nu_0\nu_2 - a_0\nu_2^2 - \nu_0\nu_2,\ \delta = a_2\nu_2 - \nu_1,\ \gamma = \nu_2\}$. We observe that $\alpha\delta - \beta\gamma \neq 0$, otherwise we have $\sigma(x) \in K$ and this gives a contradiction as $x \notin K$. □

Theorem 3.2. *Let $P \in \mathbb{P}^2$, let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d: with multiplicity m_P at P. We have $[k(C) : K_P] = d - m_P$.*

1. *If $d - m_P = 1$, then $\pi_P : C \dashrightarrow \mathbb{P}^1$ is a birational.*
2. *If $d - m_P = 2$, P is Galois if and only if the extension is separable, and if this holds, then the non-trivial element $\sigma \in G_P$ of order 2 extends to a de Jonquières map with respect to P.*
3. *If $d - m_P = 3$ and P is Galois, then there is a de Jonquières map with respect to P extending the action.*

Proof. The equality $[k(C) : K_P] = d - m_P$ follows from Lemma 2.1. We may assume $P = [1 : 0 : 0]$. Let $x = X/Z$ and $y = Y/Z$ be affine coordinates. Since the field extension $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is of degree $d - m_P$, then $k(C) = k(y)[x]/(f)$, where $f \in k[x, y]$ is the equation of C in these affine coordinates.

(1) If $d - m_p = 1$, then $k(C) = \pi_P^*(k(\mathbb{P}^1))$ and therefore $\pi_P^* : k(\mathbb{P}^1) \to k(C)$ is an isomorphism. Hence $\pi_P : C \dashrightarrow \mathbb{P}^1$ is birational.

(2) If $d - m_p = 2$, then the extension $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is of degree 2 and it is thus Galois if and only if it is separable. $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is Galois $\Leftrightarrow$ there exists an element $\sigma \in G_P$ of order 2 that permutes the roots of f $\Leftrightarrow$ f is separable $\Leftrightarrow$ the extension is separable. Furthermore, the element $\sigma \in G_P$ of order 2 is given by $x \mapsto -x$ up to a suitable change of coordinates.

(3) If $d - m_P = 3$, the equation of the curve C is given by $f = F_{d-3}(y, 1)x^3 + F_{d-2}(y, 1)x^2 + F_{d-1}(y, 1)x + F_d(y, 1)$. We apply Lemma 3.1, replacing k by $k(y)$. □

We now illustrate Theorem 3.2 in two examples.

Lemma 3.3. *Let k be a field with $\mathrm{char}(k) \neq 3$ that contains a primitive third root of unity. Let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u : v] \mapsto [uv^2 + u^2v : u^3 : v^3]$. We define $C := \overline{\phi(\mathbb{P}^1)}$ is a curve of $\mathbb{P}^2$, then the point $P =$*

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$[1:0:0]$ *is a Galois point of* C *and the extension induced by the projection* $\pi_P : C \dashrightarrow \mathbb{P}^1$ *is Galois of degree* 3. *The element of order* 3 *extends to an element of* Jonq_P.

Proof. The curve C is birational to $\mathbb{P}^1$ via ϕ, with inverse $[X:Y:Z] \mapsto [X+Y:X+Z]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$, then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^3:v^3]$, so the extension is Galois of degree 3 with Galois group G_P generated by $\sigma : x \mapsto \omega \cdot x$, where ω is a primitive cubic root of unity. By Theorem 3.2, every element of order 3 extends to an element of Jonq_P. Explicitly, σ extends to the map that is given by

$$[X:Y:Z] \mapsto [\frac{(Y-\omega Z)\,X + YZ(1-\omega)}{(\omega-1)X + Y\omega - Z} : Y : Z].$$

$\square$

Lemma 3.4. *Let* k *be a field with* $char(k) = 3$ *and* $C \subset \mathbb{P}^2$ *given by the polynomial* $f = X^3 - Y^2X + Z^3$, *then the point* $P = [1:0:0]$ *is Galois point of* C *and the extension induced by the projection* $\pi_P : C \dashrightarrow \mathbb{P}^1$ *is Galois of degree* 3.

