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RESEARCH ARTICLE

The weak Ramsey property and extreme amenability

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Abstract
We extend the Kechris–Pestov–Todorčević correspondence to weak Fraïssé categories and automorphism groups of generic objects. The new ingredient is the weak Ramsey property. We demonstrate the theory on several examples including monoid categories, the category of almost linear orders and categories of strong embeddings of trees.

Contents

1. Introduction

The main motivation for this note is the seminal work of Kechris, Pestov, Todorčević [8] exhibiting the connection between extreme amenability of automorphism groups of countable homogeneous structures and Ramsey-type properties of their finite substructures. This work has been very recently extended by

Mašulović [18] in the language of category theory. Our goal is to push it even further, namely, by making minimal assumptions both on the category of 'small' structures and weakening the homogeneity of the 'generic' object.

The phenomenon discovered by Kechris, Pestov, Todorčević [8] can be described in its simplest form as follows. We have a class $\mathcal{F}$ of finite structures of a fixed language. We assume the class has the subsequent nice properties: Every two structures in $\mathcal{F}$ can be embedded into a single one (the joint embedding property); every two extensions of a structure in $\mathcal{F}$ can be combined into a single one (the amalgamation property); every substructure of a structure in $\mathcal{F}$ is again in $\mathcal{F}$ (the class is hereditary); there are countably many isomorphic types in $\mathcal{F}$. When these conditions are met, there exists a unique countable structure U whose finite substructures are all in $\mathcal{F}$ such that every structure from $\mathcal{F}$ embeds into U, and U is ultrahomogeneous with respect to $\mathcal{F}$, that is, every isomorphism between finite substructures of U extends to an automorphism of U. These are the foundational objects of study in Fraïssé theory. Consider the group $G = \mathrm{Aut}(U)$ with the pointwise convergence topology. The *KPT correspondence* states that the group G is extremely amenable (i.e., every continuous action of G on a compact Hausdorff space has a fixed point) if and only if the class $\mathcal{F}$ has the Ramsey property and the ordering property. The ordering property ensures that all structures in $\mathcal{F}$ are rigid (i.e., have trivial automorphism groups). The Ramsey property is the structural variant of the classical Ramsey theorem, where we color structures of a fixed isomorphism type from $\mathcal{F}$ instead of coloring subsets.

For example, if $\mathcal{F}$ is the class of finite linearly ordered sets then the Ramsey property asserts in particular that, for every finite linearly ordered set X and any positive integers m and k, there exists a bigger finite linearly ordered set Y such that when we color all copies of the m-element linear ordering inside Y with at most k many colors, we can always find an embedding e of X into Y (namely, a subset with the same number of elements as X) such that all m-element subsets of $e[X]$ have the same color. This is equivalent to the classical finite Ramsey theorem. Its formulation in the language of linear orderings has two advantages: First, it allows us to talk about embeddings instead of subsets, as the domain of an embedding can be identified with its image. Second, what is perhaps more important, it is purely category-theoretic.

The last observation leads to a natural idea: Replace the class of finite structures $\mathcal{F}$ by an abstract category $\mathfrak{C}$, whose arrows are meant to be some sort of 'embeddings'. A natural assumption, made by Mašulović [18] in this categorical framework, is that every object has only finitely many arrows going into it. Note that this is immediately true in the case that the category in question is a category of finite models with embeddings for arrows. As it so happens, this assumption, and the weaker assumption asserting that there are only finitely many arrows between two prescribed objects, is not necessary to capture the KPT correspondence in a categorical framework. We can do so by instead making the Ramsey property a bit more technical, involving only finite subsets of arrows rather than all arrows between two structures.

Nevertheless, it turns out that the KPT correspondence holds in a fairly large class of categories in which the notion of 'being finite' is replaced by a factorization property with respect to a fixed sequence. Despite this subtle change, we still arrive to the same connection between extreme amenability and the Ramsey property. In order to make the theory as general as possible, we shall work with so-called *weak* Fraïssé categories [12], where the amalgamation property holds in a weaker form. We obtain the equivalence of extreme amenability of the automorphism group of the generic object with the weak version of the Ramsey property, which involves particular arrows (called *amalgamable arrows*). We also show that a weak Fraïssé category $\mathfrak{C}$ gives rise to a natural Fraïssé category $\mathrm{Am}(\mathfrak{C}^{\uparrow})$. Moreover, we show that $\mathfrak{C}$ has the *weak* Ramsey property if and only if $\mathrm{Am}(\mathfrak{C}^{\uparrow})$ has the Ramsey property.

Our main result can be roughly summarized as follows.

Theorem 1. *Assume $\mathfrak{S}$ is a weak Fraïssé category, and let U be generic over $\mathfrak{S}$. The following properties are equivalent.*

(a) $\mathrm{Aut}(U)$ *is extremely amenable.*
(b) $\mathfrak{S}$ *has the weak Ramsey property.*

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Of course, some minimal technical assumptions are needed here so that there is a good interplay between the topology of $\mathrm{Aut}(U)$ and the category $\mathfrak{S}$. In particular, U is an object of a larger category. Besides that, there are practically no further assumptions on $\mathfrak{S}$; however, the weak Ramsey property involves finite sets of arrows. The precise statement is Theorem 3.14 below.

The paper is organized as follows: After the Preliminaries section (where we introduce the setup) we prove the main results in a series of lemmas showing that the weak Ramsey property (see Definition 3.1) is equivalent to extreme amenability of the automorphism group of the generic object. The last section contains a discussion of the main results and concrete applications. This includes an analysis of monoid categories and almost linear orders, as well as finite trees under strong embeddings. The latter exhibits an interesting interplay between Milliken's theorem for trees [19] and the universal Ważewski dendrites [5], [15].

2. Preliminaries

We shall use very basic concepts from category theory. For undefined notions, we refer to MacLane [16]. Categories will be denoted by $\mathfrak{C}$, $\mathfrak{S}$, $\mathfrak{L}$ and so on. A category $\mathfrak{C}$ will be identified with its class of arrows (morphisms) and $\mathrm{Obj}(\mathfrak{C})$ will denote its class of objects. Given two $\mathfrak{C}$-objects a, b we denote by $\mathfrak{C}(a, b)$ the set of all arrows f with domain a (i.e., $\mathrm{dom}(f) = a$) and codomain b (i.e., $\mathrm{cod}(f) = b$). We sometimes write $f\colon a \to b$ instead of $f \in \mathfrak{C}(a, b)$, as long as the category $\mathfrak{C}$ is understood from the context. Composition of arrows is performed in the usual order and denoted by $\circ$, that is, $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ and $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$. The identity arrow of an object x will be denoted by id_x. All our categories are supposed to be locally small, that is, the class of all arrows from a fixed object a to a fixed object b is a set, not a proper class.

A subcategory $\mathfrak{C} \subseteq \mathfrak{S}$ is called *full* if $\mathfrak{C}(a, b) = \mathfrak{S}(a, b)$ for every $a, b \in \mathrm{Obj}(\mathfrak{C})$, that is, if we are restricting only objects, while $\mathfrak{C} \subseteq \mathfrak{S}$ is called *wide* if $\mathrm{Obj}(\mathfrak{C}) = \mathrm{Obj}(\mathfrak{S})$, that is, if we are restricting only arrows.

A *sequence* in a category $\mathfrak{C}$ is a covariant functor $F\colon \omega \to \mathfrak{C}$, where the set of natural numbers ω is treated as a poset category. Namely, F consists of a sequence of objects $\{F(n)\}_{n\in\omega}$ and a sequence of $\mathfrak{C}$-arrows $\{F(n, m)\}_{n\leqslant m<\omega}$ such that $F(n, m) \in \mathfrak{C}(F(n), F(m))$, $F(k, k) = \mathrm{id}_{F(k)}$ and $F(k, m) = F(\ell, m) \circ F(k, \ell)$ for every $k \leqslant \ell \leqslant m$. We shall use the following convention: A sequence will be denoted by $\vec{u}$ (possibly with u replaced by another letter) and in that case we denote $u_n = \vec{u}(n)$ and $u_n^m = \vec{u}(n, m)$.

A *cone* for a sequence $\vec{u}$ in $\mathfrak{C}$ is a pair $(\vec{u}^{\,\infty}, U)$, where U is a fixed object and $\vec{u}^{\,\infty}$ is a family of $\mathfrak{C}$-arrows $\{u_n^\infty \colon u_n \to U\}_{n\in\omega}$ such that $u_n^\infty = u_m^\infty \circ u_n^m$ for every $n \leqslant m$, as in Figure 1. A cone $(\vec{u}^{\,\infty}, U)$ for $\vec{u}$ is a *colimit* of $\vec{u}$ if, for every cone $(\vec{v}^{\,\infty}, V)$ for $\vec{u}$, there is a unique $\mathfrak{C}$-arrow $f\colon U \to V$ such that $v_n^\infty = f \circ u_n^\infty$ for every $n \in \omega$.

We adopt the standard convention and denote the automorphism group of an object x by $\mathrm{Aut}(x)$. Our main interest will be $\mathrm{Aut}(U)$, where U is a distinguished 'generic' object (the precise meaning is described below). So, with one exception, the objects of categories will be denoted by small letters.

2.1. The setup

Throughout the paper, we find ourselves in the following situation: We have a category of 'small' objects $\mathfrak{S}$ and a fixed 'large' object U, both living in an ambient category $\mathfrak{L}$. The main theorems relate properties of $\mathfrak{S}$, like the (weak) Ramsey property, with properties of U, like extreme amenability of its automorphism group. Note that (weak) Fraïssé theory follows the same pattern: Properties of $\mathfrak{S}$, like the (weak) amalgamation property or existence of a (weak) Fraïssé sequence, are related to properties of the 'generic' object U, like (weak) homogeneity and (weak) injectivity.

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$$\begin{tikzcd}
 & & U & & \\
u_0 \arrow[r, "u_0^1"] \arrow[urr, "u_0^\infty"] & u_1 \arrow[r, "u_1^2"] \arrow[ur, "u_1^\infty"] & u_2 \arrow[r, "u_2^3"] \arrow[u, "u_2^\infty"] & u_3 \arrow[r] \arrow[ul, "u_3^\infty"] & \dots \arrow[ull]
\end{tikzcd}$$

Figure 1. *A cone for a sequence in a category. The diagram is commutative.*

A connection between $\mathfrak{S}$ and U in $\mathfrak{L}$ is established by fixing a *coned sequence* in $(\mathfrak{S}, \mathfrak{L})$: a triple $(\vec{u}, \vec{u}^{\infty}, U)$, where $\vec{u}$ is a sequence in $\mathfrak{S}$ and $(\vec{u}^{\infty}, U)$ is a cone for $\vec{u}$ in $\mathfrak{L}$. The object U can be thought of as a 'quasi-limit' of $\vec{u}$ in $\mathfrak{L}$. Sometimes, it even is an actual colimit, but it is not necessary.

In this subsection, we impose a minimalistic set of conditions on the coned sequence $(\vec{u}, \vec{u}^{\infty}, U)$ for our theory to work. A concise list is given in the following definition. A rather long remark (that can be safely skipped) giving some insight follows.

Definition 2.1. Let $\mathfrak{S} \subseteq \mathfrak{L}$ be categories. We say that a coned sequence $(\vec{u}, \vec{u}^{\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$ is *matching* if it satisfies the following conditions.

(F1) For every $\mathfrak{L}$-arrow $f \colon x \to U$ with $x \in \mathrm{Obj}(\mathfrak{S})$, there exist n and an $\mathfrak{S}$-arrow $\tilde{f} \colon x \to u_n$ such that $f = u_n^{\infty} \circ \tilde{f}$.

(F2) For every n and every $\mathfrak{S}$-arrows $f, f' \colon x \to u_n$ such that $u_n^{\infty} \circ f = u_n^{\infty} \circ f'$ there is $n' \geqslant n$ such that $u_n^{n'} \circ f = u_n^{n'} \circ f'$.

(BF) If $\{f_n\}_{n\in\omega}, \{g_n\}_{n\in\omega}$ are sequences of $\mathfrak{S}$-arrows such that

$$f_n \colon u_{k_n} \to u_{\ell_n},\ g_n \colon u_{\ell_n} \to u_{k_{n+1}},\ g_n \circ f_n = u_{k_n}^{k_{n+1}},\ f_{n+1} \circ g_n = u_{\ell_n}^{\ell_{n+1}},$$

for some increasing cofinal sequences $\{k_n\}_{n\in\omega}, \{\ell_n\}_{n\in\omega} \subseteq \omega$, then there exists an $\mathfrak{L}$-arrow $f_\infty \in \mathrm{Aut}(U)$ such that

$$f_\infty \circ u_{k_n}^{\infty} = u_{\ell_n}^{\infty} \circ f_n \qquad \text{and} \qquad f_\infty^{-1} \circ u_{\ell_n}^{\infty} = u_{k_{n+1}}^{\infty} \circ g_n$$

for every $n \in \omega$.

(H) For every $h \in \mathrm{Aut}(U) \setminus \{\mathrm{id}_U\}$, there is n such that $h \circ u_n^{\infty} \neq u_n^{\infty}$.

The letter F stands for 'factorization', and (F1) and (F2) are called the 'factorization (existence) condition' and the 'factorization uniqueness condition', respectively. The letters BF stand for 'back-and-forth' and H stands for 'Hausdorff' (see Construction 2.6).

Remark 2.2. Let us give some insight to the conditions defining a matching sequence. Given a category $\mathfrak{S}$, let us define the induced *category of sequences* $\sigma_0\mathfrak{S}$. The objects are all sequences $\vec{u}$ in $\mathfrak{S}$. The morphisms are transformations between sequences modulo a certain equivalence. A transformation $\vec{\varphi} \colon \vec{u} \to \vec{v}$ is a sequence of $\mathfrak{S}$-arrows $\{\varphi_n \colon u_n \to v_{\varphi(n)}\}_{n\in\omega}$, where $\{\varphi(n)\}_{n\in\omega} \subseteq \omega$ is an increasing cofinal sequence such that $v_{\varphi(n)}^{\varphi(m)} \circ \varphi_n = \varphi_m \circ u_n^m$ for every $n \leqslant m \in \omega$, as in Figure 2, that is, it is a natural transformation from $\vec{u}$ to a subsequence of $\vec{v}$. The composition is obvious: $(\vec{\psi} \circ \vec{\varphi})_n = \psi_{\varphi(n)} \circ \varphi_n$ for every $n \in \omega$. We say that two transformations $\vec{\varphi}, \vec{\psi} \colon \vec{u} \to \vec{v}$ are equivalent (and we write $\vec{\varphi} \approx \vec{\psi}$) if for every $n \in \omega$ there is $m \geqslant \varphi(n), \psi(n)$ such that $v_{\varphi(n)}^m \circ \varphi_n = v_{\psi(n)}^m \circ \psi_n$. It is easy to see that this is a well-defined congruence of a category, and so defining morphisms of $\sigma_0\mathfrak{S}$ as transformations modulo this equivalence is correct.

Note that we may identify every $\mathfrak{S}$-object x with the constant sequence $\vec{\mathrm{id}}_x$ and every $\mathfrak{S}$-arrow $f \colon x \to y$ with the constant transformation $\vec{\mathrm{id}}_f : \vec{\mathrm{id}}_x \to \vec{\mathrm{id}}_y$. This way we identify $\mathfrak{S}$ with a subcategory of $\sigma_0\mathfrak{S}$. Moreover, this subcategory is full since every $\sigma_0\mathfrak{S}$-arrow $\vec{\varphi} \colon x \to \vec{u}$ from a constant identity sequence is uniquely determined (as a transformation up to the equivalence) by the $\mathfrak{S}$-arrow $\varphi_0 \colon x \to$

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$$
\begin{tikzcd}
\vec{v} & v_0 \arrow[r, "v_0^1"'] \arrow[rrrrr, bend left, "v_0^\infty"] & v_1 \arrow[r, "v_1^2"'] \arrow[rrrr, bend left, "v_1^\infty"] & v_2 \arrow[r, "v_2^3"'] \arrow[rrr, bend left, "v_2^\infty"] & v_3 \arrow[r] \arrow[rr, bend left, "v_3^\infty"] & \cdots & V \\
\vec{u} \arrow[u, "\vec{\varphi}"] & u_0 \arrow[u, "\varphi_0"] \arrow[r, "u_0^1"] \arrow[rrrrr, bend right, "u_0^\infty"'] & u_1 \arrow[ur, "\varphi_1"] \arrow[r, "u_1^2"] \arrow[rrrr, bend right, "u_1^\infty"'] & u_2 \arrow[urr, "\varphi_2"] \arrow[r, "u_2^3"] \arrow[rrr, bend right, "u_2^\infty"'] & u_3 \arrow[ur, "\varphi_3"] \arrow[r] \arrow[rr, bend right, "u_3^\infty"'] & \cdots & U \arrow[u, "\varphi_\infty"']
\end{tikzcd}
$$

Figure 2. *A morphism between sequences and a matching morphism between associated cones.*

$u_{\varphi(0)}$. Also note that every constant sequence $\vec{\mathrm{id}}_x$ admits the canonical limit cone $(\vec{\mathrm{id}}_x^{\,\infty}, x)$ in $\mathfrak{L}$ and that $(\vec{\mathrm{id}}_x, \vec{\mathrm{id}}_x^{\,\infty}, x)$ is a matching sequence in $(\mathfrak{S}, \mathfrak{L})$.

For two coned sequences $(\vec{u}, \vec{u}^{\,\infty}, U)$ and $(\vec{v}, \vec{v}^{\,\infty}, V)$ in $(\mathfrak{S}, \mathfrak{L})$ we consider the *matching relation* $\rhd$ between $\sigma_0\mathfrak{S}$-arrows $\vec{u} \to \vec{v}$ and $\mathfrak{L}$-arrows $U \to V$: We put

$$\vec{\varphi} \rhd \varphi_\infty \quad \text{if} \quad v^\infty_{\varphi(n)} \circ \varphi_n = \varphi_\infty \circ u^\infty_n \text{ for every } n \in \omega;$$

see Figure 2. The matching relation is functorial: We have $\mathrm{id}_{\vec{u}} \rhd \mathrm{id}_U$, and if $\vec{\varphi} \rhd \varphi_\infty$ and $\vec{\psi} \rhd \psi_\infty$, then $\vec{\psi} \circ \vec{\varphi} \rhd \psi_\infty \circ \varphi_\infty$.

Now, let $(\vec{u}, \vec{u}^{\,\infty}, U)$ be a coned sequence in $(\mathfrak{S}, \mathfrak{L})$. For every $x \in \mathrm{Obj}(\mathfrak{S})$, the matching relation for $(\vec{\mathrm{id}}_x, \vec{\mathrm{id}}_x^{\,\infty}, x)$ and $(\vec{u}, \vec{u}^{\,\infty}, U)$ is a function $\sigma_0\mathfrak{S}(x, \vec{u}) \to \mathfrak{L}(x, U)$: Every $\sigma_0\mathfrak{S}$-arrow $\varphi\colon x \to \vec{u}$ is determined by an $\mathfrak{S}$-arrow $f\colon x \to u_n$ for some n, and the unique $\mathfrak{L}$-map φ_∞ such that $\vec{\varphi} \rhd \varphi_\infty$ is $u^\infty_n \circ f$. The condition (F1) says that this matching function is surjective, and the condition (F2) says that the matching function is one-to-one. Together, (F1) and (F2) hold if and only if for every $x \in \mathrm{Obj}(\mathfrak{S})$ the matching relation is a bijection between $\sigma_0\mathfrak{S}(x, \vec{u})$ and $\mathfrak{L}(x, U)$.

Moreover, under (F1), for every $\mathfrak{L}$-arrow $\varphi_\infty\colon U \to U$ there is a transformation $\vec{\varphi}\colon \vec{u} \to \vec{u}$ such that $\vec{\varphi} \rhd \varphi_\infty$: For every $n \in \omega$, there is an $\mathfrak{S}$-arrow $\varphi_n\colon u_n \to u_{\varphi(n)}$ with $u^\infty_{\varphi(n)} \circ \varphi_n = \varphi_\infty \circ u^\infty_n$ and we can make sure that the sequence $\{\varphi(n)\}_{n\in\omega}$ is increasing and cofinal. Under (F2), for every $\mathfrak{L}$-arrow $\varphi_\infty\colon U \to U$ there is at most one transformation $\vec{\varphi}\colon \vec{u} \to \vec{u}$ up to the equivalence such that $\vec{\varphi} \rhd \varphi_\infty$: For a different such transformation $\vec{\psi}$ and $n \in \omega$, we have $u^\infty_{\varphi(n)} \circ \varphi_n = \varphi_\infty \circ u^\infty_n = u^\infty_{\psi(n)} \circ \psi_n$, and so by (F2) there is m such that $u^m_{\varphi(n)} \circ \varphi_n = u^m_{\psi(n)} \circ \psi_n$. Together, the matching relation for $(\vec{u}, \vec{u}^{\,\infty}, U)$ is a function $\mathfrak{L}(U, U) \to \sigma_0\mathfrak{S}(\vec{u}, \vec{u})$. By the functoriality, the function is a monoid homomorphism, and it restricts to a group homomorphism $F\colon\ \mathrm{Aut}_{\mathfrak{L}}(U) \to \mathrm{Aut}_{\sigma_0\mathfrak{S}}(\vec{u})$. Observe that $\sigma_0\mathfrak{S}$-automorphisms $\vec{u} \to \vec{u}$ correspond to the zig-zag sequences used in condition (BF), and so (BF) states that the mapping F is surjective (or even without (F1) and (F2) that for every automorphism $\vec{\varphi}\colon \vec{u} \to \vec{u}$ there is a matching automorphism $\varphi_\infty\colon U \to U$). Similarly, (H) states that the mapping F is one-to-one (here we cannot omit (F2)). Together, under (F1) and (F2), the conditions (BF) and (H) hold if and only if the matching relation is a group isomorphism between $\mathrm{Aut}_{\sigma_0\mathfrak{S}}(\vec{u})$ and $\mathrm{Aut}_{\mathfrak{L}}(U)$.

In the following lemma, we observe that the notion of a matching sequence is robust under isomorphism and that the large object U determines the sequence of small objects $\vec{u}$ uniquely.

Lemma 2.3. *Let $(\vec{u}, \vec{u}^{\,\infty}, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$.*

(i) *If $f\colon U \to V$ is an $\mathfrak{L}$-isomorphism, then $(\vec{u}, f \circ \vec{u}^{\,\infty}, V)$ is a matching sequence.*
(ii) *If $\vec{\varphi}\colon \vec{v} \to \vec{u}$ is an $\sigma_0\mathfrak{S}$-isomorphism, then $(\vec{v}, \vec{u}^{\,\infty} \circ \vec{\varphi}, U)$ is a matching sequence.*
(iii) *For every other matching sequence $(\vec{v}, \vec{v}^{\,\infty}, V)$ and every $\mathfrak{L}$-isomorphism $\varphi_\infty\colon U \to V$, there is a matching $\sigma_0\mathfrak{S}$-isomorphism $\vec{\varphi}\colon \vec{u} \to \vec{v}$.*

Proof. The proof is straightforward, though technical. For example, to obtain (BF) in (ii), for an $\sigma_0\mathfrak{S}$-automorphism $\psi\colon \vec{v} \to \vec{v}$, we consider the automorphism $(\vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1})\colon \vec{u} \to \vec{u}$ and its matching automorphism $h\colon U \to U$. For every $n \in \omega$, we have

$$u^\infty_{\varphi(\psi(\varphi^{-1}(n)))} \circ (\vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1})_n = h \circ u^\infty_n.$$

Also, $\vec{\varphi}^{-1} \circ \vec{\varphi}$ is equivalent to $\mathrm{id}_{\vec{u}}$, and so we have

$$\begin{aligned}(\vec{u}^\infty \circ \vec{\varphi})_{\psi(n)} \circ \psi_n &= (\vec{u}^\infty \circ \vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1} \circ \vec{\varphi})_n \\ &= u^\infty_{\varphi(\psi(\varphi^{-1}(\varphi(n))))} \circ (\vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1})_{\varphi(n)} \circ \varphi_n = h \circ u^\infty_{\varphi(n)} \circ \varphi_n = h \circ (\vec{u}^\infty \circ \vec{\varphi})_n,\end{aligned}$$

which we wanted.

Claim (iii) in the case of automorphisms was already discussed in Remark 2.2, and the proof for isomorphisms is analogous. □

It is not always the case that a sequence $\vec{u}$ can be completed to a matching sequence at most one way (up to an isomorphism). However, it may have at most one colimit $(\vec{u}^\infty, U)$ in $\mathfrak{L}$. Moreover, if $(\vec{u}^\infty, U)$ is a colimit of $\vec{u}$ in $\mathfrak{L}$, then $(\vec{u}, \vec{u}^\infty, U)$ satisfies (BF) and (H). In fact, for every transformation $\vec{\varphi}\colon \vec{u} \to \vec{v}$ and every coned sequence $(\vec{v}, \vec{v}^\infty, V)$ there is a unique matching $\mathfrak{L}$-arrow $\varphi_\infty\colon U \to V$ since such matching arrow is the same thing as the colimit factorizing arrow for the cone $(\vec{v}^\infty \circ \vec{\varphi}, V)$ for $\vec{u}$.

Let us continue discussing how to obtain the conditions defining a matching sequence. Observe that if $\mathfrak{L}$ consists of monomorphisms, we obtain (F2) for free: If $u^\infty_n \circ f = u^\infty_n \circ f'$, then $f = f'$ since u^∞_n is a monomorphism. Let us also recall the notion of *finitely presentable* object [1, Definition 1.1] in a category $\mathfrak{L}$: It is an object x such that for every directed colimit $\vec{u} = (u^j_i, u^\infty_i, U)_{i \leqslant j \in I}$ every $\mathfrak{L}$-arrow $f\colon x \to U$ essentially uniquely factorizes through $\vec{u}$, that is, there is an arrow $\tilde{f}\colon x \to u_i$ for some i such that $u^\infty_i \circ \tilde{f} = f$, and for every other such arrow $g\colon x \to u_j$, there is $k \geqslant i, j$ such that $u^k_i \circ \tilde{f} = u^k_j \circ g$. In other words, a finitely presented object satisfies analogues of (F1) and (F2) for every directed system with a colimit. Hence, if $\mathfrak{S} \subseteq \mathfrak{L}$ is a full subcategory whose objects are finitely presented in $\mathfrak{L}$, and $(\vec{u}, \vec{u}^\infty, U)$ is a sequence in $\mathfrak{S}$ with a colimit in $\mathfrak{L}$, then it satisfies (F1) and (F2).

We summarize this discussion in the following lemma.

Lemma 2.4. *Let $(\vec{u}, \vec{u}^\infty, U)$ be a coned sequence in $(\mathfrak{S}, \mathfrak{L})$.*

(i) *If every map u^∞_n is a monomorphism, then $(\vec{u}, \vec{u}^\infty, U)$ satisfies (F2).*
(ii) *If $(\vec{u}^\infty, U)$ is a colimit of $\vec{u}$ in $\mathfrak{L}$, then $(\vec{u}, \vec{u}^\infty, U)$ satisfies (BF) and (H).*
(iii) *If $\mathfrak{S}$ is a full subcategory of $\mathfrak{L}$, every $\mathfrak{S}$-object is finitely presentable in $\mathfrak{L}$ and $(\vec{u}^\infty, U)$ is a colimit, then $(\vec{u}, \vec{u}^\infty, U)$ satisfies also (F1) and (F2) and so is matching.*

A classical example of the described situation are categories of finitely generated L-structures in a first-order language L, with embeddings or one-to-one homomorphisms as arrows between objects. Let $\mathfrak{L}$ be the category of all L-structures and homomorphisms. Recall that a sequence $\vec{u}$ in $\mathfrak{L}$ consisting of embeddings can be without loss of generality viewed as an increasing $\subseteq$-chain of substructures. Then its colimit in $\mathfrak{L}$ is the union of the chain. In particular, the colimit maps u^∞_n can be taken to be inclusion maps. So if we consider the wide subcategory $\mathfrak{L}' \subseteq \mathfrak{L}$ of all embeddings between L-structures, then $\mathfrak{L}$-colimits of $\mathfrak{L}'$-sequences (or of any directed systems in $\mathfrak{L}'$) are also $\mathfrak{L}'$-colimits. Moreover, finitely generated L-structures are finitely presentable in $\mathfrak{L}'$: If an L-structure A is a directed union of its substructures $(A_i \subseteq A)_{i \in I}$ and $B \subseteq A$ is a substructure generated by a finite set $F \subseteq A$, then every generator $x \in F$ is contained in some A_{i_x}, and so all of them are contained in some A_i. Hence, $B \subseteq A_i$. Together, we obtain the following.

Construction 2.5 (σ-closure). Let L be a first-order language, let $\mathfrak{L}$ be the category of all L-structures and homomorphisms and let $\mathfrak{S} \subseteq \mathfrak{L}$ be a subcategory of some finitely generated L-structures and all embeddings between them. We define $\sigma\mathfrak{S}$ to be the category of all $\mathfrak{L}$-structures that are $\mathfrak{L}$-colimits of $\mathfrak{S}$-sequences with all embeddings between them as morphisms.

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We have that every $\mathfrak{S}$-sequence $\vec{u}$ has a common colimit in $\sigma\mathfrak{S}$ and $\mathfrak{L}$, that every $\sigma\mathfrak{S}$-object U is a colimit of a $\mathfrak{S}$-sequence and that every such colimit sequence $(\vec{u}, \vec{u}^{\infty}, U)$ in $(\mathfrak{S}, \sigma\mathfrak{S})$ or equivalently in $(\mathfrak{S}, \mathfrak{L})$ is matching. In fact, mapping every $\mathfrak{S}$-sequence to its colimit and every transformation of $\mathfrak{S}$-sequences to the unique matching $\mathfrak{L}$-arrow induces a functor $\sigma_0\mathfrak{S} \to \sigma\mathfrak{S}$, which is an equivalence of categories in this case. Hence, $\sigma\mathfrak{S}$ can be viewed as a concrete realization of $\sigma_0\mathfrak{S}$ in $\mathfrak{L}$.

Note $\sigma\mathfrak{S}$-objects are exactly countably generated L-structures such that every finite subset is contained in a substructure from $\mathfrak{S}$. If L is a relational language and $\mathrm{Obj}(\mathfrak{S})$ is hereditary, then $\sigma\mathfrak{S}$-objects are all countable L-structures whose finite substructures are in $\mathrm{Obj}(\mathfrak{S})$. If $\mathfrak{S}$ is the category of all finite groups, then $\sigma\mathfrak{S}$ is the category of all countable locally finite groups (meaning that every finite subset generates a finite subgroup).

The construction of $\sigma\mathfrak{S}$ can be done also with one-to-one homomorphisms instead of embeddings. In that case, the inclusion maps in the increasing chain may also refine the structure (make the relations finer).

Construction 2.6 (The topology of $\mathrm{Aut}(U)$. Let $(\vec{u}, \vec{u}^{\infty}, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$. The automorphism group $G := \mathrm{Aut}(U)$ has a natural topology defined by the following neighborhood base of the identity:

$$V_n = \left\{ g \in G \colon g \circ u_n^{\infty} = u_n^{\infty} \right\}.$$

The topology is Hausdorff, thanks to condition (H). Note that each V_n is a subgroup of G, therefore we obtain a non-Archimedean topological group. Note also that G is completely metrizable and that it may not be separable unless all the hom-sets $\mathfrak{S}(u_n, u_m)$ are countable (see the next lemma).

To show that the open subgroups V_n indeed form a base of the identity of the group topology, we need to show that for every $n \in \omega$ and $g \in G$ there is $m \in \omega$ such that $g^{-1} \circ V_m \circ g \subseteq V_n$, that is, for every $h \in G$, if $h \circ u_m^{\infty} = u_m^{\infty}$, then $h \circ g \circ u_n^{\infty} = g \circ u_n^{\infty}$. By (F1), $g \circ u_n^{\infty} = u_m^{\infty} \circ f$ for some $m \in \omega$ and an $\mathfrak{S}$-arrow $f \colon u_n \to u_m$. Hence,

$$h \circ (g \circ u_n^{\infty}) = (h \circ u_m^{\infty}) \circ f = u_m^{\infty} \circ f = g \circ u_n^{\infty}.$$

Analogously, we can show that the topology on G does not depend on the choice of a matching sequence for U: If $(\vec{v}, \vec{v}^{\infty}, U)$ is another matching sequence, then the subgroups $V'_n = \{g \in G : g \circ v_n^{\infty} = v_n^{\infty}\}$ induce the same topology. For every $n \in \omega$, we have $v_n^{\infty} = u_m^{\infty} \circ f$ for some $m \in \omega$ and an $\mathfrak{S}$-arrow $f \colon v_n \to u_m$. Hence, $V_m \subseteq V'_n$.

Note that in the case when $\mathfrak{S}$ is a category of finite first-order structures and all embeddings, and $(\vec{u}^{\infty}, U)$ is a colimit, then U is essentially the union of the chain of finite structures u_n, and the condition $g \circ u_n^{\infty} = u_n^{\infty}$ says that the automorphism g fixes the elements of u_n. Therefore, the induced topology is the topology of pointwise convergence inherited from U^U where U has the discrete topology.

Lemma 2.7. *Let $(\vec{u}, \vec{u}^{\infty}, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$.*

(i) *The topology on* $\mathrm{Aut}(U)$ *is completely metrizable.*
(ii) *If all the hom-sets $\mathfrak{S}(u_n, u_m)$ are countable, then* $\mathrm{Aut}(U)$ *is also separable and so a Polish group.*

Proof. The group $G := \mathrm{Aut}(U)$ is metrizable by Birkhoff—Kakutani theorem [2, Theorem 3.3.12] since we have a countable neighborhood base of the identity. We show that the two-sided uniformity of G is induced by a complete metric. Let $\{h_n\}_{n\in\omega}$ be a Cauchy sequence with respect to the two-sided uniformity, that is, for every $m \in \omega$ there is $\varphi(m) \in \omega$ such that $h_{n'} \in (h_n \circ V_m) \cap (V_m \circ h_n)$ for every $n, n' \geqslant \varphi(m)$, and so $h_n \circ u_m^{\infty} = h_{n'} \circ u_m^{\infty}$ and $h_n^{-1} \circ u_m^{\infty} = h_{n'}^{-1} \circ u_m^{\infty}$.

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Put $k_0 := 0$. By (F1), there is an $\mathfrak{S}$-arrow $f_0 \colon u_{k_0} \to u_{\ell_0}$ for some $\ell_0 > k_0$ such that $u^\infty_{\ell_0} \circ f_0 = h_{\varphi(k_0)} \circ u^\infty_{k_0}$. Then there is an $\mathfrak{S}$-arrow $g'_0 \colon u_{\ell_0} \to u_{k'_1}$ for some $k'_1 > \ell_0$ such that $u^\infty_{k'_1} \circ g'_0 = h^{-1}_{\varphi(\ell_0)} \circ u^\infty_{\ell_0}$. Together, we have

$$u^\infty_{k'_1} \circ g'_0 \circ f_0 = h^{-1}_{\varphi(\ell_0)} \circ u^\infty_{\ell_0} \circ f_0 = h^{-1}_{\varphi(\ell_0)} \circ h_{\varphi(k_0)} \circ u^\infty_{k_0} = u^\infty_{k_0}$$

since $h_{\varphi(k_0)} \circ u^\infty_{k_0} = h_{\varphi(\ell_0)} \circ u^\infty_{k_0}$ since $\varphi(\ell_0) \geqslant \varphi(k_0)$. By (F2), there is $k_1 \geqslant k'_1$ such that $u^{k_1}_{k'_1} \circ g'_0 \circ f_0 = u^{k_1}_{k_0}$. We put $g_0 := u^{k_1}_{k'_1} \circ g'_0$ so that $g_0 \circ f_0 = u^{k_1}_{k_0}$. We continue this way and build $\mathfrak{S}$-sequences $\{f_n\}_{n\in\omega}$, $\{g_n\}_{n\in\omega}$ as in (BF). Hence, there is an automorphism $h_\infty \in G$ such that for every $m \in \omega$ we have

$$\begin{aligned} h_\infty \circ u^\infty_{k_m} &= u^\infty_{\ell_m} \circ f_m = h_{\varphi(k_m)} \circ u^\infty_{k_m} = h_n \circ u^\infty_{k_m} && \text{for every } n \geqslant \varphi(k_m), \\ h^{-1}_\infty \circ u^\infty_{\ell_m} &= u^\infty_{k_{m+1}} \circ g_m = h^{-1}_{\varphi(\ell_m)} \circ u^\infty_{\ell_m} = h^{-1}_n \circ u^\infty_{\ell_m} && \text{for every } n \geqslant \varphi(\ell_m). \end{aligned}$$

Since the sequences $\{k_m\}_{m\in\omega}$, $\{\ell_m\}_{m\in\omega}$ are strictly increasing, it follows that $\lim_{n\to\infty} h_n = h_\infty$.

If the hom-sets $\mathfrak{S}(u_n, u_m)$ for $n \leqslant m$ are countable, then G is separable since for every n, $\{g \circ V_n : g \in G\}$ is a countable cover by basic open sets. This is because we have the one-to-one map $g \circ V_n \mapsto g \circ u^\infty_n$, and every arrow $g \circ u^\infty_n$ is of the form $u^\infty_m \circ f$ for some $f \in \mathfrak{S}(u_n, u_m)$. □

Construction 2.8 (The topological group $G(\vec{u}, \mathfrak{S})$). We observe that given matching sequences $(\vec{u}, \vec{u}^{\,\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$ and $(\vec{v}, \vec{v}^{\,\infty}, V)$ in $(\mathfrak{S}, \mathfrak{L}')$, every isomorphism of sequences $\vec{\varphi} \colon \vec{u} \to \vec{v}$ canonically induces an isomorphism of the topological groups $\operatorname{Aut}_{\mathfrak{L}}(U)$ and $\operatorname{Aut}_{\mathfrak{L}'}(V)$. Moreover, any sequence $\vec{u}$ in any category $\mathfrak{S}$ admits a canonical matching sequence $(\vec{u}, \{\vec{\imath}_n\}_{n\in\omega}, \vec{u})$ in $(\mathfrak{S}, \sigma_0\mathfrak{S})$ (see Remark 2.2). Therefore, up to a canonical isomorphism, every $\mathfrak{S}$-sequence $\vec{u}$ determines a topological group $G(\vec{u}, \mathfrak{S})$ that is the automorphism group of every associated matching sequence $(\vec{u}, \vec{u}^{\,\infty}, U)$ in every $(\mathfrak{S}, \mathfrak{L})$. Moreover, isomorphic sequences determine isomorphic topological groups.

Proof. Recall that $\vec{u}$ is both a sequence in $\mathfrak{S} \subseteq \sigma_0\mathfrak{S}$ and an object in $\sigma_0\mathfrak{S}$. For every n, we let $\vec{\imath}_n \colon u_n \to \vec{u}$ be the $\sigma_0\mathfrak{S}$-arrow corresponding to $\mathrm{id}_{u_n} \colon u_n \to u_n$, that is, $(\vec{\imath}_n)_k = u_n^{\max(k,n)}$ for $k \in \omega$. It is easy to see that $\vec{u}$ (viewed as a $\sigma_0\mathfrak{S}$-object) is the colimit of itself (viewed as an $\mathfrak{S}$-sequence). Also, for every sequences $\vec{u}, \vec{v}$ the corresponding matching relation is the identity on $\sigma_0\mathfrak{S}(\vec{u}, \vec{v})$. Hence, by Remark 2.2, $(\vec{u}, \{\vec{\imath}_n\}_{n\in\omega}, \vec{u})$ is a colimiting matching sequence in $(\mathfrak{S}, \sigma_0\mathfrak{S})$.

For every matching sequence $(\vec{u}, \vec{u}^{\,\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$ the matching relation between $\operatorname{Aut}(\vec{u})$ and $\operatorname{Aut}(U)$ is an isomorphism of groups by Remark 2.2. Recall that we write $\vec{\varphi} \approx \vec{\psi}$ for every two transformations between sequences that are equivalent, that is, representing the same $\sigma_0\mathfrak{S}$-arrow. Since for every matching pair of automorphisms $\vec{\varphi} \colon \vec{u} \to \vec{u}$ and $\varphi_\infty \colon U \to U$ we have

$$\varphi_\infty \circ u^\infty_n = u^\infty_n \iff u^\infty_{\varphi(n)} \circ \varphi_n = u^\infty_n \iff \exists m\ u^m_{\varphi(n)} \circ \varphi_n = u^m_n \iff \vec{\varphi} \circ \vec{\imath}_n \approx \vec{\imath}_n$$

for every $n \in \omega$, it follows that matching relation is also a homeomorphism between the automorphism groups. To see $\vec{\varphi} \circ \vec{\imath}_n \approx \vec{\imath}_n$, let $k \in \omega$ and $k' := \max(k, n)$, and note that for every $m \geqslant \varphi(k')$ we have $u^m_{\varphi(k')} \circ (\vec{\varphi} \circ \vec{\imath}_n)_k = u^m_{\varphi(k')} \circ (\varphi_{k'} \circ u^{k'}_n) = u^m_{\varphi(n)} \circ \varphi_n$, while $u^m_{k'} \circ (\vec{\imath}_n)_{k'} = u^m_{k'} \circ u^{k'}_n = u^m_n$.

Finally, the isomorphism $\vec{\varphi} \colon \vec{u} \to \vec{v}$ of $\mathfrak{S}$-sequences induces an isomorphism $\vec{\psi} \mapsto \vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1}$ between $\operatorname{Aut}(\vec{u})$ and $\operatorname{Aut}(\vec{v})$. This isomorphism is also a homeomorphism. Let $\{\vec{\imath}_n\}_{n\in\omega}$ and $\{\vec{\jmath}_n\}_{n\in\omega}$ be the cones of the canonical matching sequences for $\vec{u}$ and $\vec{v}$. For every $n \in \omega$, we put $m := (\vec{\varphi}^{-1})(n)$, and we have $\vec{\jmath}_{\varphi(m)} \circ \varphi_m \circ (\vec{\varphi}^{-1})_n \approx \vec{\jmath}_n$. Hence, if $\vec{\psi}$ fixes $\vec{\imath}_m$, then $\vec{\varphi} \circ \vec{\psi} \circ \vec{\varphi}^{-1}$ fixes $\vec{\varphi} \circ \vec{\imath}_m \approx \vec{\jmath}_{\varphi(m)} \circ \varphi_m$, and so it fixes also $\vec{\jmath}_{\varphi(m)} \circ \varphi_m \circ (\vec{\varphi}^{-1})_n \approx \vec{\jmath}_n$. □

Remark 2.9. As a final remark, we observe that given a matching sequence $(\vec{u}, \vec{u}^{\,\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$, it is sufficient for our considerations to use only the part $\mathfrak{L}' \subseteq \mathfrak{L}$ where $\operatorname{Obj}(\mathfrak{L}') = \operatorname{Obj}(\mathfrak{S}) \cup \{U\}$, $\mathfrak{S}$ is a full subcategory of $\mathfrak{L}'$, for every $x \in \operatorname{Obj}(\mathfrak{S})$ we have $\mathfrak{L}'(x, U) = \mathfrak{L}(x, U) = \bigcup_{n\in\omega}(u^\infty_n \circ \mathfrak{S}(x, u_n))$, and $\mathfrak{L}'(U, U) = \operatorname{Aut}_{\mathfrak{L}}(U)$. In the cases we are not interested in the generic object U, only in its automorphism group, we may ignore $\mathfrak{L}$ completely — it is enough to consider $\mathfrak{S}$, $\vec{u}$, and the topological group $G(\vec{u}, \mathfrak{S})$. On the other hand, when it comes to applications, one usually has in mind a larger subcategory $\mathfrak{L}$ consisting of all colimits of sequences in $\mathfrak{S}$.

2.2. *Weak Fraïssé theory*

In this section, we summarize the definitions and theorems of the *weak Fraïssé theory* [12] that will be used later in the paper. We recall the notion of a *weak Fraïssé category* which is a generalization of a *Fraïssé category* [13] (which is itself an abstraction of the classical notion of a *Fraïssé class* [6, §7] of first-order structures). A weak Fraïssé category $\mathfrak{S}$ can be characterized by existence of a *weak Fraïssé sequence* $\vec{u}$. First, we need to recall the concepts of *(weak) amalgamation* and *(weak) domination*.

The weak amalgamation property was introduced by Ivanov [7] and later independently by Kechris and Rosendal [14] in connection with existence of generic automorphisms of homogeneous first-order structures. Since then, ideas of weak Fraïssé theory have been developed and used in many works; see, for example, [11], [4], [9], [12], [10], [23], [17].

Definition 2.10 (Amalgamable arrows). An arrow $e\colon z \to z'$ is called *amalgamable* if for every arrows $f\colon z' \to x$, $g\colon z' \to y$ there exist arrows $f'\colon x \to w$, $g'\colon y \to w$ satisfying

$$f' \circ f \circ e = g' \circ g \circ e.$$

An object z is *amalgamable* if id_z is an amalgamable arrow. A category $\mathfrak{S}$ has the *amalgamation property* if all identity arrows are amalgamable. A category with a weak Fraïssé sequence has the *weak amalgamation property*; namely, for every object a there exists an amalgamable arrow with domain a.

Definition 2.11 (Weak Fraïssé sequence). A sequence $\vec{u}$ in a category $\mathfrak{S}$ is called a *weak Fraïssé sequence* if is satisfies the following conditions.

(W0) (*Cofinality*) For every $x \in \mathrm{Obj}(\mathfrak{S})$, there is $n \in \omega$ such that $\mathfrak{S}(x, u_n) \neq \emptyset$.
(W1) (*Weak absorption*) For every n, there is $m \geqslant n$ such that for every $\mathfrak{S}$-arrow $f\colon u_m \to y$ there are $\ell > m$ and an $\mathfrak{S}$-arrow $g\colon y \to u_\ell$ such that $g \circ f \circ u_n^m = u_n^\ell$.

A weak Fraïssé sequence $\vec{u}$ is called *normalized* if $m = n+1$ works in condition (W1). It is called a *Fraïssé sequence* [13] if one can always take $m = n$ in condition (W1). Note that every cofinal subsequence of a weak Fraïssé sequence is weak Fraïssé; therefore, we may restrict attention to normalized weak Fraïssé sequences.

Remark 2.12. Note that a number $m \geqslant n$ works in (W1) if and only if the arrow u_n^m is amalgamable. To prove that u_n^m is amalgamable one can consider $f_0\colon u_m \to y_0$, $f_1\colon u_m \to y_1$, obtaining $g_0\colon y_0 \to u_\ell$, $g_1\colon y_1 \to u_\ell$ satisfying

$$g_0 \circ f_0 \circ u_n^m = u_n^\ell = g_1 \circ f_1 \circ u_n^m.$$

The other implication is essentially [12, Lemma 3.10].

Remark 2.13. By [12, Proposition 3.6 and Corollary 3.12], a sequence isomorphic to a weak Fraïssé sequence is itself a weak Fraïssé sequence, and every two weak Fraïssé sequences are isomorphic.

The next definition is analogous to the definition of weak Fraïssé sequence but applies to subcategories. In fact, both concepts could be unified by a notion of a *weakly dominating functor*.

Definition 2.14 (Weakly dominating subcategory). A subcategory $\mathfrak{C} \subseteq \mathfrak{S}$ is called *weakly dominating* if it satisfies the following conditions.

(D0) (*Cofinality*) For every $x \in \mathrm{Obj}(\mathfrak{S})$, there is $y \in \mathrm{Obj}(\mathfrak{C})$ such that $\mathfrak{S}(x, y) \neq \emptyset$.
(D1) (*Weak absorption*) For every $x \in \mathrm{Obj}(\mathfrak{C})$, there is a $\mathfrak{C}$-arrow $e\colon x \to y$ such that for every $\mathfrak{S}$-arrow $f\colon y \to z$ there is an $\mathfrak{S}$-arrow $g\colon z \to w$ such that $g \circ f \circ e \in \mathfrak{C}$.

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$\mathfrak{C}$ is called *dominating* if additionally $e = \mathrm{id}_x$ works in (D1). Note that every full cofinal subcategory is dominating.

Lemma 2.15 [12, Lemma 3.4]. *Let $\mathfrak{C} \subseteq \mathfrak{S}$ be a weakly dominating subcategory. Then every weak Fraïssé sequence in $\mathfrak{C}$ is also a weak Fraïssé sequence in $\mathfrak{S}$.*

The following lemma, which we shall use later, was essentially proved in the proof of [12, Proposition 2.7].

Lemma 2.16. *Let $\mathfrak{C} \subseteq \mathfrak{S}$ be a dominating subcategory, and let $e\colon z \to z'$ be a $\mathfrak{C}$-arrow. If e is amalgamable in $\mathfrak{C}$, then e is amalgamable in $\mathfrak{S}$.*

Proof. Let $f\colon z' \to x$ and $g\colon z' \to y$ be $\mathfrak{S}$-maps. By domination, there are $\mathfrak{S}$-arrows $f'\colon x \to x'$ and $g'\colon y \to y'$ such that $f' \circ f$, $g' \circ g \in \mathfrak{C}$. Hence, there are $\mathfrak{C}$-arrows $f''\colon x' \to w$ and $g''\colon y' \to w$ such that $(f'' \circ f') \circ f \circ e = (g'' \circ g') \circ g \circ e$. □

Definition 2.17 (Weak Fraïssé category). We say that $\mathfrak{S}$ is a *weak Fraïssé category* if it is *directed* (i.e., for every $x, y \in \mathrm{Obj}(\mathfrak{S})$ there is $z \in \mathrm{Obj}(\mathfrak{S})$ such that $\mathfrak{S}(x, z) \neq \emptyset$ and $\mathfrak{S}(y, z) \neq \emptyset$), has the weak amalgamation property and is weakly dominated by a countable subcategory.

The category $\mathfrak{S}$ is a *Fraïssé category* if it additionally has the amalgamation property. In this case, it is even dominated by a countable subcategory.

In the classical case of categories of structures and embeddings, being directed is often called the *joint embedding property*. Note that a category is identified with its collection of morphisms, so a category is countable if it has countably many morphisms (as opposed to just having countably many objects). However, for a category of finite structures, having countably many isomorphism types is sufficient for being dominated by a countable subcategory.

Theorem 2.18. *The following conditions are equivalent for a category $\mathfrak{S}$.*

(i) *$\mathfrak{S}$ is a weak Fraïssé category.*
(ii) *$\mathfrak{S}$ is weakly dominated by a countable weak Fraïssé category.*
(iii) *$\mathfrak{S}$ has a weak Fraïssé sequence.*

This is [12, Theorem 3.7] together with the fact (implicitly used in the proof) that every countable weakly dominating subcategory of a weak Fraïssé category can be extended by countably many arrows to become directed and to have the weak amalgamation property.

Construction 2.19 (the topological group $G(\mathfrak{S})$). Let $\mathfrak{S}$ be a weak Fraïssé category. By Construction 2.8 and Remark 2.13, we have that every two weak Fraïssé sequences $\vec{u}$, $\vec{v}$ are isomorphic, and so $G(\vec{u}, \mathfrak{S})$ are $G(\vec{v}, \mathfrak{S})$ are isomorphic topological groups. Hence, we may denote this topological group determined uniquely up to isomorphism by $G(\mathfrak{S})$. Recall that it is isomorphic to $\mathrm{Aut}(U)$ for every matching weak Fraïssé sequence $(\vec{u}, \vec{u}^\infty, U)$ in every $(\mathfrak{S}, \mathfrak{L})$.

Next, we recall the key properties of a *weak Fraïssé limit*, generalizing the extension property / injectivity and homogeneity from Fraïssé theory. Again, see [12] for details.

Definition 2.20. Let $\mathfrak{S} \subseteq \mathfrak{L}$ be categories. An $\mathfrak{L}$-object U is

- *cofinal* in $(\mathfrak{S}, \mathfrak{L})$ if $\mathfrak{L}(x, U) \neq \emptyset$ for every $\mathfrak{S}$-object x,
- *weakly injective* in $(\mathfrak{S}, \mathfrak{L})$ if for every $\mathfrak{L}$-arrow $f\colon x \to U$ from an $\mathfrak{S}$-object there is an $\mathfrak{S}$-arrow $e\colon x \to x'$ such that for every $\mathfrak{S}$-arrow $g\colon x' \to y$ there is an $\mathfrak{L}$-arrow $h\colon y \to U$ such that $f = h \circ g \circ e$,
- *weakly homogeneous* if $(\mathfrak{S}, \mathfrak{L})$ if for every $\mathfrak{L}$-arrow $f\colon x \to U$ from and $\mathfrak{S}$-object there is an $\mathfrak{S}$-arrow $e\colon x \to x'$ and an $\mathfrak{L}$-arrow $f'\colon x' \to U$ with $f = f' \circ e$ such that for every $\mathfrak{L}$-arrow $g\colon x' \to U$ there is $h \in \mathrm{Aut}(U)$ such that $f = h \circ g \circ e$.

Note that every cofinal weakly homogeneous object is weakly injective and that the arrow e witnessing the weak homogeneity for f has the property that for every $\mathfrak{L}$-arrows $g, g'\colon x' \to U$ there is $h \in \mathrm{Aut}(U)$

such that $h \circ g = g'$. We say that U is *homogeneous at e* in that case. So U is weakly homogeneous if every $\mathfrak{L}$-arrow from an $\mathfrak{S}$-object to U factorizes through an $\mathfrak{S}$-arrow U is homogeneous at.