Proof. Define the birational map $\phi : \mathbb{P}^1 \dashrightarrow C$ by $\phi : [u:v] \mapsto [v^3 : u^3 : u^2v - v^3]$ with inverse $[X:Y:Z] \mapsto [Y:Z+X]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$, then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^3 : u^2v - v^3]$, so the extension is Galois of degree 3 with Galois group G_P generated by $\sigma : [u:v] \mapsto [u : u+v]$. By Theorem 3.2, we know that every element of order 3 extends to an element of Jonq_P. Explicitly σ extends to the map that is given by σ that is given by $[X:Y:Z] \mapsto [X+Y:Y:Z]$. $\square$

4. Curves that are Cremona equivalent to a line

Definition 4.1. *Let* X *be a smooth projective variety and* D *a divisor in* X. *Let* K_X *denote a canonical divisor of* X. *We define the Kodaira dimension of* $D \subset X$, *written* $\mathcal{K}(D, X)$ *to be the dimension of the image of* $X \mapsto P(H^0(m(D + K_X)))$ *for* $m >> 0$. *By convention we say that the Kodaira dimension is* $\mathcal{K}(D, X) = -\infty$ *if* $|m(D + K_X)| = \varnothing \ \forall\, m > 0$.

Definition 4.2. [7] *Let* $C \subset \mathbb{P}^2$ *be an irreducible curve. If* C *is a smooth curve, we define* $\bar{\mathcal{K}}(C, \mathbb{P}^2)$ *to be* $\mathcal{K}(C, \mathbb{P}^2)$. *If* C *is a singular curve, we take* $X \to \mathbb{P}^2$ *to be an embedded resolution of singularities of* C *in* $\mathbb{P}^2$ *where* $\tilde{C}$ *is the strict transform of* C, *then we define* $\bar{\mathcal{K}}(C, \mathbb{P}^2)$ *to be* $\mathcal{K}(\tilde{C}, X)$. *This does not depend on the choice of the resolution.*

Definition 4.3. *Let* C *be an irreducible smooth plane curve. The curve* C *is said to be Cremona equivalent to a line if there is a birational map* $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *that sends* C *to a line.*

Theorem 4.4. (Coolidge) [7, Theorem 2.6] *Let* $C \hookrightarrow \mathbb{P}^2$ *be an irreducible rational curve. Then, there exists a Cremona transformation* σ *of* $\mathbb{P}^2$ *such that* $\sigma(C)$ *is a line if and only if* $\overline{\mathcal{K}}(C, \mathbb{P}^2) = -\infty$.

Lemma 4.5. *If* $C \hookrightarrow \mathbb{P}^2$ *is an irreducible rational curve of degree* $d < 6$, *then* C *is equivalent to a line.*

Proof. Let $\pi_1 : X_1 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at P_1, and let $\pi_i : X_i \to X_{i-1}$ the blow-up of X_{i-1} at $P_i \in X_{i-1}$ for $i \geq 2$, $\mathcal{E}_i = \pi_i^{-1}(P_i)$ is a (-1)-curve, where $\mathcal{E}_i^2 = -1$ and $\mathcal{E}_i \cong \mathbb{P}^1$. After blowing up n points, let $\pi : Y \mapsto \mathbb{P}^2$ be the composition of the blow-ups π_i, where $Y = X_n$ we choose enough points such that the strict transform of C is smooth. By induction, we have $E_i = (\pi_{i+1} \circ \ldots\ldots \circ \pi_n)^*(\mathcal{E}_i)$,