Theorem 2.21 (Characterization of the weak Fraïssé limit). *For a matching sequence $(\vec{u}, \vec{u}^{\infty}, U)$ in a pair of categories $(\mathfrak{S}, \mathfrak{L})$, the following conditions are equivalent:*

(i) *$\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{S}$.*
(ii) *U is a cofinal and weakly injective object in $(\mathfrak{S}, \mathfrak{L})$.*
(iii) *U is a cofinal and weakly homogeneous object in $(\mathfrak{S}, \mathfrak{L})$.*

Moreover, given the conditions above hold, we have the following.

(a) *There exists an $\mathfrak{L}$-arrow $X \to U$ for every $\mathfrak{L}$-object X that is an $\mathfrak{S}$-object or is an $\mathfrak{L}$-colimit of a sequence of amalgamable arrows in $\mathfrak{S}$.*
(b) *U is homogeneous at an $\mathfrak{S}$-arrow e if and only if e is amalgamable. Hence, e works for $f : x \to U$ in the weak homogeneity of U if and only if e is amalgamable and f factorizes through e.*
(c) *An $\mathfrak{S}$-arrow $e : x \to x'$ works for $f : x \to U$ in the weak injectivity of U if and only if e is amalgamable and f factorizes through e. In particular, id_x for an amalgamable object x works for every arrow f.*

The theorem is essentially proved in [12]: see Theorem 4.2, Corollary 4.5, Theorem 4.6 and Theorem 5.1. However, we use weaker assumptions here — we neither assume that $(\vec{u}^{\infty}, U)$ is a colimit of $\vec{u}$ nor that $\mathfrak{S}$-sequences have colimits in $\mathfrak{L}$ nor that $\mathfrak{L}$-arrows are monic. The only assumptions that are used in the original proof are covered by the notion of a matching sequence. Given all the necessary definitions and conditions, the proof is quite direct.

Remark 2.22. In the case that $\mathfrak{S}$ is a category of some finitely generated first-order structures and all embeddings (or all one-to-one homomorphism) and $\mathfrak{L} = \sigma\mathfrak{S}$ (as defined in Construction 2.5), then $\mathfrak{L}$-objects are exactly colimits of $\mathfrak{S}$-sequences, and every such colimit sequence $(\vec{u}, \vec{u}^{\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$ is matching, so Theorem 2.21 applies. Moreover, the weak Fraïssé limit U is unique up to isomorphism in $\mathfrak{L}$ in this case, and every $\mathfrak{S}$-sequence with colimit U is weak Fraïssé. If additionally $\mathfrak{S}$ has the amalgamation property (and so is a Fraïssé category), there is an $\mathfrak{L}$-map $X \to U$ for every $\mathfrak{L}$-object X, that is, the Fraïssé limit U is cofinal in $\mathfrak{L}$.

3. Main results

The purpose of this section is to prove the announced characterization of extreme amenability. We first define the weak Ramsey property of $\mathfrak{S}$ (see Definition 3.1 below) and prove that it is equivalent to a Ramsey-like property for $\mathfrak{L}$-arrows into U. We also note that the weak Ramsey property implies the weak amalgamation property. Next, we show that these properties are equivalent to the extreme amenability of $\mathrm{Aut}(U)$. Together, the weak Ramsey property of a weak Fraïssé category $\mathfrak{S}$ is equivalent to the extreme amenability of $G(\mathfrak{S})$.

3.1. The weak Ramsey property

For an arrow $\alpha : a \to a'$ and an object b in a category $\mathfrak{C}$, we shall write $\mathfrak{C}(\alpha, b)$ as a shortcut for $\mathfrak{C}(a', b) \circ \alpha$. Below, we introduce the main definition.

Definition 3.1. We say that a category $\mathfrak{S}$ has the *weak Ramsey property* if for every $a \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $\alpha : a \to a'$ satisfying:

(wR) For every $b \in \mathrm{Obj}(\mathfrak{S})$, for every $k \in \omega$, for every finite $F \subseteq \mathfrak{S}(\alpha, b)$ there is $v \in \mathrm{Obj}(\mathfrak{S})$ such that for every $\varphi : \mathfrak{S}(\alpha, v) \to k$ there exists $e : b \to v$ such that φ is constant on $e \circ F$.

We call such α a *Ramsey arrow*. We say that $\mathfrak{S}$ has the *Ramsey property* if for every $a \in \mathrm{Obj}(\mathfrak{S})$ the identity id_a is a Ramsey arrow.

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Recall that a category $\mathfrak{S}$ is *locally finite* if $\mathfrak{S}(a, b)$ is finite for every $a, b \in \mathrm{Obj}(\mathfrak{S})$. Note that for a locally finite category $\mathfrak{S}$, the condition (wR) simplifies since we may consider just $F = \mathfrak{S}(\alpha, b)$. Hence, our notion of Ramsey property simplifies to the standard one (e.g., [18, §3]).

The following lemma generalizes [21, Theorem 4.2 (i)].

Lemma 3.2. *Let $\mathfrak{S}$ be a directed category, and let $\alpha: a \to a'$ be an $\mathfrak{S}$-arrow. If α is a Ramsey arrow, then α is amalgamable. Hence, a directed category with the weak Ramsey property has the weak amalgamation property.*

Proof. Suppose $\alpha: a \to a'$ is a Ramsey arrow, and let $k = 2$. Fix $f_0, f_1 \in \mathfrak{S}$ with $\mathrm{dom}(f_0) = a' = \mathrm{dom}(f_1)$. Using directedness, choose $b \in \mathrm{Obj}(\mathfrak{S})$ and $g_0, g_1 \in \mathfrak{S}$ such that $g_i \circ f_i \in \mathfrak{S}(a', b)$ for $i = 0, 1$. Of course, g_0, g_1 are independent of f_0, f_1 and there is no reason for the equality $g_0 \circ f_0 = g_1 \circ f_1$. Let $F = \{g_0 \circ f_0 \circ \alpha, g_1 \circ f_1 \circ \alpha\}$.

Now, find $v \in \mathrm{Obj}(\mathfrak{S})$ from the weak Ramsey property applied to F. Define $\varphi: \mathfrak{S}(\alpha, v) \to 2$ by setting $\varphi(g) = 1$ if and only if $g = g' \circ f_1 \circ \alpha$ for some $g' \in \mathfrak{S}$. By (wR), there exists $e: b \to v$ such that φ is constant on $e \circ F$. Note that $\varphi(e \circ (g_1 \circ f_1) \circ \alpha) = 1$ by associativity. Thus, also $\varphi(e \circ (g_0 \circ f_0) \circ \alpha) = 1$, which means that there exists h such that

$$e \circ g_0 \circ f_0 \circ \alpha = h \circ f_1 \circ \alpha.$$

We are done, because $e \circ g_0$ and h witness the weak amalgamation. □

Recall that if one of arrows α, β is amalgamable, then so is $\beta \circ \alpha$; see [12, Lemma 2.5]. The same composition behavior is true also for Ramsey arrows.

Lemma 3.3. *Let $\alpha: a \to a'$ and $\beta: a' \to a''$ be arrows in a category $\mathfrak{S}$. If α or β is Ramsey, then $\beta \circ \alpha$ is Ramsey.*

Proof. Suppose α is Ramsey, and let $b \in \mathrm{Obj}(\mathfrak{S})$, $k \in \omega$, and $F \subseteq \mathfrak{S}(\beta \circ \alpha, b)$ finite. Since $\mathfrak{S}(\beta \circ \alpha, b) \subseteq \mathfrak{S}(\alpha, b)$, we may take $v \in \mathrm{Obj}(\mathfrak{S})$ for α, b, k, F. Let $\varphi: \mathfrak{S}(\beta \circ \alpha, v) \to k$ be a coloring. We extend it to $\varphi': \mathfrak{S}(\alpha, v) \to k$. By the choice of v, there is $e: b \to v$ such that φ' is constant on $e \circ F$. But $e \circ F \subseteq \mathfrak{S}(\beta \circ \alpha, v)$, so we are done.

Next, suppose that β is Ramsey. Again, let $b \in \mathrm{Obj}(\mathfrak{S})$, $k \in \omega$, and $F \subseteq \mathfrak{S}(\beta \circ \alpha, b)$ finite. There is finite $F' \subseteq \mathfrak{S}(\beta, b)$ such that $F' \circ \alpha = F$. Take $v \in \mathrm{Obj}(\mathfrak{S})$ for β, b, k, F'. Let $\varphi: \mathfrak{S}(\beta \circ \alpha, v) \to k$ be a coloring, and define $\varphi': \mathfrak{S}(\beta, v) \to k$ by $\varphi'(\xi) := \varphi(\xi \circ \alpha)$. By the choice of v, there is $e: b \to v$ such that φ' is constant on $e \circ F'$. Hence, φ is constant on $e \circ F' \circ \alpha = e \circ F$. □

Lemma 3.4. *Let $\alpha: a \to a'$ and $\beta: a' \to a''$ be arrows in a category $\mathfrak{S}$. If α is amalgamable and $\beta \circ \alpha$ is Ramsey, then α is Ramsey.*

Proof. Let $b \in \mathrm{Obj}(\mathfrak{S})$, $k \in \omega$ and $F \subseteq \mathfrak{S}(\alpha, b)$ finite. First, we prove that there is $b' \in \mathrm{Obj}(\mathfrak{S})$ and $g: b \to b'$ such that $F' := g \circ F \subseteq \mathfrak{S}(\beta \circ \alpha, b')$. Enumerate F as $\{f_i : i < n\}$. Since α is amalgamable, there are $\mathfrak{S}$-arrows g_0 and h_0 such that $g_0 \circ f_0 = h_0 \circ \beta \circ \alpha$. Then there are g_1 and h_1 such that $g_1 \circ (g_0 \circ f_1) = h_1 \circ (h_0 \circ \beta \circ \alpha)$. Note that there is no reason why $g_1 \circ g_0 \circ f_0$ and $g_1 \circ g_0 \circ f_1$ should be equal, but they are certainly both factorizing through $\beta \circ \alpha$. We continue the same way and finally put $g := g_{n-1} \circ \cdots \circ g_0$ and $b' := \mathrm{cod}(g)$.

Since $\beta \circ \alpha$ is Ramsey, there is $v \in \mathrm{Obj}(\mathfrak{S})$ for b', k, F'. Let $\varphi: \mathfrak{S}(\alpha, v) \to k$ be a coloring. Since $\mathfrak{S}(\beta \circ \alpha, v) \subseteq \mathfrak{S}(\alpha, v)$, there is $e': b' \to v$ such that φ is constant on $e' \circ F' = e' \circ g \circ F =: e \circ F$. □

Corollary 3.5. *Let $\mathfrak{S}$ be a directed category with the weak Ramsey property. An $\mathfrak{S}$-arrow $\alpha: a \to a'$ is Ramsey if and only if it is amalgamable. Hence, $\mathfrak{S}$ has the weak amalgamation property, and it has the Ramsey property if and only if it has the amalgamation property.*

Proof. Every Ramsey arrow is amalgamable by Lemma 3.2. Let $\alpha: a \to a'$ be amalgamable. By the weak Ramsey property, there is a Ramsey arrows $\beta: a' \to a''$. By Lemmma 3.3, $\beta \circ \alpha$ is Ramsey, and by Lemma 3.4, α is Ramsey. □

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Proposition 3.6.

(i) *Let $F\colon \mathfrak{S} \to \mathfrak{S}'$ be a full cofinal functor (where cofinal means that for every $\mathfrak{S}'$-object x there is a $\mathfrak{S}$-object y and an $\mathfrak{S}'$-arrow $f\colon x \to F(y)$). If $\alpha\colon a \to a'$ is a Ramsey arrow in $\mathfrak{S}$, then $F(\alpha)$ is a Ramsey arrow in $\mathfrak{S}'$.*

(ii) *Let $\mathfrak{S} \subseteq \mathfrak{S}'$ be a full cofinal subcategory. An $\mathfrak{S}$-arrow $\alpha\colon a \to a'$ is Ramsey in $\mathfrak{S}$ if and only if it is Ramsey in $\mathfrak{S}'$.*

Proof. (i) Let $b' \in \mathrm{Obj}(\mathfrak{S}')$, $H' \subseteq \mathfrak{S}'(F(\alpha), b')$ finite, and $k \in \omega$. By the cofinality, there is $b \in \mathrm{Obj}(\mathfrak{S})$ and $f \in \mathfrak{S}'(b', F(b))$, and by the fullness there is finite $H \subseteq \mathfrak{S}(\alpha, b)$ such that $F[H] = f \circ H'$. There is also $v \in \mathrm{Obj}(\mathfrak{S})$ witnessing that α is Ramsey for b, H, k. For every coloring $\varphi'\colon \mathfrak{S}'(F(\alpha), F(v)) \to k$, there is the coloring $\varphi := (\varphi' \circ F)\colon \mathfrak{S}(\alpha, v) \to k$, and $g \in \mathfrak{S}(b, v)$ such that φ is constant on $g \circ H$, and so φ' is constant on $F[g \circ H] = (F(g) \circ f) \circ H'$.

The forward implication of (ii) follows from (i). For the backward implication, let $b \in \mathrm{Obj}(\mathfrak{S})$, $H \subseteq \mathfrak{S}(\alpha, b)$ finite and $k \in \omega$. Let $v' \in \mathrm{Obj}(\mathfrak{S}')$ be the corresponding witnessing object, and let $f \in \mathfrak{S}'(v', v)$ for some $v \in \mathrm{Obj}(\mathfrak{S})$. Every coloring $\varphi\colon \mathfrak{S}(\alpha, v) \to k$ induces the coloring $\varphi'\colon \mathfrak{S}'(\alpha, v') \to k$ defined by $\varphi'(g) := \varphi(f \circ g)$. There is $g \in \mathfrak{S}(b, v')$ such that φ' is constant on $g \circ H$, and so φ is constant on $(f \circ g) \circ H$. $\square$

Corollary 3.7. *Let $\mathfrak{S} \subseteq \mathfrak{S}'$ be a full cofinal subcategory. $\mathfrak{S}$ has the weak Ramsey property if and only if $\mathfrak{S}'$ has the weak Ramsey property.*

Definition 3.8. Let $\mathfrak{S} \subseteq \mathfrak{L}$ be categories, and let U be a fixed $\mathfrak{L}$-object. We say that an $\mathfrak{S}$-arrow $\alpha\colon a \to a'$ satisfies

(wA) if for every $k \in \omega$, for every finite $F \subseteq \mathfrak{L}(\alpha, U)$, for every $\varphi\colon \mathfrak{L}(\alpha, U) \to k$ there is $e \in \mathrm{Aut}(U)$ such that φ is constant on $e \circ F$,

(wB) if the map e in (wA) is required to be only an endomorphism instead of an automorphism of U, that is, if for every $k \in \omega$, for every finite $F \subseteq \mathfrak{L}(\alpha, U)$, for every $\varphi\colon \mathfrak{L}(\alpha, U) \to k$ there is $e \in \mathfrak{L}(U, U)$ such that φ is constant on $e \circ F$.

We say that U has the *weak finitary big Ramsey property* in $(\mathfrak{S}, \mathfrak{L})$ if for every $a \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $\alpha\colon a \to a'$ satisfying (wB). Similarly, we say that U has the *finitary big Ramsey property* if every id_a, $a \in \mathrm{Obj}(\mathfrak{S})$, satisfies (wB).

Note that for (wA) and (wB), given α, k, φ there is a constant value $i \in k$ that works for every finite $F \subseteq \mathfrak{L}(\alpha, U)$. Otherwise, there would be a counterexample set F_i for every $i \in k$, and so $F := \bigcup_{i \in k} F_i$ would be a counterexample for (wA) or (wB).

Remark 3.9. The name '(weak) finitary big Ramsey property' was chosen to stress the formal similarity to the standard *big Ramsey property*, that is, *big Ramsey degree* [8, p. 176] equal to one. In that (rare) case, e does not depend on F, and φ is constant on the whole set $e \circ \mathfrak{L}(\alpha, U)$.

Theorem 3.10. *Let $(\vec{u}, \vec{u}^\infty, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$ for some categories $\mathfrak{S} \subseteq \mathfrak{L}$, and let $\alpha\colon a \to a'$ be an $\mathfrak{S}$-arrow. If $\vec{u}$ is a weak Fraïssé sequence, then the following conditions are equivalent:*

(i) *α is a Ramsey arrow, that is, it satisfies (wR),*

(ii) *α satisfies (wA),*

(iii) *α satisfies (wB).*

It follows that $\mathfrak{S}$ has the (weak) Ramsey property if and only if U has the (weak) finitary big Ramsey property in $(\mathfrak{S}, \mathfrak{L})$.

Proof. By (F1), for every $f \in \mathfrak{L}(\alpha, U)$ there exists $n_f \in \omega$ and $f' \in \mathfrak{S}(\alpha, u_n)$ such that $u_n^\infty \circ f' = f$. For every $n \geqslant n_f$, let us put $\Delta^n(f) := u_{n_f}^n \circ f'$, so we have $f = u_n^\infty \circ \Delta^n(f)$ and $\Delta^{n'}(f) = u_n^{n'} \circ \Delta^n(f)$ for every $n' \geqslant n \geqslant n_f$. Note that by (F2) we have also $\Delta^n(u_m^\infty \circ f) = u_m^n \circ f$ for every compatible $\mathfrak{S}$-arrow f and every sufficiently large n.

The implication (wA) $\Longrightarrow$ (wB) is trivial.

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Next, we prove (wB) $\Longrightarrow$ (wR) by contradiction. Suppose α fails (wR) and it is witnessed by $k \in \omega$, $b \in \operatorname{Obj}(\mathfrak{S})$ and a finite $F \subseteq \mathfrak{S}(\alpha, b)$. Specifically, for every $n \in \omega$ there is $\varphi_n \colon \mathfrak{S}(\alpha, u_n) \to k$ such that $\varphi_n[e \circ F]$ is not a singleton whenever $e \in \mathfrak{S}(b, u_n)$. Fix a nonprincipal ultrafilter p on ω, and define $\varphi \colon \mathfrak{L}(\alpha, U) \to k$ by setting

$$\varphi(f) = i \iff \{n \in \omega \colon \varphi_n(\Delta^n(f)) = i\} \in p.$$

Note that $\Delta^n(f)$ is defined for all but finitely many numbers n. Since $\vec{u}$ is a weak Fraïssé sequence, there is an $\mathfrak{S}$-map $e_0 \colon b \to u_{n_0}$ for some $n_0 \in \omega$. We consider $F' := u_{n_0}^\infty \circ e_0 \circ F \subseteq \mathfrak{L}(\alpha, U)$. Since α satisfies (wB), there is $j \in k$ and $e' \in \mathfrak{L}(U, U)$ such that $\varphi[e' \circ F'] = \{j\}$. By (F1), we find $m \in \omega$ and $e \in \mathfrak{S}(b, u_m)$ such that $e' \circ u_{n_0}^\infty \circ e_0 = u_m^\infty \circ e$, as in the diagram below.

$$\begin{tikzcd}
u_0 \arrow[r] & \cdots \arrow[r] & u_{n_0} \arrow[r] & \cdots \arrow[r] & u_m \arrow[r] & \cdots \arrow[r] & U \arrow[dd, "e'"] \\
& b \arrow[ur, "e_0"] \arrow[drrr, "e"] & & & & & \\
u_0 \arrow[r] & \cdots \arrow[r] & u_{n_0} \arrow[r] & \cdots \arrow[r] & u_m \arrow[r] & \cdots \arrow[r] & U.
\end{tikzcd}$$

Given $f \in F$, we have $\Delta^n(u_m^\infty \circ e \circ f) = u_m^n \circ e \circ f$ for every sufficiently large n, and hence

$$j = \varphi(e' \circ u_{n_0}^\infty \circ e_0 \circ f) = \varphi(u_m^\infty \circ e \circ f) = \lim_{n \to p} \varphi_n(\Delta^n(u_m^\infty \circ e \circ f)) = \lim_{n \to p} \varphi_n(u_m^n \circ e \circ f).$$

The last limit along p means that the set $A_f := \{n \geqslant m \colon \varphi_n(u_m^n \circ e \circ f) = j\} \in p$. Since F is finite, we may find $\ell > m$ such that $\varphi_\ell(u_m^\ell \circ e \circ f) = j$ for every $f \in F$. It is enough to choose $\ell \in \bigcap_{f \in F} A_f$. Together, φ_ℓ restricted to $(u_m^\ell \circ e) \circ F$ is constant, which is a contradiction.

To prove (wR) $\Longrightarrow$ (wA), let $\alpha \colon a \to a'$ be a Ramsey $\mathfrak{S}$-arrow. Fix $k \in \omega$, fix finite $F \subseteq \mathfrak{L}(\alpha, U)$ and fix $\varphi \colon \mathfrak{L}(\alpha, U) \to k$. Our goal is to find $e \in \operatorname{Aut}(U)$ such that φ is constant on $e \circ F$.

Since F is finite, there is $m \in \omega$ such that $f = u_m^\infty \circ \Delta^m(f)$ for every $f \in F$. Since $\vec{u}$ is a weak Fraïssé sequence, there is $m' \geqslant m$ such that $\beta := u_m^{m'}$ works in (W1), or equivalently is amalgamable. Let $b := u_{m'}$ and $F' := \{\beta \circ \Delta^m(f) \colon f \in F\} \subseteq \mathfrak{S}(\alpha, b)$. Using (wR) with F' and b, we obtain $v \in \operatorname{Obj}(\mathfrak{S})$ such that for every $\psi \colon \mathfrak{S}(\alpha, v) \to k$ there is $e' \colon b \to v$ with ψ constant on the set $e' \circ F'$. Note that there exists at least one ψ as above (unless $k = 0$, in which case (wA) is trivially true). Consequently, there exists an $\mathfrak{S}$-arrow $\gamma \colon b \to v$. Recalling that $\beta = u_m^{m'}$ is amalgamable, we find an $\mathfrak{L}$-arrow $\delta \colon v \to U$ such that

$$u_m^\infty = \delta \circ \gamma \circ \beta.$$

The following diagram should clarify the situation.

$$\begin{tikzcd}
a \arrow[rrrr, "f"] \arrow[dr, "\Delta^m(f)"'] & & & & U \arrow[dd, "e"] \\
& u_m \arrow[r, "\beta"'] \arrow[urrr, "u_m^\infty"] & u_{m'} = b \arrow[r, "\gamma"'] \arrow[dr, "e'"'] & v \arrow[ur, "\delta"'] & \\
& & & v \arrow[r, "\delta"] & U.
\end{tikzcd}$$

Define $\tilde{\varphi} \colon \mathfrak{S}(\alpha, v) \to k$ by

$$\tilde{\varphi}(\xi) = \varphi(\delta \circ \xi) \qquad \text{for every } \xi \in \mathfrak{S}(\alpha, v).$$

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The weak Ramsey property gives $e' \colon b \to v$ such that

$$\tilde{\varphi}(e' \circ \beta \circ \Delta^m(f)) = j$$

for every $f \in F$, where $j \in k$ is fixed. Now, we use the weak homogeneity of U (Theorem 2.21), knowing that β is amalgamable. Namely, there exists $e \in \mathrm{Aut}(U)$ such that

$$e \circ \delta \circ \gamma \circ \beta = \delta \circ e' \circ \beta.$$

Finally, given $f \in F$, we have

$$\begin{aligned}\varphi(e \circ f) &= \varphi(e \circ u_m^\infty \circ \Delta^m(f)) = \varphi(e \circ \delta \circ \gamma \circ \beta \circ \Delta^m(f)) \\ &= \varphi(\delta \circ e' \circ \beta \circ \Delta^m(f)) = \tilde{\varphi}(e' \circ \beta \circ \Delta^m(f)) = j.\end{aligned}$$

This completes the proof. □

3.2. Extreme amenability

Recall that an action $G \curvearrowright X$ of a group G on a set X is a group homomorphism $\eta \colon G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the the group of all bijections of X. We shall write gx or $g \cdot x$ instead of $\eta(g)(x)$. This way an action can be equivalently viewed as a map $G \times X \to X$ such that $1x = x$ and $g(hx) = (gh)x$ for $g, h \in G$ and $x \in X$. Given $G \curvearrowright X$, the *orbit* of $x_0 \in X$ is $Gx_0 := \{gx_0 \colon g \in G\}$. The action is *transitive* if $Gx_0 = X$ for some x_0 (equivalently: for every x_0).

A morphism $\pi \colon \eta \to \eta'$ of actions $\eta \colon G \curvearrowright X$ and $\eta' \colon G \curvearrowright Y$ is a mapping $\pi \colon X \to Y$ such that $\pi(gx) = g\pi(x)$ for every $g \in G$ and $x \in X$, or equivalently $\pi \circ \eta(g) = \eta'(g) \circ \pi$ for every $g \in G$.

An action $G \curvearrowright X$ of a topological group G on a topological space X is *continuous* if it is continuous when viewed as a mapping $G \times X \to X$ with respect to the product topology on $G \times X$. Recall that a topological group G is called *extremely amenable* if every continuous action $G \curvearrowright X$ on a compact space X has a *fixed point*, that is, there is a point $x_0 \in X$ such that $gx_0 = x_0$ for every $g \in G$.

Definition 3.11. An action $G \curvearrowright X$ of a group G on a set X is *finitely oscillation stable* if

(FS) for every $k \in \omega$, for every $\varphi \colon X \to k$, and for every finite set $F \subseteq X$ there exists $g \in G$ such that φ is constant on gF.

This is an equivalent formulation of the standard finite oscillation stability [24, 1.1] of a discrete space X; see [24, Theorem 1.1.18 (7)]. Note that in the situation of Definition 3.1 an $\mathfrak{S}$-arrow α satisfies (wA) if and only if the action $\mathrm{Aut}(U) \curvearrowright \mathfrak{L}(\alpha, U)$ satisfies (FS).

Proposition 3.12. *Let G be a topological group with a neighborhood base $\mathcal{V}$ of its unit, consisting of open subgroups. The following properties are equivalent.*

(a) *G is extremely amenable.*
(b) *For every $V \in \mathcal{V}$, the action $G \curvearrowright G/V$ on left cosets ($g \cdot hV = ghV$) satisfies (FS).*

The proof can be found essentially in [8, Prop. 4.2], where it is assumed that G is a closed subgroup of S_∞; however, the proof uses exclusively the existence of a neighborhood base $\mathcal{V}$ as above (see also the remarks after [8, Prop. 4.2]). Recall that a topological group G embeds into S_∞ as a closed subgroup if and only if it is a non-Archimedean Polish group, that is, if it has a countable neighborhood base $\mathcal{V}$ of the unit consisting of open subgroups and is separable.