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$Pic(Y) = \pi^*(Pic(\mathbb{P}^2)) \oplus \mathbb{Z}E_1 \oplus \ldots \oplus \mathbb{Z}E_n$, and $E_i^2 = -1$ for every $i = 1, \ldots, n$, and $E_i \cdot E_j = 0$ for every $i \neq j$. Moreover,

$$K_Y = \pi_n^* \ldots . \pi_1^*(K_{\mathbb{P}^2}) + \Sigma_{i=1}^n \pi_n^* \ldots . \pi_{i+1}^*(\epsilon_i) = \pi^*(K_{\mathbb{P}^2}) + \Sigma_{i=1}^n E_i = -3\pi^*(L) + \Sigma_{i=1}^n E_i.$$

The strict transform $\tilde{C} \subset C$ is equivalent to $\tilde{C} = d \cdot \pi^*(L) - \Sigma_{i=1}^n m_{P_i}(C)E_i$. Hence we have $2K_Y + \tilde{C} = (-6 + d) \cdot \pi^*(L) + \Sigma_{i=1}^n (2 - m_{P_i}(C))E_i$, so $\pi^*(L) \cdot (2K_Y + \tilde{C}) = -6 + d$, thus $|2K_Y + \tilde{C}| = \phi$ for every curve of degree $d < 6$. [7, Corollary 2.4] shows that $|2K_Y + \tilde{C}| = \emptyset$ is equivalent to $\bar{\mathcal{K}}(C, \mathbb{P}^2) = -\infty$, so C is equivalent to a line by Theorem 4.4. □

Lemma 4.6. *If C is a Cremona equivalent to a line $L \subseteq \mathbb{P}^2$ and P is a Galois point, then every non-trivial element in G_P extends to an element in* $\mathrm{Bir}(\mathbb{P}^2)$.

Proof. Let $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$ that sends C onto a line L. For each $g \in G_P$, $\varphi|_C : C \dashrightarrow L$ conjugates g to an element of $\mathrm{Aut}(L)$, that extends to $\hat{g} \in \mathrm{Aut}(\mathbb{P}^2)$. Hence, g extends to $\varphi^{-1}\hat{g}\varphi \in \mathrm{Bir}(\mathbb{P}^2)$. □

Remark 4.7. *Let C be the smooth conic given by $C = \{[X : Y : Z] | Y^2 = XZ\} \subset \mathbb{P}^2$, then the natural embedding of* $\mathrm{Aut}(\mathbb{P}^2, C) = \{g \in \mathrm{Aut}(\mathbb{P}^2) | g(C) = C\} = \mathrm{PGL}_2$ *in* $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$ *is the one induced from the injective group homomorphism*

$$\mathrm{GL}_2(k) \to \mathrm{GL}_3(k), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{ad - bc} \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$

where $\rho : [u : v] \mapsto [u^2 : uv : v^2]$, and the following diagram commutes.

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, dashed, "\rho"] \arrow[d, "g"] & \tilde{C} \subset \mathbb{P}^2 \arrow[d, "\tilde{g}"] \\ \mathbb{P}^1 \arrow[r, "\rho"] & \mathbb{P}^2 \end{tikzcd}$$

Lemma 4.8. *Let k be a field of characteristic $char(k) \neq 2$ containing a primitive fourth root of unity, and let C be the irreducible curve defined by the equation*

$$X^4 - 4ZYX^2 - ZY^3 + 2Z^2Y^2 - YZ^3 = 0, \tag{2}$$

then the point $P = [1 : 0 : 0]$ is an outer Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 4. *Furthermore, the group G_P extends to* $\mathrm{Bir}(\mathbb{P}^2)$ *but not to* Jonq_P.