Remark 3.13. Let $(\vec{u}, \vec{u}^\infty, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$. Recall that a basic neighborhood of the identity $\mathrm{id}_U \in G := \mathrm{Aut}(U)$ is of the form

$$V_m = \{g \in G \colon g \circ u_m^\infty = u_m^\infty\},$$

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where $m \in \omega$. This is obviously a subgroup of G. In fact, it is the stabilizer of u_m^∞ of the action $G \curvearrowright \mathfrak{L}(u_m, U)$. Hence, the map $\pi\colon G/V_m \to G \circ u_m^\infty$ defined by $h \circ V_m \mapsto h \circ u_m^\infty$ is an isomorphism of the action $G \curvearrowright G/V_m$ on left cosets and the action $G \curvearrowright G \circ u_m^\infty$ of the automorphism group on the orbit $G \circ u_m^\infty \subseteq \mathfrak{L}(u_m, U)$. Moreover, if $m' \geqslant m$ is such that $u_m^{m'}$ is amalgamable, then by the weak homogeneity the orbit $G \circ u_m^\infty$ is the whole $\mathfrak{L}(u_m^{m'}, U)$.

Theorem 3.14. *Let $(\vec{u}, \vec{u}^{\,\infty}, U)$ be a matching sequence in $(\mathfrak{S}, \mathfrak{L})$ for some categories $\mathfrak{S} \subseteq \mathfrak{L}$. If $\vec{u}$ is a weak Fraïssé sequence, then the following conditions are equivalent.*

(i) $\mathrm{Aut}(U)$ *is extremely amenable.*
(ii) *U has the weak finitary big Ramsey property in $(\mathfrak{S}, \mathfrak{L})$.*
(iii) *$\mathfrak{S}$ has the weak Ramsey property.*

Proof. Put $G := \mathrm{Aut}(U)$. Let $m \in \omega$ and $m' \geqslant m$ such that $u_m^{m'}$ is amalgamable. By the previous remark, we have an isomorphism of the actions $G \curvearrowright G/V_m$ and $G \curvearrowright \mathfrak{L}(u_m^{m'}, U)$. Therefore, $G \curvearrowright G/V_m$ satisfies (FS) if and only if $u_m^{m'}$ satisfies (wA), or by Theorem 3.10 equivalently (wR).

Suppose G is extremely amenable. For every $a \in \mathrm{Obj}(\mathfrak{S})$, there is an $\mathfrak{S}$-arrow $f\colon a \to u_m$ for some $m \in \omega$, and there is $m' \geqslant m$ such that $u_m^{m'}$ is amalgamable. By Proposition 3.12 and the claim above, $u_m^{m'}$ is a Ramsey arrow, and so $\alpha := u_m^{m'} \circ f$ is a Ramsey arrow as well by Lemma 3.3. Hence, $\mathfrak{S}$ has the weak Ramsey property.

Suppose $\mathfrak{S}$ has the weak Ramsey property. For every $m \in \omega$, there is $m' \geqslant m$ such that $u_m^{m'}$ amalgamable. By Lemma 3.4, $u_m^{m'}$ is a Ramsey arrow, and so by the claim above, $G \curvearrowright G/V_m$ satisfies (FS). It follows from Proposition 3.12 that G is extremely amenable.

We have (ii) $\iff$ (iii) already by Theorem 3.10. □

Recalling that the topological group $\mathrm{Aut}(U)$ does not depend on the choice of $\mathfrak{L}$ and of weak Fraïssé matching sequence $(\vec{u}, \vec{u}^{\,\infty}, U)$ in $(\mathfrak{S}, \mathfrak{L})$ (see Construction 2.19), we obtain the following.

Corollary 3.15. *A weak Fraïssé category $\mathfrak{S}$ has the weak Ramsey property if and only if the topological group $G(\mathfrak{S})$ is extremely amenable.*

In the classical case when $\mathfrak{S}$ is a family of finite first-order structures with all embeddings, our category is locally finite, which allows us to simplify the weak Ramsey property. In this case, when expanded, we obtain the following corollary. Also, recall that the topology on $\mathrm{Aut}(U)$ does not depend on the choice of a weak Fraïssé sequence.

Corollary 3.16. *Let L be a first-order language, let $\mathfrak{L}$ be the category of all L-structures and all homomorphisms and let $\mathfrak{S} \subseteq \mathfrak{L}$ be a subcategory of some finite L-structures and all embeddings between them. If $\mathfrak{S}$ is a weak Fraïssé category with a generic object U, then the following properties are equivalent.*

(a) $\mathrm{Aut}(U)$ *is extremely amenable.*
(b) *For every $a \in \mathrm{Obj}(\mathfrak{S})$, there is an $\mathfrak{S}$-arrow $\alpha\colon a \to a'$ such that for every $b \in \mathrm{Obj}(\mathfrak{S})$, for every $k \in \omega$ there exists $v \in \mathrm{Obj}(\mathfrak{S})$ such that for every coloring $\varphi\colon \mathfrak{S}(\alpha, v) \to k$ there exists $e\colon b \to v$ with φ constant on $e \circ \mathfrak{S}(\alpha, b)$.*

3.3. Amalgamation extension and arrow extension

We briefly discuss the phenomenon of weak versions of certain notions like the amalgamation property and the Ramsey property from theoretical perspective. In both situations, the core property is localized at individual objects of a category and then generalized from objects to arrows. Here, we note that amalgamable/Ramsey arrows may be viewed as arrows factorizing through amalgamable/Ramsey objects in a certain extension category.

First, observe that for a full cofinal subcategory $\mathfrak{C} \subseteq \mathfrak{C}'$ we have that a $\mathfrak{C}$-arrow is amalgamable in $\mathfrak{C}$ if and only if it is amalgamable in $\mathfrak{C}'$. In this section, we shall work with full subcategories, and they will

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be sometimes identified with their classes of objects, for example, $\mathfrak{C}' \setminus \mathfrak{C}$ denotes the full subcategory of $\mathfrak{C}'$ consisting of objects in $\mathrm{Obj}(\mathfrak{C}') \setminus \mathrm{Obj}(\mathfrak{C}')$.

Let $\mathfrak{C}$ be a category. By $\mathrm{Am}(\mathfrak{C})$, we denote the full subcategory of $\mathfrak{C}$ consisting of all amalgamable objects. Suppose that both $\mathrm{Am}(\mathfrak{C})$ and $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$ are cofinal. Then $\mathrm{Am}(\mathfrak{C})$ has the amalgamation property, $\mathfrak{C}$ has the cofinal amalgamation property [12] but not the amalgamation property and $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$ has the weak amalgamation property but not the cofinal amalgamation property. In fact, $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$ has no amalgamable objects. So sometimes we may obtain a category with the weak amalgamation property without any amalgamable objects simply by removing the amalgamable objects. Sometimes, it even happens that every amalgamable arrow in $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$ factorizes through an object in $\mathrm{Am}(\mathfrak{C})$. Let us capture this situation by a definition.

Definition 3.17. By an *amalgamation extension* of a category $\mathfrak{C}$, we mean a category $\mathfrak{C}' \supseteq \mathfrak{C}$ such that $\mathfrak{C}$ is a full cofinal subcategory of $\mathfrak{C}'$, every object of $\mathfrak{C}' \setminus \mathfrak{C}$ is amalgamable in $\mathfrak{C}'$, and every amalgamable $\mathfrak{C}$-arrow factorizes through an amalgamable object in $\mathfrak{C}'$. It follows that a $\mathfrak{C}$-arrow is amalgamable if and only if it factorizes through an amalgamable object in $\mathfrak{C}'$.

Proposition 3.18. *Let $\mathfrak{S} \subseteq \mathfrak{S}'$ be an amalgamation extension.*

(i) *$\mathfrak{S}$ has the weak amalgamation property if and only if $\mathrm{Am}(\mathfrak{S}')$ is cofinal in $\mathfrak{S}'$.*
(ii) *Under the conditions in (i), $\mathfrak{S}$ is a weak Fraïssé category if and only if $\mathrm{Am}(\mathfrak{S}')$ is a Fraïssé category. Moreover, an $\mathfrak{S}$-sequence $\vec{u}$ is weak Fraïssé in $\mathfrak{S}$ if and only if it is weak Fraïssé in $\mathfrak{S}'$, and an $\mathrm{Am}(\mathfrak{S}')$-sequence $\vec{v}$ is Fraïssé in $\mathrm{Am}(\mathfrak{S}')$ if and only if it is weak Fraïssé in $\mathfrak{S}'$. In this case, the sequences $\vec{u}$ and $\vec{v}$ are isomorphic, and so the topological groups $G(\mathfrak{S})$ and $G(\mathrm{Am}(\mathfrak{S}'))$ are isomorphic as well.*
(iii) *Under the conditions in (i), $\mathfrak{S}$ has the weak Ramsey property if and only if $\mathrm{Am}(\mathfrak{S}')$ has the Ramsey property.*

Proof. Claim (i) is clear from the fact that amalgamable arrows in $\mathfrak{S}$ are exactly arrows factorizing through an $\mathrm{Am}(\mathfrak{S}')$-object. Claim (ii) follows from the fact both $\mathfrak{S}$ and $\mathrm{Am}(\mathfrak{S}')$ are full and cofinal in $\mathfrak{S}'$, and so directedness, weak domination by a countable subcategory, and the property of being a weak Fraïssé sequence is translated between $\mathfrak{S}'$ and the subcategories. For the rest, see Construction 2.19. Claim (iii) follows from two applications of Corollary 3.7. □

Example 3.19. Let $\mathfrak{C}$ be the category of all finite acyclic graphs and all embeddings. Then amalgamable objects are exactly connected graphs in $\mathfrak{C}$, that is, finite trees. Moreover, $\mathfrak{C}$ is an amalgamation extension of $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$. In other words, an embedding $e\colon G \to G'$ in the category of disconnected finite acyclic graphs $\mathfrak{C} \setminus \mathrm{Am}(\mathfrak{C})$ is amalgamable if and only if $e[G]$ lies in a single component of G'. This is because the only obstruction to amalgamation in $\mathfrak{C}$ is when two components of a given graph are connected by incompatible paths (e.g., of different lengths) in different extensions — the amalgamation would contain a cycle, which is forbidden.

As seen in the example, amalgamation extensions may arise naturally. On the other hand, every category admits at least the following ‘artificial’ amalgamation extension.

Definition 3.20. For every category $\mathfrak{C}$, we define its *arrow extension* $\mathfrak{C}^{\uparrow}$ as follows. A $\mathfrak{C}^{\uparrow}$-object is a $\mathfrak{C}$-arrow $\alpha\colon a \to a'$, sometimes written as a pair (a, α). A $\mathfrak{C}^{\uparrow}$-arrow $f\colon (a, \alpha) \to (b, \beta)$ is a $\mathfrak{C}$-arrow $f\colon a \to b$ that factorizes through α (i.e., such that there is a $\mathfrak{C}$-arrow f' with $f' \circ \alpha = f$) or id_a if $\alpha = \beta$, so we have identities in $\mathfrak{C}^{\uparrow}$. The composition in $\mathfrak{C}^{\uparrow}$ defined by the composition in $\mathfrak{C}$ is correct. In fact, for every $\mathfrak{C}^{\uparrow}$-arrow $f\colon (a, \alpha) \to (b, \beta)$, every $\mathfrak{C}$-arrow $g\colon b \to c$, and every $\mathfrak{C}^{\uparrow}$-object (c, γ) we have that $g \circ f$ is a $\mathfrak{C}^{\uparrow}$-arrow $(a, \alpha) \to (c, \gamma)$.

Note that the natural functor $F\colon \mathfrak{C} \to \mathfrak{C}^{\uparrow}$ mapping $a \mapsto (a, \mathrm{id}_a)$ is fully faithful, and so $\mathfrak{C}$ may be identified with the full subcategory of $\mathfrak{C}^{\uparrow}$ consisting of identities. On the other hand, we also have the faithful functor $U\colon \mathfrak{C}^{\uparrow} \to \mathfrak{C}$ mapping $(a, \alpha) \mapsto a$. Moreover, since $U \circ F = \mathrm{id}_{\mathfrak{C}}$, $\mathfrak{C}$ is a retract of $\mathfrak{C}^{\uparrow}$. Also, note that $\alpha\colon (a, \alpha) \to (a', \mathrm{id}_{a'})$ for every $\mathfrak{C}^{\uparrow}$-object $\alpha\colon a \to a'$, so $\mathfrak{C} \subseteq \mathfrak{C}^{\uparrow}$ is cofinal. Finally, note

$$
\begin{tikzcd}
 & (b,\beta) \arrow[dr, dashed, "f'\circ\beta"] & \\
(a,\alpha) \arrow[ur, "f"] \arrow[dr, "g"'] & & d \\
 & (c,\gamma) \arrow[ur, dashed, "g'\circ\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & b \arrow[r, "\beta"] & b' \arrow[dr, dashed, "f'"] & \\
a \arrow[urr, "f"] \arrow[r, "\alpha"] \arrow[drr, "g"'] & a' \arrow[ur, dashed, "\tilde{f}"'] \arrow[dr, dashed, "\tilde{g}"] & & & d \\
 & & c \arrow[r, "\gamma"] & c' \arrow[ur, dashed, "g'"'] &
\end{tikzcd}
$$

Figure 3. *A span in $\mathfrak{C}^{\uparrow}$ and the corresponding span in $\mathfrak{C}$.*

that the notation $\mathfrak{C}(\alpha, b)$ used as a shortcut for $\mathfrak{C}(a', b) \circ \alpha$ in the previous sections really corresponds to the actual hom-set $\mathfrak{C}^{\uparrow}(\alpha, \beta)$, where $\mathrm{dom}(\beta) = b$.

Proposition 3.21. *A $\mathfrak{C}$-arrow $\alpha\colon a \to a'$ is amalgamable if and only if it is an amalgamable object in $\mathfrak{C}^{\uparrow}$. Hence, $\mathfrak{C} \cup \mathrm{Am}(\mathfrak{C}^{\uparrow})$ is an amalgamation extension of $\mathfrak{C}$.*

Proof. For the first part, it is enough to translate amalgamation spans (i.e., diagrams of the form $b \leftarrow a \to c$) and their solutions between $\mathfrak{C}$ and $\mathfrak{C}^{\uparrow}$, as shown in Figure 3 for the first implication. If α is an amalgamable arrow in $\mathfrak{C}$, $\beta\colon b \to b'$ and $\gamma\colon c \to c'$ are $\mathfrak{C}^{\uparrow}$ objects, and $f\colon \alpha \to \beta$ and $g\colon \alpha \to \gamma$ are $\mathfrak{C}^{\uparrow}$-arrows, then there are $\mathfrak{C}$-arrows $\tilde{f}\colon a' \to b$ and $\tilde{g}\colon a' \to c$ such that $f = \tilde{f} \circ \alpha$ and $g = \tilde{g} \circ \alpha$ (or one of f, g is an identity arrow, in which case the amalgamation is trivial), and there are $\mathfrak{C}$-arrows f' and g' to a $\mathfrak{C}$-object d such that $f' \circ \beta \circ \tilde{f} \circ \alpha = g' \circ \gamma \circ \tilde{g} \circ \alpha$.

If α is an amalgamable object in $\mathfrak{C}^{\uparrow}$ and $f\colon a' \to b$ and $g\colon a' \to c$ are $\mathfrak{C}$-arrows, then $f \circ \alpha\colon \alpha \to b$ and $g \circ \alpha\colon \alpha \to c$, and so there is a $\mathfrak{C}^{\uparrow}$-object (d, δ) and $\mathfrak{C}^{\uparrow}$-arrows $f'\colon b \to \delta$ and $g'\colon c \to \delta$ such that $f' \circ f \circ \alpha = g' \circ g \circ \alpha$. Since f' and g' may be also viewed as $\mathfrak{C}$-arrows $b \to d$ and $c \to d$, we are done.

It follows that $\mathfrak{C} \cup \mathrm{Am}(\mathfrak{C}^{\uparrow})$ is and amalgamation extension of $\mathfrak{C}$. $\mathfrak{C}$ is full and cofinal in $\mathfrak{C} \cup \mathrm{Am}(\mathfrak{C}^{\uparrow})$ since it is so in $\mathfrak{C}^{\uparrow}$. Every object in $\mathrm{Am}(\mathfrak{C}^{\uparrow})$ is amalgamable in $\mathfrak{C} \cup \mathrm{Am}(\mathfrak{C}^{\uparrow})$ since $\mathfrak{C} \cup \mathrm{Am}(\mathfrak{C}^{\uparrow})$ is full and cofinal in $\mathfrak{C}^{\uparrow}$. Finally, every amalgamable arrow $\alpha\colon a \to a'$ in $\mathfrak{C}$ factorizes as $g \circ f$, where $f\colon a \to \alpha$ corresponds to id_a and $g\colon \alpha \to a'$ corresponds to α. □

Proposition 3.22. *A $\mathfrak{C}$-arrow $\alpha\colon a \to a'$ is Ramsey if and only if it is a Ramsey object in $\mathfrak{C}^{\uparrow}$, that is, if id_α is a Ramsey arrow in $\mathfrak{C}^{\uparrow}$.*

Proof. We have already observed that $\mathfrak{C}(\alpha, b) = \mathfrak{C}^{\uparrow}(\alpha, \beta)$ for every $\mathfrak{C}$-arrow β with domain b. Now, the difference between the situation in $\mathfrak{C}$ and $\mathfrak{C}^{\uparrow}$ is that in $\mathfrak{C}^{\uparrow}$ more objects are allowed for b as well as for the Ramsey witnessing object v. But that may be overcome by the fact that $\mathfrak{C}$ is cofinal in $\mathfrak{C}'$.

Suppose α is a Ramsey arrow in $\mathfrak{C}$. Let $\beta\colon b \to b'$ be a $\mathfrak{C}^{\uparrow}$-object, let $F \subseteq \mathfrak{C}^{\uparrow}(\alpha, \beta)$ be finite and let $k \in \omega$. There is a $\mathfrak{C}^{\uparrow}$-arrow $f\colon \beta \to b'$ and a $\mathfrak{C}$-object v such that for every coloring $\varphi\colon \mathfrak{C}(\alpha, v) \to k$ there is a $\mathfrak{C}$-arrow $g\colon b' \to v$ such that φ is constant on $g \circ f \circ F$.

On the other hand, suppose that α is a Ramsey object in $\mathfrak{C}^{\uparrow}$. For every $\mathfrak{C}$-object b, finite $F \subseteq \mathfrak{C}(\alpha, b)$ and $k \in \omega$ there is a witnessing $\mathfrak{C}^{\uparrow}$-object $\gamma\colon v \to v'$ and a $\mathfrak{C}^{\uparrow}$-arrow $f\colon \gamma \to v'$. Every coloring $\varphi\colon \mathfrak{C}(\alpha, v') \to k$ induces the coloring $\psi\colon \mathfrak{C}^{\uparrow}(\alpha, \gamma) \to k$ by $\psi(g) := \varphi(f \circ g)$. Hence, there is a $\mathfrak{C}^{\uparrow}$-arrow $g\colon b \to \gamma$ such that ψ is constant on $g \circ F$, and so φ is constant on $f \circ g \circ F$, where $(f \circ g)\colon b \to v'$. □

Together, we obtain the following.

Corollary 3.23. *Let $\mathfrak{C}$ be a category with the weak Ramsey property and the weak amalgamation property. Then $\mathrm{Am}(\mathfrak{C}^{\uparrow})$ has Ramsey property and the amalgamation property, and both $\mathfrak{C}$ and $\mathrm{Am}(\mathfrak{C}^{\uparrow})$ are full cofinal subcategories of $\mathfrak{C}^{\uparrow}$.*

4. Applications

We demonstrate the theory developed in the previous sections on several examples. Two extreme kinds of categories – in the sense of having degenerate hom-sets and degenerate class of objects, respectively – are posets and monoids. Recall that every poset $(P, \leqslant)$ may be regarded as a category $\mathfrak{C}$ with $\mathrm{Obj}(\mathfrak{C}) = P$

and $\mathfrak{C}(x, y)$ being a singleton if $x \leqslant y$ and empty otherwise. In general, a category $\mathfrak{C}$ such that every hom-set $\mathfrak{C}(a, b)$ is empty or a singleton is called *thin*. Clearly, every thin category has the Ramsey property.

Definition 4.1. We say that a category $\mathfrak{C}$ is *weakly thin* if for every $\mathfrak{C}$-object a there is a $\mathfrak{C}$-arrow $\alpha\colon a \to a'$ such that for every $\mathfrak{C}$-object b we have $|\mathfrak{C}(a', b) \circ \alpha| \leqslant 1$, that is, there is at most one arrow $a \to b$ that factorizes through α.

Clearly, every weakly thin category has the weak Ramsey property.

Example 4.2. Let $\mathfrak{C}$ be the category of all finite graphs whose all cycles have pairwise disjoint sets of vertices and have different lengths, with all embeddings as morphisms. Then $\mathfrak{C}$ is a weakly thin (and so having the weak Ramsey property) hereditary class without the Ramsey property, though it is not weak Fraïssé since it is not directed and does not have the weak amalgamation property.

Proof. For every graph G in $\mathfrak{C}$, we describe an inclusion $\alpha\colon G \to G'$ to a bigger graph G' in $\mathfrak{C}$ such that every vertex of G is definable (by an existential formula without parameters) in G' as well as in every extension of G' in $\mathfrak{C}$ (by the same formula across the extensions). Every cycle is definable (as a set) because of unique lengths. Since cycles are disjoint, there is at most one edge connecting two fixed cycles, and so the endpoints of that edge are definable. Finally, if at least two nonantipodal vertices on a cycle are definable, all vertices on the cycle are definable. Hence, to form G' it is enough to add cycles and paths so that every vertex of G is covered by one of the cases above.

The category $\mathfrak{C}$ does not have the Ramsey property since its objects are not rigid — for example, a cycle C has nontrivial automorphisms and for every G in $\mathfrak{C}$ we can color $\mathfrak{C}(C, G)$ so that every two different embeddings with the same image get different colors.

The category $\mathfrak{C}$ is not directed and does not have the weak amalgamation property: Let C_n denote a cycle of length $n \geqslant 3$, and let $G_{a,b,c}$, $a \neq b \neq c \geqslant 3$, be a graph consisting of C_a and C_c both joined by an edge to the same vertex in C_b. Then $G_{a,b,c}$ and $G_{b,c,a}$ can never be jointly embedded into a graph in $\mathfrak{C}$. Since every $\mathfrak{C}$-object H can be extended to $H \cup G_{a,b,c}$ and $H \cup G_{b,c,a}$ for suitably large a, b, c, $\mathfrak{C}$ does not have the weak amalgamation property. □

We shall look at monoids in the next section.

4.1. Monoids as categories

Recall that a *monoid* is a triple $(M, \cdot, 1)$, where $\cdot$ is an associative operation on a set M, and $1 \in M$ is the unit: We have $x \cdot 1 = 1 \cdot x = x$ for every $x \in M$. A monoid M can be viewed as a category with a single object: The elements of M become the endomorphisms, the multiplication $\cdot$ becomes the composition, and the unit 1 becomes the identity on the single unnamed object.

Then, an element $\alpha \in M$ corresponds to a Ramsey arrow if and only if for every $k \in \omega$, for every finite $F \subseteq M\alpha$, and for every $\varphi\colon M\alpha \to k$ there exists $e \in M$ such that $\varphi \restriction eF$ is constant. The monoid M has the weak Ramsey property if and only if there exists a Ramsey arrow, and M has the Ramsey property if and only if the unit (and so every element) is a Ramsey arrow.

Definition 4.3. We say that an element $\alpha \in M$ of a monoid satisfies the *left equalization condition* (LE) if for every finite $F \subseteq M\alpha$ there exists $e \in M$ such that eF is a singleton. Since we may assume that $\alpha \in F$, this is equivalent to $eF = \{e\alpha\}$. Also, by induction, (LE) is equivalent to the property that for every $x, y \in M\alpha$ there is $e \in M$ such that $ex = ey$. This is because for $x, y, z \in M\alpha$ there is $e \in M$ such that $ex = ey$ and $e' \in M$ such that $e'(ey) = e'(ez)$. It follows that $e'e\{x, y, z\}$ is a singleton.

Note that if an element $\alpha \in M$ satisfies (LE), then it is a Ramsey arrow. At several cases, the converse is true as well.

Proposition 4.4. *Let M be a monoid and $\alpha \in M$. Let F be the submonoid $\{f \in M : M\alpha f \subseteq M\alpha\}$, and let G be the graph of the right action of F on $M\alpha$, that is, the set of vertices is $M\alpha$, and we put an edge $x \mapsto_f xf$ for every $x \in M\alpha$ and $f \in F$. If G has finitely many undirected components, then α is a*

Ramsey arrow if and only if it satisfies (LE). This includes the following cases: (i) M is finite, (ii) M is commutative, (iii) α is left-invertible, in particular if α is the unit or M is a group.

Proof. Suppose that α is a Ramsey arrow. First, we show that if x, y lie in the same component of G, then there is $e \in M$ such that $ex = ey$. By induction, it is enough to consider the case $y = xf$ for some $f \in F$. Let G_f be the subgraph of G of the right action of f on $M\alpha$, that is, we consider only the edges $x \mapsto xf$ for every $x \in M\alpha$. For every $x \in M\alpha$, the path $(xf^n)_{n\in\omega}$ in G_f is an infinite ray or finite cycle with a finite initial segment attached. We may inductively define a coloring $\varphi\colon M\alpha \to 3$ such that $\varphi(x) \neq \varphi(xf)$ unless $x = xf$. We need the third color only because of possible cycles of odd length. Since α is a Ramsey arrow, there is $e \in M$ such that $\varphi(ex) = \varphi(exf)$, and so $ex = exf$.

Now, let $x, y \in M\alpha$ be arbitrary. Since G has finitely many components, there is a coloring $\psi\colon M\alpha \to k$ for some $k \in \omega$, such that points from different components take different colors. Since α is a Ramsey arrow, there is $e \in M$ such that $\psi(ex) = \psi(ey)$, and so ex and ey lie in the same component. So by the previous claim, there is $e' \in M$ such that $e'ex = e'ey$.

It remains to show that G has finitely many components in the cases (i), (ii), (iii). This is clear if M is finite. If M is commutative, then G has only one component: Every $x \in M\alpha$ is of the form $f\alpha = \alpha f$ for some $f \in M = F$, and hence x is in the component of α. Similarly, if $M\alpha = M \ni 1$, then $F = M$ and so $M\alpha = 1F$, and again G has only one component. □

Corollary 4.5. *A monoid M has the Ramsey property if and only if for every $x, y \in M$ there is $e \in M$ such that $ex = ey$.*

Example 4.6. A monoid with a *left zero*, that is, an element 0 such that $0 \cdot x = 0$ for every x, has the Ramsey property. A monoid with a *right zero* has the weak Ramsey property since a right zero is a Ramsey arrow. In fact, a monoid with a right zero is a weakly thin category.