Proof. Define the birational map $\phi : \mathbb{A}^1 \dashrightarrow C$ by $\phi : t \mapsto [t + t^3 : t^4 : 1]$ with inverse $[X : Y : Z] \mapsto (X\,(Y + Z))\,/(X^2 - YZ + Z^2)$. Hence C is a rational irreducible curve of degree 4 and therefore, C is equivalent to a line by Lemma 4.5. Furthermore, every non-trivial element in G_P extends to an element in $\mathrm{Bir}(\mathbb{P}^2)$ by Lemma 4.6. We will also prove it explicitly below. We have $K(C) = k(t)$ and define the projection by $\pi_P : [X : Y : Z] \mapsto [Y : Z]$. Let $x = X/Z$ and $y = Y/Z$ be affine coordinates, so the affine equation $x^4 - 4yx^2 - y^3 + 2y^2 - y = 0$ is defining the extension field $k(y)[x]/k(y) = k(t)/k(t^4)$. Since k contains the 4*th* root of unity, the extension is Galois of degree 4 with basis $\{1, t, t^2, t^3\}$, and we have the following diagram

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, dashed, "\phi"] \arrow[dr, "\psi"'] & C \subseteq \mathbb{P}^2 \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

where ψ is given by $\psi : t \mapsto t^4$. By contradiction, we prove that there is no de Jonquières map f extending the action. Let us assume that there exists a de Jonquières map g that extends the action, *i.e* there exists $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in k(y)$ with $\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} \neq 0$ such that $g : (x, y) \mapsto (\frac{\tilde{\alpha}x + \tilde{\beta}}{\tilde{\gamma}x + \tilde{\delta}}, y)$. Since $g \circ \phi = \phi \circ \sigma$, writing

$\alpha = \tilde{\alpha}(t^4)$, $\beta = \tilde{\beta}(t^4)$, $\gamma = \tilde{\gamma}(t^4)$ and $\delta = \tilde{\delta}(t^4)$ where $\alpha, \beta, \gamma, \delta \in k(t^4)$. We obtain the equation

$$\mathrm{i}t - \mathrm{i}t^3 = \frac{\alpha(t+t^3)+\beta}{\gamma(t+t^3)+\delta}.$$

This implies that $\beta = \beta(t) = -(\mathrm{i}t^6 - \mathrm{i})\gamma t^2 - (\mathrm{i}\delta + \alpha)t^3 + (\delta\mathrm{i} - \alpha)t \in k(t^4)$ and is then $(\mathrm{i}t^4 - \mathrm{i})\gamma = 0$, $\mathrm{i}\delta + \alpha = 0$ and $\delta\mathrm{i} - \alpha = 0$. This gives $\alpha = 0, \gamma = 0$ which is a contradiction. Viewing C as an irreducible curve in $\mathbb{P}^2$ of degree 4, there are three singular points on the curve $[0:1:1]$, $\left[\mathrm{i}\sqrt{2} : -1 : 1\right]$, $\left[\mathrm{i}\sqrt{2} : 1 : -1\right]$. After suitable change of coordinates, $\sigma : \mathbb{P}^2 \to \mathbb{P}^2$ is given by $[X:Y:Z] \mapsto \left[-\mathrm{i}\sqrt{2}X - \mathrm{i}\sqrt{2}Z : 2X + Y - Z : 2X - Y + Z\right]$, this map sends the curve C to $\tilde{C}$, which is given by $\tilde{f} = X^2Y^2 + 6X^2YZ + X^2Z^2 + 4Y^2Z^2 = 0$ and this new equation has $\{[1:0:0], [0:1:0], [0:0:1]\}$ as multiple points of order 2. After blowing up the three points $\{[1:0:0], [0:1:0], [0:0:1]\}$ in $\mathbb{P}^2$ and contract again, the strict transform curve is of degree $d' = 2 \cdot d - m_1 - m_2 - m_3 = 2 \cdot 4 - 2 - 2 - 2 = 2$, so it is a conic given by the equation $F = 4X^2 + Y^2 + 6YZ + Z^2 = 0$. We change the coordinates using the following matrix

$$\begin{bmatrix} 4\mathrm{I} & 0 & -\mathrm{I} \\ 0 & 2\sqrt{2} & 0 \\ 8 & -6\sqrt{2} & 2 \end{bmatrix}$$

to send the conic to $Y^2 - XZ = 0$ and we extend G_P explicitly using Remark 4.7. □