But not every monoid with the Ramsey property has a left zero, and not every monoid with the weak Ramsey property has a right zero. For example, consider the commutative monoid on ω given by $a \cdot b = \max\{a, b\}$. It is clear this monoid satisfies the condition of the above corollary; however, there is no left or right zero. The infinitude of ω is a necessity for this which we see in the next observation.

Observation 4.7. A finite monoid is Ramsey if and only if it has a left zero.

Proof. It suffices to show that a finite monoid with the Ramsey property has a left zero. There is are $e, s \in M$ such that $eM = \{s\}$. Since M has the unit, we have that $e = s$ is a left zero. □

Example 4.8. Let M be the monoid $\{x^n, 0x^n : n \in \omega\}$ where the operation is the concatenation of words with discarding everything to the left of an occurrence of 0, that is, M is the free monoid generated by x with a right zero 0 freely added. The monoid M has the weak Ramsey property since it has a right zero, but M does not have the Ramsey property since there is no $e \in M$ such that $e = ex$.

Example 4.9. In a left-cancellative monoid M, that is, a monoid whose elements are monomorphisms, an element α satisfies (LE) if and only if it is a right zero. This is because if $x\alpha \neq \alpha$ for some $x \in M$, then there is no $e \in M$ such that $ex\alpha = e\alpha$. At the same time, M has no idempotents but the unit, and a right zero is an idempotent. It follows that no nontrivial left-cancellative monoid has an element satisfying (LE). In particular, it cannot have the Ramsey property, and if M is a group or is commutative, then it has not even the weak Ramsey property since in those cases Ramsey arrows satisfy (LE).

Question 4.10. Is there a nontrivial left-cancellative monoid with the weak Ramsey property?

Example 4.11. A nontrivial free monoid M does not have the weak Ramsey property. We can explicitly construct a coloring: Let $x \in M$ be a generator, and let $\varphi\colon M \to 2$ be the parity of the number of occurrences of x in a given word. Then for every $\alpha, e \in M$ we have $\varphi(ex\alpha) \neq \varphi(e\alpha)$, and so α is not a Ramsey arrow.

Definition 4.12. For every monoid M we may define the *left absorption relation* $x \geqslant y$ by $x = xy$ for $x, y \in M$. The relation is transitive: If $x = xy$ and $y = yz$, then $x = xy = x(yz) = (xy)z = xz$.

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We have $x \leqslant x$ if and only if x is an idempotent, so $\leqslant$ is a genuine preorder if and only if M consists of idempotents. Nevertheless, we shall use the preorder terminology even in the general case.

The unit 1 is the unique minimum: Clearly, it is a minimum, and if $x \leqslant 1$, then $1 = 1x = x$. The condition (LE) for an element $\alpha \in M$ says that $(M\alpha, \leqslant)$ is directed. Note that in a left-cancellative monoid the relation $\leqslant$ is trivial: All elements $x \neq 1$ are nonidempotent pairwise incomparable maxima. If $x \in M$ is an idempotent, then $ex \geqslant x$ for every $e \in M$ since $(ex)x = e(xx) = ex$.

Proposition 4.13. *An element α of a monoid M of idempotents is a Ramsey arrow if and only if it satisfies (LE) if and only if it is amalgamable.*

Proof. The left absorption relation is a preorder, and for every $x, y \in M\alpha$ we have that Mx and My are $\leqslant$-upper subsets of $M\alpha$. Suppose that α is a Ramsey arrow. We prove that $Mx \cap My \neq \emptyset$. Otherwise, there is a coloring $\varphi\colon M\alpha \to 2$ such that Mx and My are colored by different colors. Since α is Ramsey, there is $e \in M$ such that $\varphi(Mx) \ni \varphi(ex) = \varphi(ey) \in \varphi(My)$, which is a contradiction. Hence, $x, y \in M\alpha$ have an upper bound, so we have (LE).

Clearly, if $\alpha \in M$ satisfies (LE), then it is amalgamable. On the other hand, if for $x, y \in M\alpha$ there are elements $x', y' \in M$ such that $x'x = y'y =: e$, then $ex = ey$. □

Example 4.14. Let M be a monoid of idempotents. If M is also commutative, then $xy = yx$ is the supremum of $\{x, y\}$: Clearly, xy is an upper bound since $(xy)x = y(xx) = yx = xy$ and similarly for y. The fact that for $x, y \leqslant z$ we have also $xy \leqslant z$ holds in general: We have $z = zx = zy$, and so $z = zy = (zx)y = z(xy)$. Moreover, $\leqslant$ is an order: If $x \leqslant y \leqslant x$, then $x = xy = yx = y$. Together, a commutative monoid of idemponents with the left absorption order becomes a join semilattice with the minimum 1. Conversely, every join semilattice with a minimum is a monoid. We shall call these *semilattice monoids*. Since the order is directed (we have even suprema), semilattice monoids have the Ramsey property.

Example 4.15. Another special class of monoids of idempotents are *left-zero monoids*, that is, monoids M such that for every $x, y \in M \setminus \{1\}$ we have $xy = x$. This means that the left absorption preorder trivializes: All elements $x \neq 1$ are $\leqslant$-equivalent maxima. Hence, left-zero monoids have the Ramsey property. Similarly, a *right-zero monoid* consists of the unit and of right zeros and so has the weak Ramsey property. In this case, if $x \geqslant y \neq 1$, then $x = xy = y$, so nonunit elements are pairwise incomparable maxima. Hence, nontrivial right-zero monoids do not have the Ramsey property.

Question 4.16. Is being a Ramsey arrow equivalent to (LE) in a general monoid?

4.2. Almost linear orders

We revisit Pouzet's example (mentioned in a note by Pabion [22]; see also [10]) of a weak Fraïssé category without the cofinal amalgamation property and observe that it also has the weak Ramsey property. The basic idea is to consider the linear order of rationals $(\mathbb{Q}, \leqslant)$, which is known to be a Fraïssé limit and known to have extremely amenable automorphism group, and to interpret it using a different language, namely the ternary relation $R(x, y, z)$ defined by $(x < y) \land (x < z) \land (y \neq z)$.

Let LO denote the category of all linear orders and all embeddings. Note that an order P is linear if and only if every two-element subset $\{x, y\} \subseteq P$ has a minimum. For the sake of this example, we say that an order P is *almost linear* if every three-element subset $\{x, y, z\} \subseteq P$ has a minimum. We define LO_3 to be the category of all almost linear orders and all one-to-one homomorphisms. Note that every one-to-one homomorphism $f\colon P \to Q$ between orders is already an embedding if P is linear: If $f(x) \leqslant f(y)$, then either $x \leqslant y$ and we are done, or $y \leqslant x$, so $f(y) \leqslant f(x)$ and $f(x) = f(y)$, and so $x = y$. It follows that LO is a full subcategory of LO_3.

Let us view ordinals as linear orders, for example, $2 = \{0 < 1\}$, and for orders X, Y let $X + Y$ denote the ordered sum, that is, the disjoint union of X and Y with $x < y$ for every $x \in X$ and $y \in Y$, so, for example, $1 + 2$ is a linear order isomorphic to 3. Let B denote an order consisting of two incomparable elements.

Observation 4.17. Every almost linear order X is either linear or of the form $L + B$ for some linear order L.

Proof. For every $x \in X$, we have that $\{y \in X : y \leqslant x\}$ is linearly ordered since for every $y, z < x$ with $y \neq z$ the set $\{x, y, z\}$ has a minimum, which cannot be x. Also, for a similar reason, for every incomparable elements $x \neq y \in X$ and $z \in X \setminus \{x, y\}$ we have $z < x, y$. Together, there may be at most one pair of incomparable elements, and in that case they are maxima and everything below them is linear. □

Every almost linear order X with two maxima x, y admits exactly two linear refinements: We decide which of x, y will be greater and extend the relation appropriately. The two corresponding homomorphisms will be called *refinements* here.

Observation 4.18. Every LO_3-arrow $f\colon X \to Y$ is either an embedding, or a refinement followed by an embedding.

Proof. As observed earlier, if X is linearly ordered, then f is an embedding. Otherwise, let x, y be the two maxima of X. If $f(x), f(y)$ are incomparable, then they are the two maxima of Y, and f is again an embedding. If $f(x), f(y)$ are comparable, then $f = f' \circ g$, where $g\colon X \to X'$ is the refinement of X corresponding to the order of $f(x), f(y)$, and the uniquely determined map f' is an embedding. □

Construction 4.19. For an almost linear order $(X, \leqslant)$, we define a ternary relation R on X by

$$R(x, y, z) \;:\Longleftrightarrow\; (x < y) \wedge (x < z) \wedge (y \neq z),$$

meaning that x is the minimum of the three-point set $\{x, y, z\}$. The relation R satisfies the following properties:

(i) $R(x, y, z) \implies |\{x, y, z\}| = 3$,
(ii) $R(x, y, z) \iff R(x, z, y)$,
(iii) $R(x, y, w) \wedge R(y, z, w') \implies R(x, z, w')$,
(iv) $|\{x, y, z\}| = 3 \implies R(x, y, z) \vee R(y, z, x) \vee R(z, x, y)$,

which are in this context called *antireflexivity*, *symmetry*, *transitivity* and *linearity*, respectively.

Let $\mathsf{LO}_{3\mathsf{R}}$ denote the category whose objects are all structures (X, R) with a ternary relation satisfying the axioms above and whose morphisms are all first-order embeddings. In a moment, we shall make it clear that $\mathsf{LO}_{3\mathsf{R}}$ can be identified with a full subcategory of LO_3 such that $\mathsf{LO}_3 = \mathsf{LO} \cup \mathsf{LO}_{3\mathsf{R}}$ (see Observation 4.21).

The construction $(X, \leqslant) \mapsto (X, R)$ induces a functor $F\colon \mathsf{LO}_3 \to \mathsf{LO}_{3\mathsf{R}}$: It maps an LO_3-arrow $f\colon (X, \leqslant_X) \to (Y, \leqslant_Y)$ to the $\mathsf{LO}_{3\mathsf{R}}$-arrow $f\colon (X, R_X) \to (Y, R_Y)$ represented by the same map between the underlying sets (i.e., it is a concrete functor). We need to show that a one-to-one homomorphism f between almost linear orders is indeed an embedding of the corresponding R-structures.

Clearly, if x is the minimum of a three-point set $\{x, y, z\}$, then $f(x)$ is the minimum of the three-point set $\{f(x), f(y), f(z)\}$. On the other hand, if $f(x)$ is the minimum of a three-point set $\{f(x), f(y), f(z)\}$, then the set $\{x, y, z\}$ is also three point and so has a minimum, which has to be x by the first implication.

Now, let (X, R) be an $\mathsf{LO}_{3\mathsf{R}}$-object. We put

$$x < y \;:\Longleftrightarrow\; \exists w\, R(x, y, w) \quad \text{and} \quad x \leqslant y \;:\Longleftrightarrow\; (x = y) \vee \exists w\, R(x, y, w).$$

The relation $<$ is antireflexive and transitive by the corresponding properties of R, and so it is a strict order. Hence, $\leqslant$ is an order and $<$ is its strict part. By the symmetry and linearity of R, we have that $\leqslant$ is an almost linear order. Moreover, if $f\colon (X, R_X) \to (Y, R_Y)$ is an embedding, then it is a one-to-one homomorphism of the induced almost linear orders. The one-to-one part is clear. Suppose $x \leqslant_X y$. We want to show that $f(x) \leqslant_Y f(y)$. This is clear if $x = y$, so we may suppose that $R_X(x, y, w)$

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for some $w \in X$. But then $R_Y(f(x), f(y), f(w))$, and so $f(x) \leqslant_Y f(y)$. Together, the construction $(X, R) \mapsto (X, \leqslant)$ induces a concrete functor $G\colon \mathsf{LO}_{3\mathsf{R}} \to \mathsf{LO}_3$.

Proposition 4.20. *We have $F \circ G = \mathrm{id}_{\mathsf{LO}_{3\mathsf{R}}}$, so G is a full embedding. Moreover, for every LO_3-object X we have that $G(F(X))$ is X with the modified order forgetting the order of the largest and the second largest element (if these exist). The corresponding arrows $\varepsilon_X\colon G(F(X)) \to X$ are universal, so we have an adjunction counit $\varepsilon\colon G \circ F \to \mathrm{id}_{\mathsf{LO}_3}$. Together, $\mathsf{LO}_{3\mathsf{R}}$ may be identified with a full coreflective subcategory of LO_3.*

Proof. Let (X, R) be an $\mathsf{LO}_{3\mathsf{R}}$-object, $(X, \leqslant) := G(X, R)$ and $(X, R') := F(X, \leqslant)$. We want to show $R'(x, y, z) \iff R(x, y, z)$. By the definition, $R'(x, y, z)$ is equivalent to $(x < y) \wedge (x < z) \wedge (y \neq z)$, which is in turn equivalent to $\exists w, w'\, R(x, y, w) \wedge R(x, z, w') \wedge (y \neq z)$. This is clearly implied by $R(x, y, z)$ since we may simply put $w := z$ and $w' := y$. On the other hand, by the linearity of R we have $R(x, y, z) \vee R(y, z, x) \vee R(z, x, y)$, so it is enough to show that $R(y, z, x) \vee R(z, x, y)$ is not possible. Also, $R(z, x, y)$ is not possible since by $R(x, z, w')$ and the transitivity we would obtain $R(z, z, w')$, which contradicts the antireflexivity. Similarly, $R(y, z, x)$, which is equivalent with $R(y, x, z)$ contradicts $R(x, y, w)$.

For the second part, let $(X, \leqslant)$ be an LO_3-object and let $(X, \leqslant') := G(F(X, \leqslant))$. Clearly, $x <' y$ if and only if $x < y$ and there is some $z \neq y$ such that $x < z$, so $\leqslant'$ only forgets the order of the two consecutive $\leqslant$-largest elements (if they exist). We want to show that $\mathrm{id}_X\colon (X, \leqslant') \to (X, \leqslant)$ is the universal arrow, that is, that every map $f\colon Y \to X$ for an almost linear order $(Y, \leqslant)$ of the form $G(Y, R)$ is an LO_3-arrow $(Y, \leqslant) \to (X, \leqslant)$ if and only if it is an LO_3-arrow $(Y, \leqslant) \to (X, \leqslant')$. The backward implication is trivial. For the forward implication, the only possible difference between $\leqslant$ and $\leqslant'$ is that $(X, \leqslant)$ is a linear order $X \setminus \{a, b\} < a < b$ for some $a, b \in X$, while $X \setminus \{a, b\} <' a, b$ with $\leqslant'$-incomparable a, b. So the only possible fail is when $a = f(a')$ and $b = f(b')$ for some comparable elements $a', b' \in Y$. Since we suppose that $f\colon (Y, \leqslant) \to (X, \leqslant)$ is a one-to-one homomorphism, necessarily $Y \setminus \{a', b'\} < a' < b'$. But that is a contradiction with the fact that $(Y, \leqslant) = G(Y, R)$. ◻

As a consequence of the previous proposition and Observation 4.17, we obtain the following summary.

Corollary 4.21. *When we view $\mathsf{LO}_{3\mathsf{R}} \subseteq \mathsf{LO}_3$, we have $\mathsf{LO}_3 = \mathsf{LO} \cup \mathsf{LO}_{3\mathsf{R}}$. Moreover, $\mathsf{LO} \setminus \mathsf{LO}_{3\mathsf{R}}$ consists exactly of linear orders of the form $L + 2$, while $\mathsf{LO}_{3\mathsf{R}} \setminus \mathsf{LO}$ consists exactly of almost linear orders of the form $L + B$. For every linear order L, we have $F(L + 2) = L + B$.*

Let FinLO, FinLO_3 and $\mathsf{FinLO}_{3\mathsf{R}}$ denote the full subcategories of LO, LO_3 and $\mathsf{LO}_{3\mathsf{R}}$, consisting of all finite structures. We will show that $\mathsf{FinLO}_{3\mathsf{R}}$, which is a hereditary class of first-order structures, is a weak Ramsey category with almost no amalgamable objects (only the empty and singleton structures are amalgamable). Recall the notion of amalgamation extension from Section 3.3.

Proposition 4.22. *Both FinLO and $\mathsf{FinLO}_{3\mathsf{R}}$ are full cofinal subcategories of $\mathsf{FinLO}_3 = \mathsf{FinLO} \cup \mathsf{FinLO}_{3\mathsf{R}}$. A FinLO_3-object is amalgamable if and only if it is a linear order, that is, a FinLO-object. A $\mathsf{FinLO}_{3\mathsf{R}}$-arrow is amalgamable if and only if it factorizes through a linear order, which happens if and only if it is not of the form $f + B\colon L + B \to L' + B$ for an FinLO-arrow $f\colon L \to L'$. Together, FinLO_3 is an amalgamation extension of $\mathsf{FinLO}_{3\mathsf{R}}$.*

Proof. Both FinLO and $\mathsf{FinLO}_{3\mathsf{R}}$ are cofinal: A nonlinear almost linear order $L + B$ admits a refinement $L + B \to L + 2$; a linear order L admits an embedding $L \to L + B$. The rest of the first claim follows from Corollary 4.21.

It is well known that FinLO has the amalgamation property, and so every FinLO-object is amalgamable in FinLO_3 since FinLO is a full cofinal subcategory. On the other hand, if a FinLO_3-arrow $\alpha\colon X \to Y$ does not factorize through a linear order, then X has two maxima that are mapped to the two maxima of Y. The two linear refinements $f, g\colon Y \to Y'$ are not amalgamable over α, and so α is not an amalgamable arrow. ◻

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It follows from Proposition 3.18 that FinLO_{3R} is a weak Fraïssé category with no amalgamable objects, but the degenerate ones (the empty and singleton orders). Moreover, it is well known that FinLO has the Ramsey property — this corresponds to the classical finite Ramsey theorem, and so FinLO_{3R} has the weak Ramsey property.

Let us also describe the situation with the (weak) Fraïssé limit. Let $\sigma\mathsf{FinLO}$, $\sigma\mathsf{FinLO}_3$ and $\sigma\mathsf{FinLO}_{3R}$ be the corresponding σ-closures (Construction 2.5), which in this case are the full subcategories of LO, LO_3 and LO_{3R} of all countable structures. By the general theory (Remark 2.22), each pair $(\mathsf{FinLO}, \sigma\mathsf{FinLO})$, $(\mathsf{FinLO}_3, \sigma\mathsf{FinLO}_3)$ and $(\mathsf{FinLO}_{3R}, \sigma\mathsf{FinLO}_{3R})$ has a weak Fraïssé limit. By Proposition 3.18, the corresponding weak Fraïssé sequences are isomorphic and so are their colimits, so all three pairs have a common weak Fraïssé limit, necessarily being an object of $\sigma\mathsf{FinLO} \cap \sigma\mathsf{FinLO}_{3R}$. Of course, the limit is the order of rationals $(\mathbb{Q}, \leqslant)$ as the well-known Fraïssé limit of finite linear orders. The classical KPT correspondence translates the Ramsey property of FinLO to the extreme amenability of $\mathrm{Aut}(\mathbb{Q}, \leqslant)$, which is then translated by the weak KPT correspondence (Theorem 3.14) to the weak Ramsey property of FinLO_{3R}. This is an alternative proof of the weak Ramsey property of FinLO_{3R}.

4.3. Trees

It turns out there are many notions of trees and embeddings of trees. In this section, we consider certain categories $\mathfrak{T}_M$ of structured finite trees and *strong embeddings* related to the classical Milliken's theorem [19]. We show that the categories $\mathfrak{T}_M$ are weak Fraïssé and describe the countable trees U_M that are their weak Fraïssé limits. After we forget part of the structure, namely the levels, the corresponding categories $\mathfrak{T}'_M$ are still weak Fraïssé, and moreover they have the weak Ramsey property. By the KPT correspondence (Theorem 3.14), the corresponding limit trees U'_M have extremely amenable automorphism groups.

Since we shall consider both finite and infinite trees with various extra structure, we carefully define our categories in several steps. By a *tree*, we mean a triple $(T, \leqslant, \wedge)$, where $(T, \leqslant)$ is a partially ordered set such that every set $(\leftarrow, x]_T := \{y \in T : y \leqslant x\}$ is linearly ordered and such that every pair $x, y \in T$ has the $\leqslant$-meet $x \wedge y$. By an *embedding of trees*, we mean a one-to-one map $f : S \to T$ between trees such that $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in S$ (and so $x \leqslant y \iff f(x) \leqslant f(y)$ for every $x, y \in S$), that is, a first-order embedding in the language $\{\leqslant, \wedge\}$. The category of all trees and all embeddings is denoted by Tree.

An element of a tree is often called a *node*, the minimum element (if it exists) is called the *root*, and a maximal element is called a *terminal node*. A node s is an *immediate successor* of a node t if $s > t$ and there is no x such that $s > x > t$. We use interval notation with respect to the tree order, that is, $[t, \to)_T := \{x \in T : x \geqslant t\}$, $(s, t)_T := \{x \in T : s < x < t\}$ and so on. We omit the T subscript when the tree is clear from the context. A tree T is *well founded* if every set $(\leftarrow, t]$ for $t \in T$ is well ordered. Trees considered in set-theory are often well founded, but here we consider also trees that are not well founded, for example, the set $(\leftarrow, t)$ may be isomorphic to $(\mathbb{Q}, \leqslant)$.

Definition 4.23 (Splitting degree and splitting preserving embeddings). For every tree T and $t \in T$, we may consider the *splitting equivalence* defined on $(t, \to)$ by $x \sim y \iff x \wedge y > t$. (The transitivity follows since $x \wedge z \in \{x \wedge y, y \wedge z\}$.) Let $\mathrm{Spl}(t)$ denote the set of the equivalence classes $(t, \to)/{\sim}$, which correspond to the connected parts strictly above t, and let $\mathrm{spl}(t)$ denote the *splitting degree* at a nonterminal node t defined as the cardinality of $\mathrm{Spl}(t)$. Of course, if the tree T is well ordered, $\mathrm{Spl}(t)$ corresponds to immediate successors of t, but in general it is not the case. Also, maximal antichains in $(t, \to)$ are exactly transversals of $\mathrm{Spl}(t)$. A tree is *finitely splitting* if every set $\mathrm{Spl}(t)$, $t \in T$, is finite.

For every embedding $f : T \to S$, we have that a class $[x] \in \mathrm{Spl}(t)$ gets mapped into the class $[f(x)] \in \mathrm{Spl}(f(t))$ and that different classes get mapped into different classes, so f induces an injective map $\mathrm{Spl}_T(t) \to \mathrm{Spl}_S(f(t))$. We say that f *preserves splitting* if for every nonterminal node $t \in T$ the induced map $\mathrm{Spl}_T(t) \to \mathrm{Spl}_S(f(t))$ is a bijection. For finitely splitting trees, this is equivalent to preserving the splitting degree of all nonterminal nodes of T. Let $\mathsf{SplTree} \subseteq \mathsf{Tree}$ be the wide subcategory of all trees and all splitting preserving embeddings, and let $\mathsf{FSplTree} \subseteq \mathsf{SplTree}$ be the full subcategory of finitely splitting trees.

It will be useful to consider also trees whose splitting degree has been decided on terminal nodes. Let $\overline{\mathsf{FSplTree}}$ denote the category of all trees T endowed with unary relations $\{R_n\}_{n\in\mathbb{N}^+}$ such that for every $t \in T$ at most one of $R_n(t)$ holds, and if t is nonterminal, then $R_{\mathrm{spl}(t)}(t)$ holds. The morphisms are first-order one-to-one homomorphisms in the language $\{\leqslant, \wedge\} \cup \{R_n\}_{n\in\mathbb{N}^+}$. Nodes $t \in T$ for which some $R_n(t)$ holds are called *decided*. Clearly, $\mathsf{FSplTree}$ can be identified with the full subcategory of $\overline{\mathsf{FSplTree}}$ of trees with undecided terminal nodes. On the other hand, by $\mathsf{DSplTree}$ we denote the full subcategory of $\overline{\mathsf{FSplTree}}$ of trees whose all terminal nodes are decided.

Definition 4.24 (Leveled trees). By a *leveled tree*, we mean a tree T equipped with an equivalence $\leftrightarrows$ encoding the information of which nodes are on the same level, that is, the level set $\mathrm{Lev}(T) := T/\leftrightarrows$ with the induced order is linearly ordered, and for every $t \in T$ the *level function* $\mathrm{lev}\colon T \to \mathrm{Lev}(T)$ induces an order isomorphism $(\leftarrow, t]_T \to (\leftarrow, \mathrm{lev}(t)]_{\mathrm{Lev}(T)}$. By $\mathsf{LevTree}$, we denote the category of all leveled trees and first-order embeddings (or equivalently one-to-one homomorphisms) in the language $\{\leqslant, \wedge, \leftrightarrows\}$. Note that a $\mathsf{LevTree}$-embedding $f\colon S \to T$ induces an order embedding $\mathrm{Lev}(S) \to \mathrm{Lev}(T)$.

The preorder on T induced by $\mathrm{Lev}(T)$ is denoted by $\leqslant^{\mathrm{lev}}$, that is, $x \leqslant^{\mathrm{lev}} y \iff \mathrm{lev}(x) \leqslant \mathrm{lev}(y)$. The set $\{t \in T : \mathrm{lev}(t) = \alpha\}$ is denoted by $T(\alpha)$. For every $\alpha \leqslant \mathrm{lev}(t)$, the unique node $s \leqslant t$ such that $\mathrm{lev}(s) = \alpha$ is denoted by $t \restriction \alpha$. Sometimes, we implicitly identify the level set $\mathrm{Lev}(T) = T/\leftrightarrows$ with another linearly ordered set, for example, an ordinal or the rational numbers.

Clearly, every well-ordered tree T admits a unique level structure with $\mathrm{Lev}(T)$ being an ordinal, yet a Tree-map between well-ordered trees is not necessarily level preserving. On the other hand, there are trees admitting several nonisomorphic level structures, for example, two copies of $(\mathbb{Q}, \leqslant)$ above a common root with one level structure gluing the copies of $\mathbb{Q}$ and another level structure making one copy a strict initial segment of the other. There are also trees admitting no level structure, for example, $(\mathbb{Q}, \leqslant)$ and a terminal node above a common root.

A leveled tree is called *balanced* if for every pair of nodes $t <^{\mathrm{lev}} s$, there is a node t' such that $t < t' \leftrightarrows s$. A finite tree is balanced if and only if all its terminal nodes are at the same level.

Definition 4.25 (Lexicographic trees). By a *lexicographic tree*, we mean a tree T endowed with a linear order $\leqslant^{\mathrm{lex}}$ that is coherent with the splitting structure of T, that is, for every $t \in T$ there is a linear order $\leqslant^t$ on $\mathrm{Spl}(t)$ such that $x \leqslant^{\mathrm{lex}} y$ if and only if $x \leqslant y$ or $x, y > x \wedge y$ and $[x] <^{x\wedge y} [y]$. Informally, the tree is linearly ordered from bottom to top and from left to right. By $\mathsf{LexTree}$, we denote the category of all lexicographic trees and all first-order embeddings (or equivalently one-to-one homomorphisms) in the language $\{\leqslant, \wedge, \leqslant^{\mathrm{lex}}\}$.

Definition 4.26 (Strong embeddings). By a *strong embedding*, we mean and embedding of leveled trees preserving both splitting and levels. A *strong subtree* of a leveled tree T is a subset S such that the inclusion $S \subseteq T$ is a strong embedding. By $\mathsf{StrTree}$, we denote the category of leveled trees (called *strong trees* in this context) and strong embeddings, and by $\mathsf{LexStrTree}$ we denote the category of lexicographic leveled trees and strong embeddings also preserving the lexicographic order. We add the lexicographic order in order to ensure the rigidity needed for the weak Ramsey property. By $\mathsf{LexLevTree}$ and $\mathsf{LexSplTree}$, we denote the expansions of $\mathsf{LevTree}$ and $\mathsf{SplTree}$ by the lexicographic order. We summarize the categories in Figure 4.

Remark 4.27. Observe that an inclusion of a subset $S \subseteq T$ of a finite tree with the induced order preserves meets (and splitting) if and only if for every nonterminal node $s \in S$ and every immediate successor $t \in T$ we have that $[t, \to)_T$ contains at most one (exactly one) immediate successor of s in S. Therefore, Tree-embeddings, $\mathsf{SplTree}$-embeddings, and $\mathsf{StrTree}$-embeddings correspond to weakly embedded, embedded and strongly embedded subtrees as defined in the original Milliken's paper [19]. The lexicographic order is also considered in [19] by the name extended order.

Definition 4.28 (Categories of finite trees). We finally define the categories of finite strong trees we are interested in. Let $M \subseteq \mathbb{N}^+$ be a nonempty set. $\mathfrak{T}_M$ denotes the full subcategory of $\mathsf{LexStrTree}$ consisting

$$\begin{tikzcd}
 & \mathsf{Tree} & \\
\mathsf{SplTree} \arrow[ur] & \mathsf{LevTree} \arrow[u] & \mathsf{LexTree} \arrow[ul] \\
\mathsf{StrTree} \arrow[u] \arrow[ur] & \mathsf{LexSplTree} \arrow[ul] \arrow[ur] & \mathsf{LexLevTree} \arrow[ul] \arrow[u] \\
 & \mathsf{LexStrTree} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Figure 4. *Categories of trees and forgetful functors between them.*

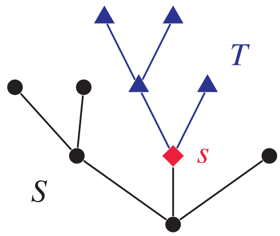

Figure 5. *A terminal planting $S \lhd_s T$.*

of finite trees such that the splitting degree takes values only in M. Note that if $M = \{1\}$, then $\mathfrak{T}_M$ consists of finite linearly ordered sets, and strong embeddings are just increasing embeddings.

Moreover, $\overline{\mathfrak{T}}_M$ denotes the variant of $\mathfrak{T}_M$ where terminal nodes may have decided splitting without actually being split at, that is, the FSplTree-component of the category is replaced by $\overline{\mathsf{FSplTree}}$. Of course, $\mathfrak{T}_M$ is identified with full a subcategory of $\overline{\mathfrak{T}}_M$. By $\mathfrak{D}_M$, we denote the full subcategory of $\overline{\mathfrak{T}}_M$ of trees where all terminal nodes are decided.

4.3.1. Extensions and amalgamations

In this section, we describe and classify arrows in $\mathfrak{T}_M$, $\overline{\mathfrak{T}}_M$ and $\mathfrak{D}_M$ for a fixed nonempty set $M \subseteq \mathbb{N}^+$ and characterize amalgamable objects and arrows.

First, observe that without loss of generality we may view a $\overline{\mathfrak{T}}_M$-arrow $f : S \to T$ as an inclusion $S \subseteq T$ (formally we compose f with an isomorphism $g : T \to T'$ such that $g \circ f : S \subseteq T'$). So instead of embeddings we may work with *extensions*, which simplifies the proofs. Second, given an extension $S \subseteq T$ there are two aspects: one, newly added nodes, and two, *deciding* the splitting degree of existing terminal nodes without adding new nodes above them, which is possible in $\overline{\mathfrak{T}}_M$ and required in $\mathfrak{D}_M$. We shall focus on the first aspect and mention the second one only when important. So from now on we keep in mind that we are working in any of $\mathfrak{T}_M$, $\overline{\mathfrak{T}}_M$, $\mathfrak{D}_M$, without necessarily specifying which one. By a *tree*, we mean an object of our category (in particular it is always a finite tree), by an *embedding* we mean a morphism and by an *extension* we mean an embedding that is also an inclusion of sets.

Definition 4.29. We say that an extension $S \subseteq T$ is *terminal* if S is a lower subset of T, that is, there is no newly added node $t \in T \setminus S$ below any existing node $s \in S$.

Construction 4.30 (Terminal planting). Let S, T be trees, let $s \in S$ be a terminal node and let r denote the root of T. We consider the tree $S \lhd_s T$ resulting from planting the tree T at the terminal node $s \in S$, like on Figure 5.

The trees S and T can be replaced by isomorphic copies, so we may suppose that $s = r$ and $S \cap T = \{s\}$. Then there is a unique lexicographic strong tree structure on $S \cup T$ that extends both S and T: For every $u \in S$ and $t \in T$, we have $u \wedge t = u \wedge_S s$; the splitting at every node is realized either in S or T, and so the lexicographic order is uniquely determined and since our trees are finite, there is a unique level structure.

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For convenience, given the original trees with the sets S, T we can take $S \lhd_s T = S \cup \{(s,t) : r \neq t \in T\}$ as the representing set so that we have an extension $S \subseteq (S \lhd_s T)$.

Note that up to a canonical isomorphism we have $(S \lhd_s T) \lhd_t R = S \lhd_s (T \lhd_t R)$ for s a terminal node of S and t a terminal node of T. Similarly, $(S \lhd_{s_1} T_1) \lhd_{s_2} T_2 = (S \lhd_{s_2} T_2) \lhd_{s_1} T_1$ for s_1, s_2 distinct terminal nodes of S. Finally, more trees can be planted at distinct terminal nodes at once: For a set of terminal nodes $A \subseteq S$ and trees $\{T_a : a \in A\}$, we consider the extension $S \lhd_{a \in A} T_a$.

Proposition 4.31. *For every set $A \subseteq S$ of terminal nodes and for nonempty trees T_a, we have that $S \lhd_{a \in A} T_a$ is a terminal extension of S. On the other hand, every terminal extension $S \subseteq T$ is canonically of the form $S \lhd_{a \in A} T_a$ with nondegenerate trees T_a (unless $S = \emptyset$).*

Proof. The first claim is clear from the definition. For the second claim, let $\emptyset \neq S \subseteq T$ be a terminal extension. For a node $t \in T \setminus S$, let $a = s(t)$ be the maximum of $(\leftarrow, t] \cap S$. Note that $T_a := [a, \rightarrow)$ is a strong subtree of T. Moreover, we have that a is a terminal node of S and so $T_a \cap S = \{a\}$. Otherwise, a would have an immediate successor from S by the terminality of $S \subseteq T$ and also an immediate successor from $T \setminus S$ by the definition of a. That would contradict $\mathrm{spl}_S(a) = \mathrm{spl}_T(a)$. Together, we have $T = (S \lhd_{a \in A} T_a)$ for $A := \{s(t) : t \in T \setminus S\}$. □

Definition 4.32. By a *bush*, we mean a tree with exactly two levels: the root and the nonempty set of its immediate successors.

By a *one-step terminal extension* of a tree S, we mean an extension of the form $S \lhd_s B$, where B is a bush. Note that every terminal extension $S \subseteq T$ can be realized as a finite sequence/composition of one-step terminal extensions $S \lhd_s B_1 \lhd_{b_1} \cdots \lhd_{b_{k-1}} B_k$. In particular, every nonempty tree T can be build by planting bushes and starting with a singleton tree $S = \{*\}$.

Definition 4.33. By a *nonterminal extension*, we mean the complete opposite of the terminal extension: It is an extension $S \subseteq T$ such that for every decomposition into two consecutive extensions $S \subseteq T' \subseteq T$ such that $T' \subseteq T$ is a terminal extension, we have $T' = T$.

Proposition 4.34. *Every extension $S \subseteq T$ can be canonically decomposed as $S \subseteq T' \subseteq T$ in a way that $S \subseteq T'$ is a nonterminal extension and $T' \subseteq T$ is a terminal extension.*

Proof. Let $S' := \bigcup_{s \in S} (\leftarrow, s]_T$ be the lower subset of T generated by S. Clearly, every suitable tree T' has to contain S'. But S' may not be a strong subtree. It is closed under meets and levels but typically not under splitting: We may have $\mathrm{spl}_{S'}(s) < \mathrm{spl}_T(s)$ for some $s \in S' \setminus S$, so we need to add more nodes witnessing the splitting. Fortunately, since our tree is finite and so every level but the top one has a successor level, we have canonical witnesses. It is enough to add all T-immediate successors of all nodes $s \in S' \setminus S$. Note that all T-immediate successors of nonterminal nodes of S are already in S'. So let $S'' := S' \cup \{t \in T : t \text{ an immediate successor of a node form } S' \setminus S\}$.

We have that S'' is the smallest strong subtree of T containing S'. It follows that every suitable T' has to contain S''. Also, $S'' \subseteq T$ is a terminal extension since S'' is a lower subset. Hence, $S'' \subseteq T'$ is a terminal extension, and necessarily $T' = S''$. It remains to show that $S \subseteq S''$ is a nonterminal extension. Let $N := S'' \setminus T''$ be the set of newly added nodes for some terminal extension $T'' \subseteq S''$ such that $S \subseteq T''$. Clearly, no node from S' is in N. Similarly, no node from $S'' \setminus S'$ is in N since its predecessor is in $S' \setminus S \subseteq T''$ and so would have to be a terminal node of T'', which it is not. So $N = \emptyset$ and $T'' = S''$. □

From the previous proof, we obtain the following.

Corollary 4.35. *An extension $S \subseteq T$ is nonterminal if and only if every node $t \in T \setminus S$ is below an S-node or its immediate predecessor is a node from $T \setminus S$ below an S-node.*

Construction 4.36 (Tree surgery). By a *pointed bush* B^b, we mean a pair (B, b), where B is a bush and b is one of its top-level nodes.

For a tree S, its level α, and a family of pointed bushes $(B_s^{b_s})_{s \in S(\alpha)}$, we consider the extension $S \rhd_\alpha (B_s^{b_s})_{s \in S(\alpha)}$ obtained by performing a (one-step) *tree surgery* on S. Again, for simplicity we may

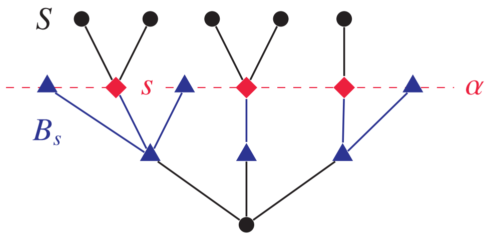

Figure 6. *A tree surgery* $S \rhd_\alpha (B_s^s)_{s\in S(\alpha)}$.

suppose that $b_s = s$ and $S \cap B_s = \{s\}$ for every $s \in S(\alpha)$ and that the bushes B_s are pairwise disjoint. There is a unique lexicographic strong tree structure on $S \cup \bigcup_{s\in S(\alpha)} B_s$ that is an extension of S and of every B_s. The roots of the bushes B_s form a new level immediately preceding α, while the level α is extended by the top levels of the bushes B_s with the nodes b_s being identified with the old level $S(\alpha)$; see Figure 6. In the case that α is the root level, and so we have a single pointed bush B^b, we write just $S \rhd B^b$. Note that this nonterminal extension of S is also the terminal extension $B \lhd_b S$ of B.

We may iterate one-step tree surgeries and consider

$$\Big(\underbrace{\big(S \rhd_\alpha (B_{1,s}^{b_{1,s}})_{s\in S(\alpha)}\big)}_{T_1} \rhd_{\beta_1} \cdots\Big) \rhd_{\beta_{k-1}} (B_{k,s}^{b_{k,s}})_{s\in T_{k-1}(\beta_{k-1})},$$

where β_i is the new level in T_i for $1 \leqslant i \leqslant k$. We may simplify this by introducing another helper notion. A *bush-column* is a tree of the form

$$C = \big(((B_0 \rhd B_1^{b_1}) \rhd \cdots) \rhd B_k^{b_k}\big) = \big(B_k \lhd_{b_k} B_{k-1} \lhd_{b_{k-1}} \cdots \lhd_{b_1} B_0\big),$$

and a *pointed bush-column* C^c is a pair (C, c), where C is a bush-column and c is a top-level node of C. Iterated tree surgery may be written as $S \rhd_\alpha (C_s^{c_s})_{s\in S(\alpha)}$ where $C_s^{c_s}$ for $s \in S(\alpha)$ are pointed bush-columns of the same height. Finally, we may perform tree surgery simultaneously on several levels and use the notation $S \rhd_{\alpha\in A} (C_{\alpha,s}^{c_{\alpha,s}})_{s\in S(\alpha)}$ or just $S \rhd_{\alpha\in A} C_\alpha$ for a set $A \subseteq \mathrm{Lev}(S)$ and a sequence of pointed bush-columns of the same height for every $\alpha \in A$.

Proposition 4.37. *Every extension* $S \subseteq \big(S \rhd_{\alpha\in A} (C_{\alpha,s}^{c_{\alpha,s}})_{s\in S(\alpha)}\big)$ *is nonterminal. On the other hand, for every nonterminal extension* $S \subseteq T$, T *can be canonically written as* $S \rhd_{\alpha\in A} (C_{\alpha,s}^{c_{\alpha,s}})_{s\in S(\alpha)}$. *It follows that every nonterminal extension can be realized as a finite sequence/composition of one-step tree surgeries.*

Proof. The first part is clear from the definition of tree surgery and Corollary 4.35. For the second part, let $S \subseteq T$ be a nonterminal extension. Let N be the set of all nodes $t \in T \setminus S$ such that there is a node $s(t) \in S$ above t. Moreover, let $s(t)$ denote the least such node, which exists since S is closed under meets. Also, let $N' := T \setminus (S \cup N)$, so we have a decomposition $T = S \cup N \cup N'$. By Corollary 4.35, every $t \in N'$ has no S-node above and its predecessor is in N. It follows that t is a terminal node of T (its immediate successor would have to be in N', which would contradict $t \in N'$).

For every $t \in N$, let $B(t) \subseteq T$ be the bush consisting of t and its immediate successors in T. We have $N' \subseteq \bigcup_{t\in N} B(t)$. Also, the top level of $B(t)$ may contain at most one node not in N' – either $s(t)$ or another $t' \in N$ with $s(t') = s(t)$. This follows from the fact that S is closed under meets.

Next, observe that for every level α if $T(\alpha) \cap N \neq \emptyset$, then $T(\alpha) \subseteq N \cup N'$. This is because $t \leqslant s(t)$, so if $t \leftrightarrows s' \in S$, then $t \in S$ as S is closed under levels. So immediately below every level $\alpha \in A := \{\mathrm{lev}_T(s(t)) : t \in N\}$, there are T-levels $\beta_{\alpha,1} > \beta_{\alpha,2} > \cdots > \beta_{\alpha,k_\alpha}$ consisting of new nodes. We have $N = \{s \restriction \beta_{\alpha,i} : \alpha \in A, s \in S(\alpha), 1 \leqslant i \leqslant k_\alpha\}$.

For every $\alpha \in A$ and $s \in S(\alpha)$, let $C_{\alpha,s} := \bigcup\{B(t) : t \in N \wedge s(t) = s\}$. Now, it is clear that every $C_{\alpha,s}$ is a bush-column, every two bush-columns $C_{\alpha,s}, C_{\alpha,s'}$ have the same height, and $T = \big(S \rhd_{\alpha\in A} (C_{\alpha,c}^s)_{s\in S(\alpha)}\big)$. □

Definition 4.38. By a *one-step extension*, we mean either a one-step terminal extension or a one-step tree surgery. It follows that a *one-step nonterminal extension* is just a different name for a one-step tree surgery.

Corollary 4.39. *Every extension $S \subseteq T$ is canonically of the form*

$$\left(S \triangleright_{\alpha \in A} C_\alpha\right) \triangleleft_{s \in B} T_s,$$

where A is a set of S-levels, B is a set of terminal nodes of the nonterminal extension tree, C_α are suitable sequences of pointed bush-columns and T_s are suitable trees. Moreover, every extension is a finite composition of one-step extensions.

Next, we characterize amalgamable arrows in the categories $\mathfrak{T}_M$, $\overline{\mathfrak{T}}_M$ and $\mathfrak{D}_M$. We are in a situation as in Section 3.3. The situation can be summarized as follows.

Theorem 4.40. *Let $M \subseteq \mathbb{N}^+$ be nonempty. Both $\mathfrak{T}_M$ and $\mathfrak{D}_M$ are full cofinal subcategories of $\overline{\mathfrak{T}}_M$. $\mathfrak{D}_M$ has the amalgamation property, so $\mathfrak{D}_M \subseteq \operatorname{Am}(\overline{\mathfrak{T}}_M)$. Moreover, every amalgamable $\overline{\mathfrak{T}}_M$-arrow factorizes through an amalgamable $\overline{\mathfrak{T}}_M$-object, and so (the full subcategory generated by) $\mathfrak{T}_M \cup \mathfrak{D}_M$ is an amalgamation extension of $\mathfrak{T}_M$.*

(i) *If $M = \{m\}$ for some $m \in \mathbb{N}^+$, then forgetting the decided splitting degree at terminal nodes is an isomorphism of categories $\mathfrak{D}_M \to \mathfrak{T}_M$, and the whole category $\overline{\mathfrak{T}}_M$ has the amalgamation property.*

(ii) *Otherwise, if $|M| \geqslant 2$, then a $\overline{\mathfrak{T}}_M$-object is amalgamable if and only if it is fully decided, that is, $\operatorname{Am}(\overline{\mathfrak{T}}_M) = \mathfrak{D}_M$. In particular, a $\mathfrak{T}_M$-arrow $f\colon S \to T$ is amalgamable if and only if for every terminal node $s \in S$, $f(s)$ is not terminal in T.*

The goal is to prove the theorem above.

Definition 4.41. Let $f_1\colon S \to T_2$ and $f_2\colon S \to T_2$ be $\overline{\mathfrak{T}}_M$-arrows. By a *node of incompatibility* for f_1, f_2, we mean a node $s \in S$ such that $f_1(s)$ and $f_2(s)$ are decided, but $\overline{\operatorname{spl}}(f_1(s)) \neq \overline{\operatorname{spl}}(f_2(s))$. Necessarily, s is an undecided terminal node of S. The pair of embeddings f_1, f_2 is *compatible* if there is no node of incompatibility.

Proposition 4.42. *A pair of $\overline{\mathfrak{T}}_M$-embeddings $f_1\colon S \to T_1$ and $f_2\colon S \to T_2$ is amalgamable (i.e., there are embeddings $g_i\colon T_i \to T$, $i \in \{1, 2\}$ such that $g_1 \circ f_1 = g_2 \circ f_2$) if and only if the pair is compatible.*

Proof. Clearly, if $s \in S$ is a node of incompatibility, then $\overline{\operatorname{spl}}(g_1(f_1(s))) = \overline{\operatorname{spl}}(f_1(s)) \neq \overline{\operatorname{spl}}(f_2(s)) = \overline{\operatorname{spl}}(g_2(f_2(s)))$, so f_1, f_2 cannot be amalgamated. The other, nontrivial, implication follows from Construction 4.43 below. □

Proof of Theorem 4.40. We already know that $\mathfrak{T}_M, \mathfrak{D}_M \subseteq \overline{\mathfrak{T}}_M$ are full. The cofinality of $\mathfrak{D}_M$ is clear — we just refine the relations by adding $R_m(s)$ for a fixed $m \in M$ and every undecided terminal node s of a given tree S. For the cofinality of $\mathfrak{T}_M$, let S be any $\overline{\mathfrak{T}}_M$-tree. We consider the terminal extension $S \triangleleft_{s \in D} B_{\overline{\operatorname{spl}}(s)}$, where D is the set of decided terminal nodes of S, and for every $m \in M$, B_m is an undecided bush whose root has the splitting degree m.

Clearly, every pair $f_i\colon S \to T_i$, $i \in \{1, 2\}$, of $\overline{\mathfrak{T}}_M$-arrows is compatible if S is decided or if $M = \{m\}$, so every $\mathfrak{D}_M$-object is amalgamable, and in the case (i), $\overline{\mathfrak{T}}_M$-has the amalgamation property.

The map $F\colon \mathfrak{D}_M \to \mathfrak{T}_M$ that forgets the decided splitting degrees at terminal nodes on objects and that acts as identity on arrows is always a faithful functor surjective on objects. But if $M = \{m\}$, then F is also one-to-one on objects and full since there is only one way how to decide a splitting degree, and so F is an isomorphism.

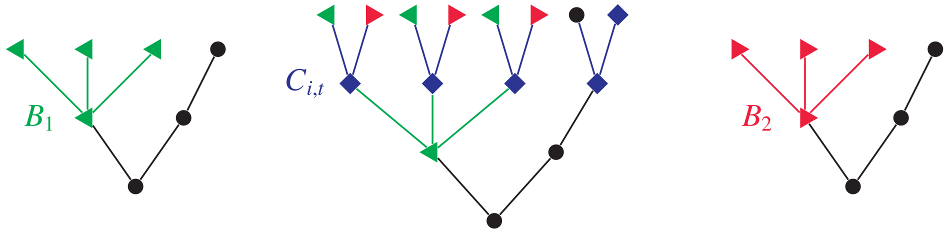

Figure 7. *An amalgamation of two one-step terminal extensions.*

Finally, if $m_1 \neq m_2 \in M$ and $\alpha\colon S \to S'$ is a $\overline{\mathfrak{T}}_M$-arrow such that $\alpha(s)$ is not decided for a node $s \in S$, then $f_1 \circ \alpha, f_2 \circ \alpha$, where $f_i\colon S' \to T_i$ is the refinement by $R_{m_i}(\alpha(s))$ are not amalgamable, and so α is not an amalgamable arrow. Hence, an amalgamable tree is necessarily decided in (ii), and every amalgamable arrow factorizes through an amalgamable object (which is trivial in (i)). □

It remains to construct an amalgamation $g_i\colon T_i \to T$, $i \in \{1,2\}$, for a pair $f_i\colon S \to T$ of compatible embeddings. As before, we may suppose for simplicity that f_i is an extension $S \subseteq T_i$. We will be able to construct an amalgamation such that $g_1(t_1) = g_2(t_2)$ if and only if $t_i = f_i(s)$ for a node $s \in S$ and $i \in \{1,2\}$. In this case, the embeddings g_i may be replaced by extensions $T_i \subseteq T$, so we would have $T_1 \cup T_2 \subseteq T$ with $T_1 \cap T_2 = S$. We shall call such situation a *nongluing amalgamation*.

We have seen in Corollary 4.39 that the extension $S \subseteq T_i$ may be viewed as a result of a certain construction: $T_i = E_i(S)$. It may happen that the same construction E_i can be applied also to an extended tree $S' \supseteq S$. A nongluing amalgamation is a *free amalgamation* if $E_1(E_2(S)) = E_2(E_1(S)) = T_1 \cup T_2 = T$. This happens for example if $T_i = S \lhd_{s\in A_i} V_s$ for $A_1 \cap A_2 = \emptyset$.

Construction 4.43. Let $S \subseteq T_i$, $i \in \{1,2\}$, be compatible extensions. We shall construct a nongluing amalgamation $T \supseteq T_1 \cup T_2$. We consider several cases.

(i) Suppose $T_i = S \lhd_s B_i$ for $i = 1,2$ are one-step terminal extensions planting at the same terminal node $s \in S$. Clearly, if $\mathrm{spl}_{T_1}(s) \neq \mathrm{spl}_{T_2}(s)$, then there is no amalgamation. But this is not the case since the extensions are compatible. It follows that the bushes B_i are isomorphic. Of course, we could amalgamate simply by identifying the bushes B_i, but we are interested in a nongluing amalgamation since the bushes B_i may be just parts of bigger extensions we consider in later cases.

We may suppose $B_1 \cap B_2 = \{s\}$. Let α denote the top level of the bushes B_i as well as the corresponding level of T_i. Hence, $T_i(\alpha)$ is the disjoint union $S(\alpha) \cup B_i(\alpha)$ with the possibility that $S(\alpha)$ is empty. Let $(b_{i,j} : j < k)$ for $i = 1,2$ be the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $B_i(\alpha)$. Furthermore, for every $s \in S(\alpha)$ let $C_{1,s} = C_{2,s}$ be a bush containing s at the top level, and for every $j < k$ let $C_{1,b_{1,j}} = C_{2,b_{2,j}}$ be a bush containing $b_{1,j}$ and $b_{2,j}$ as distinct top-level nodes (here, we suppose $M \neq \{1\}$, otherwise the amalgamation is done as for linear orders). Suppose the bushes are otherwise disjoint with all the other bushes considered. Our amalgamation is

$$T := \underbrace{(S \lhd_s B_1)}_{T_1} \rhd_\alpha (C^t_{1,t})_{t\in T_1(\alpha)} = \underbrace{(S \lhd_s B_2)}_{T_2} \rhd_\alpha (C^t_{2,t})_{t\in T_2(\alpha)}$$

as in Figure 7. Note that the extensions $T_i \subseteq T$ for $i = 1,2$ are one-step nonterminal.

(ii) To amalgamate general terminal extensions $S \lhd_s V_i$ at the same node $s \in S$ for $i = 1,2$, we write V_i as $B_i \lhd_{t\in A_i} V'_t$, where B_i is the root bush of V_i and $A_i \subseteq B_i$ is a subset of the top level of the bush (we suppose that $V_1 \cap V_2 = \{s\}$, and so $A_1 \cap A_2 = \emptyset$). Then we consider the free amalgamation $S' \lhd_{t\in A_1\cup A_2} V'_t$ of the extensions $S' \lhd_{t\in A_i} V'_t$ for $i = 1,2$, where S' is a nongluing amalgamation of $S \lhd_s B_1$ and $S \lhd_s B_2$ from (i).