5. Example where G_P cannot be extended to $\mathrm{Bir}(\mathbb{P}^2)$

Lemma 5.1. *Let k be an algebraically closed field, $C \subset \mathbb{P}^2$ be an irreducible curve, $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map sends the curve C to itself, and $X \to \mathbb{P}^2$ is an embedded resolution of singularities of C in $\mathbb{P}^2$ where $\tilde{C}$ is the strict transform of C. If all singular points of C have a multiplicity $m_P(C) < deg(C)/3$, then f is an automorphism of $\mathbb{P}^2$.*

Proof. Let $deg(f) = d$, and assume for contradiction that $d > 1$. We take a commutative diagram where π and η are sequences of blow-ups

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "\eta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

and we can assume that the strict transform of C is smooth. As in Lemma 4.5 $\eta^*(L) = d \cdot \pi^*(L) - \Sigma m_i E_i$, $K_X = -3\pi^*(L) + \Sigma E_i$, and $\eta^*(C) = \tilde{C} = deg(C) \cdot \pi^*(L) - \Sigma m_{P_i}(C) E_i$. Since $deg(C) = C \cdot L = \eta^*(C) \cdot \eta^*(L) = d \cdot deg(C) - \Sigma m_i \cdot m_{P_i}(C)$, then $deg(C)(d-1) = \Sigma m_i \cdot m_{P_i}(C) < \Sigma m_i \cdot deg(C)/3$ and therefore $3(d-1) < \Sigma m_i$ Noether equality, which is a contradiction as $\Sigma m_i = 3(d-1)$: this equation follows from $\eta^*(L)^2 = L^2 = 1$ and from the adjunction formula $\eta^*(L) \cdot (\eta^*(L) + K_X) = -2$, which gives $\eta^*(L) \cdot K_X = -3$. □

Lemma 5.2. *Let k be a field with $char(k) \neq 5$ that contains a primitive 5th root of unity, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ given by $\phi : [u:v] \mapsto [uv^6 - u^7 : u^5(u^2 + v^2) : v^5(u^2 + v^2)]$. We define $C := \overline{\phi(\mathbb{P}^1)}$, then the point $P = [1:0:0]$ is an inner Galois point of C and the extension induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ is Galois of degree 5. Moreover, there is no birational map f extending the action of the generator of the Galois group.*