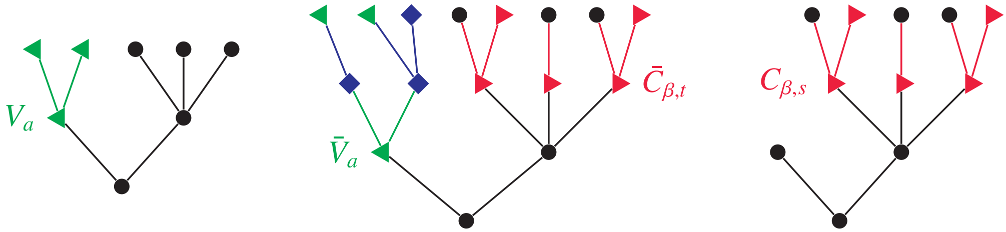

Figure 8. *An amalgamation of a terminal and a nonterminal extension.*

(iii) To amalgamate completely general terminal extensions $S \lhd_{s\in A_i} V_{i,s}$ for $i = 1, 2$, we consider the free amalgamation $T := S \lhd_{s\in A_1\cup A_2} V'_s$ of the extensions $S \lhd_s V'_s$ for $s \in A_1 \cup A_2$, where

$$S \lhd_s V'_s = \begin{cases} S \lhd_s V_1 & \text{if } s \in A_1 \setminus A_2, \\ S \lhd_s V_2 & \text{if } s \in A_1 \setminus A_2, \\ \text{the amalgamation of } S \lhd_s V_1 \text{ and } S \lhd_s V_2 \text{ from (ii)} & \text{if } s \in A_1 \cap A_2. \end{cases}$$

This is a free amalgamation of the original extensions $S \lhd_{s\in A_i} V_{i,s}$ if and only if $A_1 \cap A_2 = \emptyset$.

(iv) If $T_1 = (S \lhd_{a\in A} V_a)$ is a terminal extension and $T_2 = \big(S \rhd_{\beta\in B} (C^s_{\beta,s})_{s\in S(\beta)}\big)$ is a nonterminal extension, then there is a nongluing amalgamation T such that $T_1 \subseteq T$ is nonterminal and $T_2 \subseteq T$ is terminal.

For every $a \in A$, let $B_a := \{\beta \in B : V_a \cap T_1(\beta) \neq \emptyset\}$, and for every $\beta \in B$, let $A_\beta := \{a \in A : V_a \cap T_1(\beta) \neq \emptyset\}$. We have $T_1(\beta) = S(\beta) \cup \bigcup_{a\in A_\beta} V_a(\beta)$. For every $\beta \in B$ and $t \in T_1(\beta)$, let $\bar{C}_{\beta,t}$ be the bush-column $C_{\beta,t}$ if $t \in S(\beta)$, or a new bush-column of the same height containing t as a top-level node. For every $a \in A$, we put $\bar{V}_a := V_a \rhd_{\beta\in B_a} (\bar{C}^t_{\beta,t})_{t\in V_a(\beta)}$. We consider the amalgamation

$$T := \underbrace{(S \lhd_{a\in A} V_a)}_{T_1} \rhd_{\beta\in B} (\bar{C}^t_{\beta,t})_{t\in T_1(\beta)} = \underbrace{\big(S \rhd_{\beta\in B} (C^s_{\beta,s})_{s\in S(\beta)}\big)}_{T_2} \lhd_{a\in A} \bar{V}_a$$

as in Figure 8. The amalgamation is free if and only if every V_a is disjoint with every $T_1(\beta)$.

(v) Suppose $T_1 = \big(S \rhd_\alpha (C^s_s)_{s\in S(\alpha)}\big)$ and $T_2 = \big(S \rhd_\alpha (D^s_s)_{s\in S(\alpha)}\big)$ are two nonterminal extensions extending the same level α.

Let $\beta_{i,0} < \beta_{i,1} < \cdots < \beta_{i,k_i}$, $i \in \{0, 1\}$, be the enumeration of the levels of the bush-columns C_s and D_s, respectively, so in T_i the newly added levels are $\beta_{i,j}$, $j < k_i$, while $\beta_{i,k_i} = \alpha$. In our amalgamation T of T_1 and T_2, we shall have $\beta_{2,0} < \beta_{2,1} < \cdots < \beta_{2,k_2-1} < \beta_{1,0} < \beta_{1,1} < \cdots < \beta_{1,k_1} = \alpha = \beta_{2,k_2}$.

For every $t \in T_2(\alpha) = \bigcup_{s\in S(\alpha)} D_s(\alpha)$, let $\bar{C}_t$ denote the bush-column C_t if $t \in S(\alpha)$, or a completely new bush-column of the same height with t being a top-level node. Let $r(t)$ denote the root of $\bar{C}_t$. Moreover, for every $s \in S(\alpha)$ let $\bar{D}_s$ be an isomorphic copy of the bush-column D_s with every top-level node t replaced by $r(t)$. We suppose all the bush-columns are as disjoint as needed.

We shall write the amalgamation T in two ways so it is clear that it is an extension of both T_1 and T_2 agreeing on S. Let $\beta := \beta_{1,0}$. First, we consider the extension $T' := T_1 \rhd_\beta (\bar{D}^r_{s(r)})_{r\in T_1(\beta)}$, where $s(r)$ denotes the unique $s \in S(\alpha)$ with $r(s) = r$. Hence, β becomes the top level of the bush-columns $\bar{D}_s$ in T'. For every $r \in T'(\beta) = \bigcup_{s\in S(\alpha)} \bar{D}_s(\beta)$, let $t(r)$ denote the unique $t \in T_2(\alpha)$

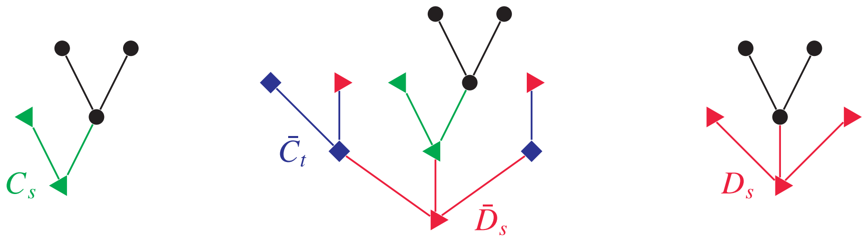

Figure 9. *An amalgamation two nonterminal extensions.*

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & T & & & \\
 & T_1' \arrow[ur] & & & & \\
T_1 \arrow[ur] & & S' \arrow[ul] & & T_2'' \arrow[uull] & \\
 & S_1 \arrow[ul] \arrow[ur] & & S'' \arrow[ul] \arrow[ur] & & T_2 \arrow[ul] \\
 & & & & S_2 \arrow[ul] \arrow[ur] & \\
 & & & S \arrow[uull] \arrow[ur] & &
\end{tikzcd}
$$

Figure 10. *The composition of immediate amalgamations for decomposed extensions.*

with $r(t) = r$. Our amalgamation is

$$
\begin{aligned}
T &:= \overbrace{\Big(\underbrace{\big(S \rhd_\alpha (C_s^s)_{s\in S(\alpha)}\big)}_{T_1} \rhd_\beta (\bar{D}^r_{s(r)})_{r\in T_1(\beta)}\Big)}^{T'} \lhd_{r\in T'(\beta)\setminus T_1(\beta)} \bar{C}_{t(r)} \\
&= \underbrace{\big(S \rhd_\alpha (D_s^s)_{s\in S(\alpha)}\big)}_{T_2} \rhd_\alpha (\bar{C}_t^t)_{t\in T_2(\alpha)}
\end{aligned}
$$

as in Figure 9. Note that the expansion $T_2 \subseteq T$ is nonterminal and that the amalgamation $T \supseteq T_1 \cup T_2$ is nongluing.

(vi) To amalgamate general nonterminal extensions $T_i = (S \rhd_{\alpha\in A_i} C_{i,\alpha})$ for $i = 1, 2$, we consider the free amalgamation $T := (S \rhd_{\alpha\in A_1\cup A_2} \bar{C}_\alpha)$ of the extensions $S \rhd_\alpha \bar{C}_\alpha$ for $\alpha \in A_1 \cup A_2$, where

$$
S \rhd_\alpha \bar{C}_\alpha = \begin{cases} S \rhd_\alpha C_{1,\alpha} & \text{if } \alpha \in A_1 \setminus A_2, \\ S \rhd_\alpha C_{2,\alpha} & \text{if } \alpha \in A_1 \setminus A_2, \\ \text{the amalgamation of } S \rhd_\alpha C_{1,\alpha} \text{ and } S \rhd_\alpha C_{2,\alpha} \text{ from (v)} & \text{if } \alpha \in A_1 \cap A_2. \end{cases}
$$

This is always a nongluing amalgamation of the original extensions $S \rhd_{\alpha\in A_i} C_{i,\alpha}$, and it is a free amalgamation if and only if $A_1 \cap A_2 = \emptyset$. Note that by (v) one of $T_i \subseteq T$ is nonterminal.

(vii) Finally, for general extensions $S \subseteq T_i$ for $i = 1, 2$ we consider the canonical decompositions $S \subseteq S_i \subseteq T_i$ such that $S \subseteq S_i$ is nonterminal and $S_i \subseteq T_i$ is terminal, and we perform several immediate amalgamations as in Figure 10. By (vi), we take an amalgamation $S_1 \cup S_2 \subseteq S'$ such that $S_1 \subseteq S'$ is nonterminal, and we consider the canonical decomposition $S_2 \subseteq S'' \subseteq S'$, so $S_2 \subseteq S''$ is nonterminal and $S'' \subseteq S'$ is terminal. By (iv), we take an amalgamation $T_1 \cup S' \subseteq T_1'$, so $S' \subseteq T_1'$ and $S'' \subseteq T_1'$ are terminal. Again, by (iv) we take an amalgamation $S'' \cup T_2 \subseteq T_2''$, so $S'' \subseteq T_2''$ is terminal. We obtain T as the amalgamation of $T_1' \cup T_2''$ performed by (iii). Note that since all the immediate amalgamations are nongluing, the compatibility of $S \subseteq T_1$ and $S \subseteq T_2$ implies the compatibility of $S'' \subseteq T_1'$ and $S'' \subseteq T_2''$. Also, the resulting amalgamation is nongluing.

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4.3.2. Countable trees and the generic tree

We have shown that $\mathfrak{T}_M$ is a weak Fraïssé category, that $\mathfrak{D}_M$ is a Fraïssé category and that both are full cofinal subcategories of $\overline{\mathfrak{T}}_M$. Let $\omega\mathfrak{T}_M$, $\omega\mathfrak{D}_M$, and $\omega\overline{\mathfrak{T}}_M$ be the corresponding categories of countable strong trees. Note that the objects of $\omega\mathfrak{T}_M \cap \omega\mathfrak{D}_M$ are exactly trees from $\omega\overline{\mathfrak{T}}_M$ with no terminal nodes.

Since $\omega\overline{\mathfrak{T}}_M$ is a category of first-order structures and all one-to-one homomorphisms (not necessarily embeddings since deciding a splitting degree of a terminal node is allowed), colimits of sequences essentially correspond to unions of increasing chains. Note that not every countable tree is a colimit of a sequence finite trees — the colimits are exactly *locally finite* trees, that is, trees T such that for every finite subset $F \subseteq T$ there is a subtree $S \subseteq T$ containing F. In our case of lexicographic strong subtrees, we must close F under the meet operation and under levels, which is trivial but also add witnessing nodes for splitting at nonterminals, which may complicate things.

Example 4.44. Consider the tree $T = \{0, 0', 1, 1', \ldots, \omega\}$, where every node $n \in \omega$ has two immediate successors: $n+1$ and n', every node n' is terminal and ω is a terminal node above the chain $\{0 < 1 < \ldots\}$. The subset $F := \{0, \omega\}$ is not contained in any finite strong subtree $S \subseteq T$. This is because S would need to contain $0'$ as the witness of the splitting at 0, and then 1 as the node below ω on the level of $0'$, and then $1'$ as the witness of the splitting at 1, and so on.

Lemma 4.45. *Balanced trees from $\omega\overline{\mathfrak{T}}_M$ are locally finite. More precisely, let T be an $\omega\overline{\mathfrak{T}}_M$-object. For every finite $F \subseteq T$, there is a $\overline{\mathfrak{T}}_M$-object S with $F \subseteq S \subseteq T$. Moreover, if T is in $\omega\mathfrak{T}_M$ or $\omega\mathfrak{D}_M$, then S can be taken in $\mathfrak{T}_M$ or $\mathfrak{D}_M$, respectively.*

Proof. We proceed in several steps. Let S_0 be the closure of F under the meet operation in T. Clearly, S_0 is finite, it becomes a tree when endowed with the $\{\leqslant, \wedge\}$-structure inherited from T and the inclusion $S_0 \subseteq T$ is a Tree-arrow.

Let S_1 be the closure of S_0 under T-levels, that is, $S_1 := \{s' \restriction \mathrm{lev}_T(s) : s, s' \in S_0 \text{ with } \mathrm{lev}_T(s) \leqslant \mathrm{lev}_T(s')\}$. It is easy to see that S_1 is finite, that it becomes a leveled tree with the inherited $\{\leqslant, \wedge, \between\}$-structure and that the inclusion $S_1 \subseteq T$ is a $\mathsf{LevTree}$-arrow.

For every nonterminal node of $s \in S_1$, let $\alpha_s \in \mathrm{Lev}(S_1) \subseteq \mathrm{Lev}(T)$ denote the level corresponding to immediate successors of s in S_1 and let $A_s \subseteq T(\alpha_s)$ be a transversal of $\mathrm{Spl}_T(s)$ extending $S_1(\alpha_s)$. Such transversal exists since T is balanced. We put $S := S_1 \cup \bigcup_{s \in S_1} A_s$ and endow with the inherited $\{\leqslant, \wedge, \between, \leqslant^{\mathrm{lex}}, \{R_m\}_{m \in M}\}$-structure. We have that S is a $\overline{\mathfrak{T}}_M$-object and that the inclusion $S \subseteq T$ is an $\omega\overline{\mathfrak{T}}_M$-arrow. If T is an $\omega\mathfrak{D}_M$-object, then S is a $\mathfrak{D}_M$-object. If T is an $\omega\mathfrak{T}_M$-object, then S is a $\mathfrak{T}_M$-object after forgetting the relations R_m. □

Let $\sigma\mathfrak{T}_M$, $\sigma\overline{\mathfrak{T}}_M$ and $\sigma\mathfrak{T}_M$ denote the full subcategories of all locally finite trees in $\omega\mathfrak{T}_M$, $\omega\overline{\mathfrak{T}}_M$ and $\omega\mathfrak{T}_M$, respectively. Every sequence in $\mathfrak{T}_M$ has a colimit in $\sigma\mathfrak{T}_M$, and every $\sigma\mathfrak{T}_M$-object is a colimit of a $\mathfrak{T}_M$-sequence. Also, every coned sequence in $(\mathfrak{T}_M, \sigma\mathfrak{T}_M)$ is a matching sequence; see Construction 2.5. The same is true for $(\overline{\mathfrak{T}}_M, \sigma\overline{\mathfrak{T}}_M)$ and $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$. It is not hard to see that both $\sigma\mathfrak{T}_M$ and $\sigma\mathfrak{D}_M$ are cofinal in $\sigma\overline{\mathfrak{T}}_M$. Therefore, by the general theory, there is a unique (up to isomorphism) object U_M in $\sigma\overline{\mathfrak{T}}_M$ that is a weak Fraïssé limit in $(\overline{\mathfrak{T}}_M, \sigma\overline{\mathfrak{T}}_M)$ as well as in $(\mathfrak{T}_M, \sigma\mathfrak{T}_M)$ and a Fraïssé limit in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$. Moreover, by Theorem 3.14 $\mathrm{Aut}(U_M)$ is extremely amenable if and only if $\mathfrak{T}_M$ has the weak Ramsey property if and only if $\mathfrak{D}_M$ has the Ramsey property.

Proposition 4.46. *The generic tree U_M is cofinal in $\omega\overline{\mathfrak{T}}_M$.*

Proof. The tree U_M is cofinal in $\sigma\mathfrak{D}_M$ as a Fraïssé limit in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$. Moreover, it is cofinal in $\sigma\overline{\mathfrak{T}}_M$ since $\sigma\mathfrak{D}_M$ is cofinal in $\sigma\overline{\mathfrak{T}}_M$. It is enough to prove that every tree S in $\omega\overline{\mathfrak{T}}_M$ can be strongly embedded into a balanced countable tree T since by Lemma 4.45 T is in $\sigma\overline{\mathfrak{T}}_M$.

By saying that S is *balanced at a node* s, we mean that for every level $\alpha > \mathrm{lev}(s)$ there is $s' > s$ such that $\mathrm{lev}(s') = \alpha$. Suppose $a \in S$ is a terminal node at which S is not balanced and let $Y := [\mathrm{lev}(a), \rightarrow)_{\mathrm{Lev}(S)}$. By Construction 4.49, there is a balanced tree T_a in $\omega\mathfrak{T}_M$ with $\mathrm{Lev}(T_a) = Y$ (just take $V_{m,Y}$ where $M \ni m = \{0 < \cdots < m-1\}$). By planting the tree T_a at $a \in S$, that is, by considering

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an infinite version of a terminal extension $S \lhd_a T_a$, we obtain a strong supertree of S balanced at a as well as at every newly added node.

Similarly, suppose that $B \subseteq S$ is an unbounded branch (i.e., a maximal linearly ordered subset without maximum) such that the upper set $Y := \mathrm{Lev}(S) \setminus \mathrm{lev}[B]$ is nonempty (where $\mathrm{lev}[B] = \{\mathrm{lev}(b) : b \in B\}$). We shall again consider a balanced tree T_B in $\omega\mathfrak{T}_M$ with $\mathrm{Lev}(T_s) = Y$ and plant it above B, that is, all nodes of T_B will be strictly above all nodes of B. The resulting tree $S \lhd_B T_B$ is balanced at every node in B as well as at every newly added node.

Together, there is a set $A \subseteq S$ of terminal nodes and a countable set $\mathcal{B}$ of unbounded branches such that every node $s \in S$ at which S is not balanced is below a node $a \in A$ or contained in a branch $B \in \mathcal{B}$. The tree $T := S \lhd_{a \in A} T_a \lhd_{B \in \mathcal{B}} T_B$ is the desired balanced extension. □

Next, we shall characterize U_M, but first consider the following definition.

Definition 4.47. For every countable set of colors C, let $\mathbb{Q}_C$ denote $(\mathbb{Q}, \leqslant)$ endowed with a *generic C-coloring* $\varphi \colon \mathbb{Q} \to C$, meaning that every monochromatic set $\varphi^{-1}(c)$ is $\leqslant$-dense. Such structure $\mathbb{Q}_C$ is unique up to isomorphism as a Fraïssé limit of all C-colored finite linear orders and color-preserving embeddings.

For every $M \subseteq \mathbb{N}^+$, we put $M^< := \{(k, m) \in \mathbb{N}_0 \times M : k < m\}$, so we can use $\mathbb{Q}_M$ and $\mathbb{Q}_{M^<}$ later. For every $s < t \in T \in \mathrm{Obj}(\omega\overline{\mathfrak{T}}_M)$, let $\mathrm{spl}(s, t)$ denote the pair $(k, \mathrm{spl}(s)) \in M^<$ such that $t \in C_k$, where $(C_i : i < \mathrm{spl}(s))$ is the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $\mathrm{Spl}(s)$.

Theorem 4.48. *U_M is the unique (up to isomorphism) $\omega\overline{\mathfrak{T}}_M$-object satisfying*

(T0) *U_M is balanced with* $\mathrm{Lev}(U_M)$ *being isomorphic to* $(\mathbb{Q}, \leqslant)$,
(T1) *for every $\alpha \in \mathrm{Lev}(U_M)$, every $t' <^{lex} t''$ in $U_M(\alpha)$, and $m \in M$ there is $t \in U_M(\alpha)$ such that $t' <^{lex} t <^{lex} t''$ and $\mathrm{spl}(t) = m$,*
(T2) *for every $\beta < \alpha \in \mathrm{Lev}(U_M)$, finite $F \subseteq U_M(\alpha)$ and $\varphi \colon F \to M^<$, there is $\alpha' \in (\beta, \alpha)$ such that $\mathrm{spl}(t \restriction \alpha', t) = \varphi(t)$ for every $t \in F$.*

It follows that U_M satisfies also

(T3) *every branch B with the coloring $\varphi \colon t \mapsto \mathrm{spl}(t) \in M$ is isomorphic to $\mathbb{Q}_M$,*
(T4) *for every $t \in U_M$, the interval $(\leftarrow, t)$ with the coloring $\varphi \colon s \mapsto \mathrm{spl}(s, t) \in M^<$ is isomorphic to $\mathbb{Q}_{M^<}$,*
(T5) *for every $t \in U_M$, $\alpha > \mathrm{lev}(t)$ and $C \in \mathrm{Spl}(t)$, the sets $U_M(\alpha) \cap C$, $U_M(\alpha) \cap [t, \rightarrow)$ and $U_M(\alpha)$ with the lexicographic order and with the coloring $\varphi \colon t \mapsto \mathrm{spl}(t) \in M$ are all isomorphic to $\mathbb{Q}_M$ unless $M = \{1\}$,*
(T6) *for every finite $F \subseteq U_M(\alpha)$, there is $\beta < \alpha$ such that (β, α) with the coloring $\varphi \colon \gamma \mapsto (\mathrm{spl}(t \restriction \gamma, t))_{t \in F} \in (M^<)^F$ is isomorphic to $\mathbb{Q}_{(M^<)^F}$.*

Proof. U_M is characterized by being a Fraïssé limit of $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$. Since we have the initial object (the empty tree), cofinality of U_M in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$ already follows from injectivity. So U_M is characterized by being injective in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$. Recall that the injectivity means that for every inclusion $S \subseteq U_M$ that is a $\sigma\mathfrak{D}_M$-arrow from a $\mathfrak{D}_M$-object (i.e., a finite decided strong subtree) and every (one-step) $\mathfrak{D}_M$-extension $S \subseteq T$ we find a $\sigma\mathfrak{D}_M$-arrow $f \colon T \to U_M$ such that $f \restriction S = \mathrm{id}$. Note that without loss of generality we may always suppose $\mathrm{Lev}(S) \subseteq \mathrm{Lev}(U_M)$.

Let U be an $\omega\overline{\mathfrak{T}}_M$-object. First, we show that if U is injective in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$, then it satisfies (T0), (T1), (T2). U is balanced since for every $s \in U$ and $\alpha > \mathrm{lev}(s)$ we may consider a finite strong subtree $S \subseteq U$ such that $s \in S$ and $S \cap U(\alpha) \neq \emptyset$ (since U is a $\sigma\mathfrak{D}_M$-object and so locally finite) with a node $t \in [s, \rightarrow)_S$ such that $\mathrm{lev}_S(t)$ is as high as possible below α. If $\mathrm{lev}_S(t) = \alpha$, we are done. Otherwise, we consider an extension $T := S \lhd_t B$ for a bush B and the extending map $f \colon T \to U$. The image $S' := f[T] \supseteq S$ contains a node $t' > t$, whose level is strictly closer to α.

U has no maximal level since if $s \in U$ was a node at a maximal level, we could consider an extending map $f\colon \{s\} \lhd_s B \to U$ for a bush B. Similarly, U has no minimal level since if $r \in U$ was the root, we could consider an extending map $f\colon \{r\} \rhd B^b \to U$ for a pointed bush B^b. It follows from (T2) (to be proved) that the order of $\mathrm{Lev}(U)$ is dense. Together, $\mathrm{Lev}(U)$ is isomorphic to $(\mathbb{Q}, \leqslant)$ and to we have (T0).

We shall prove (T1) and (T2) together. Let $\beta < \alpha$ be levels of U, let $F \subseteq U(\alpha)$ be finite, and let $\varphi\colon F \to M^<$. In the case we are proving (T1), we make sure that $t', t'' \in F$ and that $\varphi(t') = (0, m')$ for some $m' > 1$ (note that $M \neq \{1\}$ since $t' <^{\mathrm{lex}} t''$). Since U is locally finite, there is a finite strong subtree $S \subseteq U$ such that $F \subseteq S$ and $S \cap U(\beta) \neq \emptyset$. For every $s \in S(\alpha)$, let (k_s, m_s) be an element of $M^<$ that is equal to $\varphi(s)$ if $s \in F$. We consider an extending map $f\colon S \rhd_\alpha (B_s^{b_{s,k_s}})_{s \in S(\alpha)} \to U$, where B_s are bushes with a root r_s and $\leqslant^{\mathrm{lex}}$-enumerated terminal nodes $(b_{s,i} : i < m_s)$ such that $b_{s,k_s} = s$ and $\overline{\mathrm{spl}}(b_{t',1}) = m$ in the case we are proving (T1). Let $\alpha' \in \mathrm{Lev}(U)$ be the level containing images of the roots $\{f(r_s) : s \in S(\alpha)\}$. We have $\beta < \alpha' < \alpha$ since $S \cap U(\beta) \neq \emptyset$, and $\mathrm{spl}(s \restriction \alpha', s) = \varphi(s)$ for every $s \in F$ since $s \restriction \alpha' = f(r_s)$. Hence, we have proved (T2). In the case we were also proving (T1), we have $b_{t',0} = t' <^{\mathrm{lex}} t := f(b_{t',1}) <^{\mathrm{lex}} t''$ and $\mathrm{spl}(t) = \overline{\mathrm{spl}}(b_{t',1}) = m$.

We have proved that the injectivity of U implies (T0), (T1), (T2). Next, we shall prove that the conditions (T0), (T1), (T2) imply (T3), (T4), (T5), (T6) for any $U \in \mathrm{Obj}(\omega\overline{\mathfrak{T}}_M)$. The condition (T3) follows from the fact that $\mathrm{Lev}(U)$ is isomorphic to $(\mathbb{Q}, \leqslant)$ and from (T2). (T4) follows similarly. To prove (T5), we first show that $U(\alpha) \cap C$ is isomorphic to $(\mathbb{Q}, \leqslant)$. Let $\beta := \mathrm{lev}(t) < \alpha$ and $t' <^{\mathrm{lex}} t'' \in C$. Let also $1 \neq m \in M$. By putting $\varphi(t') := (1, m)$ and $\varphi(t'') := (0, m)$, by applying (T2) and using the fact that U is balanced, we obtain nodes $s', s'' \in C$ with $s' <^{\mathrm{lex}} t' <^{\mathrm{lex}} t'' <^{\mathrm{lex}} s''$. By putting $\varphi(t') := (0, m)$ instead, we obtain $t' < s''' < t'' \in C$. Hence, $U(\alpha) \cap C$ is isomorphic to $(\mathbb{Q}, \leqslant)$. It follows that $U(\alpha) \cap [t, \to)$ and $U(\alpha)$ are isomorphic to $(\mathbb{Q}, \leqslant)$ as well since they are covered by the sets of the form $U(\alpha) \cap C$. We conclude (T5) by applying (T1). To prove (T6), we consider a level $\beta < \alpha$ such that every meet of elements from F is strictly below the level β, so that the map $\gamma \in (\beta, \alpha) \mapsto (t \restriction \gamma : t \in F)$ is one-to-one. By (T0), the interval (β, α) is isomorphic to $(\mathbb{Q}, \leqslant)$ and by (T2), the coloring $\varphi\colon (\beta, \alpha) \to (M^<)^F$ is generic.