Proof. The curve C is birational to $\mathbb{P}^1$ via ϕ, with inverse $[X:Y:Z] \mapsto [X^4Y + 4X^3Y^2 - 2X^3Z^2 + 6X^2Y^3 - 2X^2YZ^2 + 4XY^4 + XZ^4 + Y^5 : Z(X^4 + 2X^3Y + X^2Y^2 - 3X^2Z^2 - 3XYZ^2 + Z^4)]$. Define the projection by $\pi_P : [X:Y:Z] \mapsto [Y:Z]$. Let $\psi = \pi_P \circ \phi$ then $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ maps $[u:v]$ to $[u^5 : v^5]$, hence the extension is Galois of degree 5 with Galois group G_P generated by $\sigma : x \mapsto \zeta \cdot x$, where ζ is the 5th root of unity. We now prove that the curve C does not have a point P of multiplicity $m_P(C) \geq 3$. By contradiction, we take a point $P = [P_0 : P_1 : P_2]$ of multiplicity 3, and then we take two distinct lines $a_1x + a_2y - a_3z = 0$ and $b_2y - b_3z = 0$ passing through the point P. We take the preimage in $\mathbb{P}^1$, so we get a common factor of degree at least 3. Let $f_1(u, v) = a_1\left(-u^7 + u\,v^6\right) + a_2u^5\left(u^2 + v^2\right) + a_3v^5\left(u^2 + v^2\right)$ and $f_2(u, v) = \left(u^2 + v^2\right)\left(b_2u^5 + b_3v^5\right)$. We check now that it is not possible for the polynomials f_1 and f_2 to have a factor of degree 3 in common. Assume first that $u + \mathrm{i}v$ divides both polynomials, so we should have $f_1(1, \mathrm{i}) = f_2(1, \mathrm{i}) = 0$ implies to $a_1 = 0$, hence $P = [1 : 0 : 0]$ is a smooth point and this gives a contradiction. If we assume that $u - \mathrm{i}v$ divides both polynomials, then we should have $f_1(1, -\mathrm{i}) = f_2(1, -\mathrm{i}) = 0$, again we have $a_1 = 0$, so the factor of degree 3 must divide $b_2u^5 + b_3v^5$. If we assume that u divides the polynomial f_2, then $b_3 = 0$ and u^3 should divide f_1, but this is not true as $f_1 = -u(u^6 - v^6)$. If we assume that v divides the polynomial f_2, then $b_2 = 0$ and v^3 should divide f_1, but this is not true as $f_1 = uv^2(u^4 + v^4)$. So the factor of degree 3 must divide $b_2u^5 + b_3v^5$, $b_2 \neq 0$ and $b_3 \neq 0$. Hence we can assume that $a_1 = 1$ and replace f_1 by $f_1 - a_3f_2/b_3$ so we can put $a_3 = 0$ and up to multiple, we can assume that $b_3 = 1$, $a_1 = 1$ and $b_2 = -\xi^5$. So $f_2 = -u^5\xi^5 + v^5$, $f_1 = \left(-u^7 + u\,v^6\right) + a_2u^5\left(u^2 + v^2\right)$ this implies that the roots of f_2 are $(u, v) = (1, \xi\zeta^i)$, where ζ is a 5th root of unity. Since f_1 and f_2 should have three roots in common, therefore let $\{(1, \xi), (1, \xi\zeta), (1, \xi\rho)\}$ are the three roots in common where $\rho^5 = 1$ and $\zeta^5 = 1$, and $\rho^5 \neq \zeta$ and they are not equal to 1, so f_1 should vanish on these three roots. This gives three equations $q_1 = \xi^6 - 1 + a_2\left(\xi^2 + 1\right) = 0$, $q_2 = a_2\left(\xi^2\zeta^2 + 1\right) + \xi^6\zeta - 1 = 0$, $q_3 = a_2\left(\rho^2\xi^2 + 1\right) + \rho\,\xi^6 - 1 = 0$, by solving this system in a_2, ζ and ρ, we found that $\zeta = \rho = (\xi^4 + 1)/(\xi^6 - 1)$, which is a contradiction as $\zeta \neq \rho$. Finally, f_1 and f_2 cannot have a factor of degree $d \geq 3$. Since $m_P(C) < 3$ for each $P \in \mathbb{P}^2$, let us assume that there exists a birational map g that extends the generator of the Galois group, then by Lemma 5.1, g is a linear automorphism of $\mathbb{P}^2$, so it is given by a matrix let us say $A \in \mathrm{PGL}_3$, Since $g \circ \phi = \phi \circ \sigma$, so we have

$$\begin{bmatrix} UV^6 - U^7 \\ U^5(U^2 + V^2) \\ V^5(U^2 + V^2) \end{bmatrix} = A \cdot \begin{bmatrix} \zeta UV^6 - \zeta^2U^7 \\ U^5(\zeta^2U^2 + V^2) \\ V^5(\zeta^2U^2 + V^2) \end{bmatrix}$$

where ζ is the $5th$ root of unity. Since UV^6, U^7, U^5V^2, and V^7 are linearly independent, after checking the calculation we found that A should be diagonal, but $U^5(\zeta^2U^2 + V^2)$ is not a multiple of $U^5(\zeta^2U^2 + V^2)$, then we have a contradiction. □

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