In the last part of the proof, we suppose that $U \in \mathrm{Obj}(\omega\overline{\mathfrak{T}}_M)$ satisfies (T0), (T1), (T2), and we show that U is injective in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$, and hence it is isomorphic to the Fraïssé limit U_M. By (T0), U is has no terminal nodes, so $U \in \mathrm{Obj}(\omega\mathfrak{D}_M)$, and it is balanced, so $U \in \mathrm{Obj}(\sigma\mathfrak{D}_M)$ by Lemma 4.45. To prove the injectivity, for every inclusion $S \subseteq U$ of a finite strong subtree and every one-step $\mathfrak{D}_M$-extension $S \subseteq T$, we find an extending $\sigma\mathfrak{D}_M$-arrow $f\colon T \to U$.

Let $T = S \lhd_s B$ be a terminal one-step extension by a decided bush B with $(b_i : i < m)$ being the $\leqslant^{\mathrm{lex}}$-increasing enumeration of the top level of B. We have $m = \overline{\mathrm{spl}}_S(s) = \mathrm{spl}_U(s)$ since S is decided. Let $\alpha \in \mathrm{Lev}(U)$ be the S-successor level to $\mathrm{lev}(s)$ if $\mathrm{lev}(s) \neq \max(\mathrm{Lev}(S))$. Otherwise let α be an arbitrary U-level strictly above $\mathrm{lev}(s)$, which exists since U has no terminal nodes. Let $(C_i : i < m)$ be the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $\mathrm{Spl}_U(s)$. By (T5), there is $a_i \in U(\alpha) \cap C_i$ such that $\mathrm{spl}(a_i) = \overline{\mathrm{spl}}(b_i)$ for every $i < m$. It is enough to put $f(b_i) := a_i$, $i < m$ to obtain the desired map $f\colon T \to U$.

Next, let $T = S \rhd_\alpha (B_s^s)_{s \in S(\alpha)}$ be a nonterminal one-step extension where r_s denotes the root of the bush B_s, $(b_{s,i} : i < m_s)$ is the $\leqslant^{\mathrm{lex}}$-increasing enumeration of the top level of B_s, and $k_s < m_s$ is the index such that $b_{s,k_s} = s$, for every $s \in S(\alpha)$. Let $\beta \in \mathrm{Lev}(U)$ be the S-level immediately preceding α if it exists or any U-level $\beta < \alpha$ otherwise. By (T2), there is $\alpha' \in (\beta, \alpha)$ such that $\mathrm{spl}(s \restriction \alpha', s) = (k_s, m_s)$ for every $s \in S(\alpha)$. Fix $s \in S(\alpha)$. We put $f(r_s) := s \restriction \alpha' \in U$, and we denote the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $\mathrm{Spl}(f(r_s))$ by $(C_{s,i} : i < m_s)$. By (T5) for every $i < m_s$, there is a node $a_{s,i} \in U(\alpha) \cap C_{s,i}$ with $\mathrm{spl}(a_{s,i}) = \overline{\mathrm{spl}}(b_{s,i})$. Of course, for $i = k_s$ we take $a_{s,i} = s$. Putting $f(b_{s,i}) := a_{s,i}$, $i < m_s$, concludes the construction of the desired map $f\colon T \to U$.

Altogether, we have proved that any $\omega\overline{\mathfrak{T}}_M$-object U is injective in $(\mathfrak{D}_M, \sigma\mathfrak{D}_M)$, which is equivalent to being isomorphic to U_M, if and only if it satisfies (T0), (T1), (T2), in which case it satisfies also (T3), (T4), (T5), (T6). □

We have proved that the generic tree U_M exists and characterized it up to isomorphism. Still, it may be useful to give a concrete description of U_M.

Construction 4.49. For countable linearly ordered sets S, Y and a distinguished point $0 \in S$ we describe a countable balanced strong lexicographic tree $V_{S,Y}$ with the set of levels Y and every $\mathrm{Spl}(t)$ with the lexicographic order isomorphic to S.

Let X be the countable family of all total maps $x\colon Y \to S$ with finite support $\mathrm{supp}(x) := \{y \in Y : x(y) \neq 0\}$. We endow X with the lexicographic order, that is, $x \leqslant x'$ if $x = x'$ or $x(y) < x'(y)$, where $y = \min\{y' \in Y : x(y') \neq x'(y')\}$ (note the minimum exists as the supports are finite). We put $V_{S,Y} := \{x \restriction (\leftarrow, y) : x \in X \text{ and } y \in Y\}$, that is, the family of all partial maps $t\colon (\leftarrow, y) \to S$, $y \in Y$, with finite support. The idea is that 0 corresponds to the canonical splitting direction and that a node is a code for the splitting path from the 'trunk' to the node itself. The tree order $\leqslant$ is the extension of maps, that is, $\subseteq$ when maps are viewed as sets, and it admits meets since $\mathrm{dom}(\bigcup\{v \in V_{S,Y} : v \subseteq t \cap t'\})$ is of the form $(\leftarrow, y)$ because of the finite supports.

For every $t \in V_{S,V}$ let $x(t) \in X$, denote the zero-extension of t to a map $Y \to S$, and let $y(t) := \sup(\mathrm{dom}(t)) \in Y$. The map $x \times y\colon V_{S,V} \to X \times Y$ is one-to-one since $t = x(t) \restriction (\leftarrow, y(t))$. For every incomparable t', t'', we put $t' <^{\mathrm{lex}} t''$ if $x(t') < x(t'')$. This defines the lexicographic order on $V_{S,V}$. Clearly, for every $t \in V_{S,V}$ we have that $\mathrm{Spl}(t)$ with the lexicographic order is isomorphic to S. The linearly ordered set Y serves as the level set: We put $t \leftrightarrows t'$ if $y(t) = y(t')$, so $\mathrm{lev}(t) = y(t)$ for every $t \in V_{S,V}$. Clearly, the tree $V_{S,V}$ is balanced with respect to this level structure.

Note that $V_{1,Y}$ is isomorphic to Y for every linearly ordered set Y. Also $V_{2,\omega}$ is the full binary tree $2^{<\omega}$.

Every two embeddings of linear orders $f\colon S \to S'$ and $g\colon Y \to Y'$ induce an embedding $e_{f,g}\colon V_{S,Y} \to V_{S',Y'}$ preserving meets, levels, and the lexicographic order, where every node $t\colon (\leftarrow, y)_Y \to S$ is mapped to the zero-extended lifting $t'\colon (\leftarrow, g(y))_{Y'} \to S'$, that is, $t' \circ g = f \circ t$ and $t'(y') = 0$ for every y' not in the range of g. Note that if f, g are inclusions, then t' just extends t from $(\leftarrow, y)_Y$ to $(\leftarrow, y)_{Y'}$ by zeros. If $f\colon S \to S'$ is an isomorphism, then $e_{f,g}$ also preserves splitting. This construction yields a functor $\mathsf{LO}^2 \to \mathsf{LexLevTree}$ (where LO denotes the category of all linear orders and embeddings), and for every fixed S we have a functor $\mathsf{LO} \to \mathsf{LexStrTree}$.

Let C_S denote the set of colors $\{c \subseteq S : 0 \in c\}$, that is, colors are arbitrary subsets of S containing 0 (though we will be considering mostly finite subsets later). A coloring $\varphi\colon V_{S,Y} \to C_S$ induces a pruning $V_{S,Y,\varphi} \subseteq V_{S,Y}$. We put $t \in V_{S,Y,\varphi}$ if and only if $t(y) \in \varphi(t \restriction y)$ for every $y \in \mathrm{dom}(t)$ (equivalently for every $y \in \mathrm{supp}(t)$ since $0 \in c$ for every color c). In fact, for every $t \in V_{S,Y}$ with $\mathrm{supp}(t) \neq \emptyset$ we put $y'(t) := \max(\mathrm{supp}(t))$ and consider the canonical predecessor $p(t) := t \restriction y'(t)$. We have $t \in V_{S,Y,\varphi}$ if and only if $p(t) \in V_{S,Y,\varphi}$ and $y'(t) \in \varphi(p(t))$ or $\mathrm{supp}(t) = \emptyset$. The subtree $V_{S,Y,\varphi}$ is balanced since every node can be extended by zeros. Also, we have $\mathrm{spl}(t) = |\varphi(t)|$ for every node t. The coloring φ may be given also as $\varphi(t) = \varphi'(x(t), y(t))$ for a coloring $\varphi'\colon X \times Y \to C_S$.

Construction 4.50 (A concrete model of the generic tree). The generic tree U_M may be represented as $T_\varphi := V_{\mathbb{Q},\mathbb{Q},\varphi}$ for a suitable coloring $\varphi\colon T \to C_M$, where $T := V_{\mathbb{Q},\mathbb{Q}}$ and $C_M := \{c \subseteq \mathbb{Q} : 0 \in c \text{ and } |c| \in M\}$ is the set of colors corresponding to the allowed splitting degrees. Clearly, every tree T_φ is an object of $\sigma\mathfrak{D}_M$ satisfying (T0). If $M = \{1\}$, then there is only the constant coloring φ taking the value $c_0 := \{0\}$, which works since the conditions (T1) and (T2) are trivial in this case. Otherwise, we fix $c_1 \in C_M$ with $1 \in c_1$.

Let $A := \{(t', t'', m) \in T \times T \times M : t' \leftrightarrows t'' \text{ and } t' <^{\mathrm{lex}} t''\}$ be the countable set of all situations considered in (T1). For every $a = (t', t'', m) \in A$, we choose nodes $u_a, v_a \in T$ such that $t' \wedge t'' < u_a < t'$ and $\mathrm{lev}(u_a) > \max(\mathrm{supp}(t'))$. Moreover, we let v_a be the extension of u_a defined by $v_a(\mathrm{lev}(u_a)) := 1$ and $v_a(\alpha) := 0$ for $\alpha \in (\mathrm{lev}(u_a), \mathrm{lev}(t'))$, so $t' \leftrightarrows v_a \leftrightarrows t''$ and $t' <^{\mathrm{lex}} v_a <^{\mathrm{lex}} t''$, and also u_a is the canonical predecessor $p(v_a)$. Since the set A is countable and every interval $(t \restriction \alpha, t)$ for $\alpha < \mathrm{lev}(t)$ is infinite, we may choose the points u_a so that the map $a \mapsto \mathrm{lev}(u_a)$ is one-to-one and its range is disjoint from an order-dense subset $D \subseteq \mathbb{Q}$. It follows that the map $a \mapsto v_a$ is also one-to-one since $p(v_a) = u_a$. Moreover, we may assure that $\{u_a : a \in A\} \cap \{v_a : a \in A\} = \emptyset$. Now, it is correct to define $\varphi_1(u_a) := c_1$ and $|\varphi_1(v_a)| := m$ for every $a = (t', t'', m) \in A$, and $\varphi_1(t) := c_0$ otherwise. The tree T_{φ_1} satisfies (T1) since for every $a = (t', t'', m) \in A$, if $t' \in T_{\varphi_1}$, then $u_a \in T_{\varphi_1}$, and so $v_a \in T_{\varphi_1}$. However, T_{φ_1} does not satisfy (T2): Let $a = (t', t'', m) \in A$ be such that $t' \in T_{\varphi_1}$, and let $b := (t', v_a, m)$. We have $u_b > u_a$.

There is no $\alpha \in (\mathrm{lev}(u_b), \mathrm{lev}(t'))$ such that $\mathrm{spl}(v_a \restriction \alpha) \neq 1 \neq \mathrm{spl}(v_b \restriction \alpha)$. This is because if $\varphi_1(t) \neq c_0$ for some $t \in (u_a, v_a)$, then $t = u_c$ or $t = v_c$ for some $c \in A$, and the latter is not possible since we would have $u_a = p(v_a) = p(v_c) = u_c$. The same is true for (u_b, v_b), but every level can contain at most one node of the form u_c.

To assure the condition (T2), we define $\varphi_2(t) := \psi(\mathrm{lev}(t))(x(t))$ for every $t \in T$ with $\mathrm{lev}(t) \in D$, where $x(t) \in X$ is the zero-extension of t defined in Construction 4.49 and $\psi\colon D \to H_M$ is a coloring defined as follows. H_M is the countable set of 'hypercolors' $\{h \in (C_M)^X : \mathrm{supp}(h) := \{x \in X : h(x) \neq c_0\}$ is finite$\}$. Note that D is an isomorphic copy of $\mathbb{Q}$. We let ψ be a generic H_M-coloring of D. Then for every $\beta < \alpha \in \mathrm{Lev}(T_{\varphi_2})$, every finite $F \subseteq T_{\varphi_2}(\alpha)$, and $\{(k_t, m_t) : t \in F\} \subseteq M^<$ we consider the hypercolor $h \in H_M$ defined by $h(x(t)) := c_t := \{-k_t, \cdots, m_t - k_t - 1\}$ (so that c_t contains m_t numbers and 0 is at the position k_t) for every $t \in F$, and $h(x) := c_0$ otherwise. There is $\beta' \in (\beta, \alpha)$ such that $\max(\mathrm{supp}(t)) > \beta'$ for every $t \in F$, and there is $\alpha' \in (\beta', \alpha) \cap D$ such that $\psi(\alpha') = h$, and so $\varphi_2(t \restriction \alpha') = c_t$ and $\mathrm{spl}_{T_{\varphi_2}}(t \restriction \alpha', t) = (k_t, m_t)$ for every $t \in F$. This shows (T2). On the other hand, if we define $\varphi_2(t) = c_0$ for every t with $\mathrm{lev}(t) \notin D$, we have levels α such that $\mathrm{spl}(t) = 1$ for every $t \in T_{\varphi_2}(\alpha)$, so (T1) does not hold.

To assure both (T1) and (T2), we combine the colorings φ_1 and φ_2. Observe that for a coloring $\varphi\colon T \to C_M$ to assure (T2) it is enough that $\varphi(t) = \varphi_2(t)$ for every $t \in H_\alpha$ and $\alpha \in D$ for some finite sets $H_\alpha \subseteq T(\alpha)$. This is because we can choose different witnessing levels $\alpha' \in (\beta, \alpha)$ for different (T2) situations $(\alpha, \beta, F, \{(k_t, m_t) : t \in F\})$ and put $H_{\alpha'} := \{t \restriction \alpha' : t \in F\}$. We also put $H_\alpha := \emptyset$ for every $\alpha \in \mathbb{Q} \setminus D$. So, φ defined as $\varphi(t) := \varphi_2(t)$ if $t \in H_{\mathrm{lev}(t)}$, and $\varphi(t) := \varphi_1(t)$ otherwise, assures (T2). It also assures (T1) since for $(t', t'', m) \in A$ there is $t \in (t', t'']_{\leqslant^{\mathrm{lex}}}$ with $t \leftrightarrows t'$ such that $(t', t)_{\leqslant^{\mathrm{lex}}} \cap H_{\mathrm{lev}(t')} = \emptyset$, and so $\varphi(v_{a'}) = \varphi_1(v_{a'})$ for $a' := (t', t, m) \in A$. Clearly, also $\varphi(u_a) = \varphi_1(u_a)$ since $H_{\mathrm{lev}(u_a)} = \emptyset$. This concludes the proof that T_φ is isomorphic to U_M by Theorem 4.48.

4.3.3. The weak Ramsey property and dendrites

We work with the trio of categories of finite trees $\mathfrak{T}_M$, $\overline{\mathfrak{T}}_M$, $\mathfrak{D}_M$ for a fixed nonempty set $M \subseteq \mathbb{N}^+$ of allowed splitting degrees. We have shown that $\mathfrak{T}_M, \mathfrak{D}_M \subseteq \overline{\mathfrak{T}}_M$ are full cofinal subcategories, that $\mathfrak{T}_M$ is a weak Fraïssé category, $\mathfrak{D}_M$ is a Fraïssé category and we have described the common (weak) Fraïssé limit U_M. In this situation, $\mathfrak{T}_M$ has the weak Ramsey property if and only if $\mathfrak{D}_M$ has the Ramsey property if and only if $\mathrm{Aut}(U_M)$ is extremely amenable. We were unable to show that the equivalent properties hold. In this section, we give partial results in this direction and pose the general case as an open question.

The special case $M = \{m\}$, in which $\mathfrak{T}_M$ has the amalgamation property since there is only one option for the splitting degree (Theorem 4.40 (i)), is covered by the finite version of Milliken's theorem [19, Corollary 1.5].

Theorem 4.51 (Milliken). *For every positive integers m, a, b, k, there exists a positive integer $N = N(m, a, b, k)$ with the following property. If T is a finite balanced tree of height N and splitting degrees $\leqslant m$ and all its balanced strong subtrees of height a are colored by (partitioned into) k colors, then there exists a balanced strong subtree $S \subseteq T$ of height b such that all balanced strong subtrees of S of height a have the same color.*

Corollary 4.52. *For $m \in \mathbb{N}^+$ and $M = \{m\}$, the category $\mathfrak{T}_M$ has the Ramsey property.*

Proof. For every $n \in \mathbb{N}$, let T_n denote the (unique up to isomorphism) lexicographically ordered balanced m-splitting tree of height n. The full subcategory $\{T_n : n \in \mathbb{N}^+\} \subseteq \mathfrak{T}_M$ is cofinal and so it suffices to show that for any a, the tree T_a is Ramsey. Note that for $a, b \in \mathbb{N}$, balanced strong subtrees $S \subseteq T_b$ of height a are in one-to-one correspondence with $\mathfrak{T}_M(T_a, T_b)$ by taking an embedding to its image. Fix $a, b, k \in \mathbb{N}^+$, and set $N = N(m, a, b, k)$. Take a coloring $\varphi\colon \mathfrak{T}_M(T_a, T_N) \to k$. By the Milliken's theorem, there is a balanced strong subtree $S \subseteq T_N$ of height b and the corresponding $\mathfrak{T}_M$-embedding $f\colon T_b \to T_N$ such that φ is constant on $f \circ \mathfrak{T}_M(a, b)$. $\square$

$$\begin{tikzcd}
& \overline{\mathfrak{T}}'_M & \\
\mathfrak{T}'_M \arrow[ur, hook] & & \mathfrak{D}'_M \arrow[ul, hook'] \\
& \overline{\mathfrak{T}}_M \arrow[uu, hook, dashed] & \\
\mathfrak{T}_M \arrow[uu, hook, dashed] \arrow[ur, hook] & & \mathfrak{D}_M \arrow[uu, hook, dashed] \arrow[ul, hook']
\end{tikzcd}$$

Figure 11. *Inclusions between the trios of categories. Diagonal inclusions are full cofinal; vertical inclusions are wide dominating.*

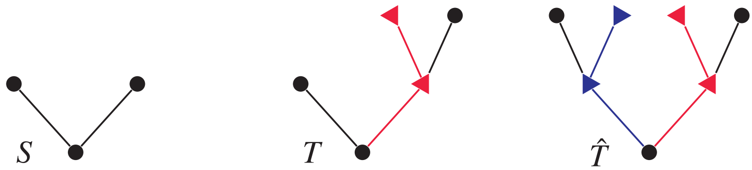

Figure 12. *Domination by level-preserving embeddings.*

The generic tree U_M shares a lot of connections with the generalized Ważewski dendrite W_{M+1}. Known related results by Duchesne [5] and Kwiatkowska [15] allow us to obtain the desired Ramsey property. However, we need to modify our categories. Namely, we need to forget the level structure.

Definition 4.53. For $M \subseteq \mathbb{N}^+$, we consider the trio $\mathfrak{T}'_M, \overline{\mathfrak{T}}'_M, \mathfrak{D}'_M$ of *leveless* variants of the categories $\mathfrak{T}_M, \overline{\mathfrak{T}}_M, \mathfrak{D}_M$, that is, the levels are not part of the structure and they are not preserved by embeddings. In particular, $\mathfrak{T}'_M$ is the full subcategory of $\mathsf{LexSplTree}$ (as opposed to $\mathsf{LexStrTree}$) consisting of finite trees with splitting degrees in M; $\overline{\mathfrak{T}}'_M$ additionally allows and $\mathfrak{D}'_M$ requires deciding the splitting degrees at terminal nodes (formally via unary relations R_m as in the definition of $\overline{\mathsf{FSplTree}}$).

Let $F\colon \overline{\mathfrak{T}}_M \to \overline{\mathfrak{T}}'_M$ be the functor forgetting the level structure. Since every tree in $\overline{\mathfrak{T}}'_M$ is finite, it admits a unique level structure, so the functor F is not only faithful but also bijective on objects. Hence, we may view $\overline{\mathfrak{T}}_M$ as a subcategory of $\overline{\mathfrak{T}}'_M$ with the same objects, but fewer morphisms. This way, we have $\mathfrak{T}_M = \overline{\mathfrak{T}}_M \cap \mathfrak{T}'_M$ and $\mathfrak{D}_M = \overline{\mathfrak{T}}_M \cap \mathfrak{D}'_M$ as in Figure 11, and $\mathfrak{T}'_M, \mathfrak{D}'_M \subseteq \overline{\mathfrak{T}}'_M$ are full cofinal subcategories as with the original trio.

We will show in Proposition 4.55 that the subcategory $\overline{\mathfrak{T}}_M \subseteq \overline{\mathfrak{T}}'_M$ (and so also $\mathfrak{T}_M \subseteq \mathfrak{T}'_M$ and $\mathfrak{D}_M \subseteq \mathfrak{D}'_M$) is dominating, that is, for an embedding $f \in \overline{\mathfrak{T}}'_M$ there is $g \in \overline{\mathfrak{T}}'_M$ such that $g \circ f \in \overline{\mathfrak{T}}_M$. It follows that every amalgamable $\overline{\mathfrak{T}}_M$-arrow is also amalgamable in $\overline{\mathfrak{T}}'_M$ (Lemma 2.16), so $\mathfrak{T}'_M$ is a weak Fraïssé category and $\mathfrak{D}'_M$ is a Fraïssé category. Moreover, a (weak) Fraïssé sequence in $\mathfrak{T}_M$ or $\mathfrak{D}_M$ is a (weak) Fraïssé sequence in $\mathfrak{T}'_M$ or $\mathfrak{D}'_M$ (Lemma 2.15), and so the common (weak) Fraïssé limit of $\mathfrak{T}'_M$ and $\mathfrak{D}'_M$ is the generic tree U_M with its level structure removed. We denote it by U'_M.

Remark 4.54. Following [8, Definition 5.1], we may call a faithful functor $F\colon \mathfrak{C} \to \mathfrak{C}'$ (e.g., representing the language reduct between classes of structures) *reasonable* if for every $\mathfrak{C}$-object A and every $\mathfrak{C}'$-arrow $f'\colon F(A) \to B'$ there is a $\mathfrak{C}$-arrow $f\colon A \to B$ with $F(f) = f'$. This may be viewed as a strong form of the absorption property (D1). However, in the case F is the inclusion of a wide subcategory (as above), F being reasonable would already imply $\mathfrak{C} = \mathfrak{C}'$. In particular, our $F\colon \overline{\mathfrak{T}}_M \to \overline{\mathfrak{T}}'_M$ is not reasonable.

Proposition 4.55. *The subcategory $\overline{\mathfrak{T}}_M$ is dominating in $\overline{\mathfrak{T}}'_M$.*

Proof. Let S be a $\overline{\mathfrak{T}}'_M$-subtree of T. The inclusion $S \subseteq T$ may not preserve levels. We find an $\overline{\mathfrak{T}}'_M$-extension $T \subseteq \hat{T}$ such that the inclusion $S \subseteq \hat{T}$ will preserve levels, as in Figure 12. Namely, for every nonterminal node $s \in S$ and an S-immediate successor $s' > s$ we let $h_{s,s'}$ denote the cardinality of $(s, s')_T$, we put $h_s := \max\{h_{s,s'} : s'$ an immediate successor of s in $S\}$, and we form $\hat{T} \supseteq T$ by adding a chain of $h_s - h_{s,s'}$ new nodes between s' and its predecessor in T for every $s \in S$ and an S-immediate successor $s' > s$. Also, for every newly added node t we decide its splitting degree $m \in M$ and add $m - 1$ new immediate successors of t. □

We can obtain a characterization of the generic leveless tree U'_M similar to Theorem 4.48. But first, we need the following observation.

Observation 4.56 (Leveless extensions). We reuse our results and notation from Section 3.3 to quickly describe how the extensions in the leveless categories $\mathfrak{T}'_M$, $\overline{\mathfrak{T}}'_M$ and $\mathfrak{D}'_M$ look like. The definition of a terminal and nonterminal extension makes sense in the leveless context. As in Proposition 4.34 every extension can be canonically decomposed into a nonterminal extension followed by a terminal extension, and as in Proposition 4.31 every terminal extension is of the form $S \lhd_{s \in A} T_a$ for a set A of terminal nodes. Also, every terminal extension can be decomposed into one-step terminal extensions, which are of the form $S \lhd_s B$ for a bush B. The situation with nonterminal extensions is different — in the leveless context we are allowed to perform tree surgery at individual nodes rather than whole levels. Similarly to Proposition 4.37, it can be shown that every nonterminal extension is canonically of the form $T = S \rhd_{s \in A} C_s^{c_s}$, where $A \subseteq S$ and every $C_s^{c_s}$ is a pointed bush-column, and so T can be decomposed into one-step nonterminal extensions of the form $S \rhd_s B^b$ for a pointed bush B^b.

Theorem 4.57. *U'_M is the unique (up to* LexSplTree*-isomorphism) countable lexicographic tree such that every branch is isomorphic to $(\mathbb{Q}, \leqslant)$, and for every $s < t \in U'_M$ and $(k, m) \in M^<$ there is $u \in (s, t)$ such that* $\mathrm{spl}(u, t) = (k, m)$.

Proof. Clearly, by Theorem 4.48, U'_M satisfies the characterizing conditions. For the converse implication, we may use Fraïssé theory and prove that a countable lexicographic tree U satisfying the conditions is injective with respect to $\mathfrak{D}'_M$. Let $S \subseteq U$ be a decided lexicographic subtree. By the previous observation, it is enough to consider one-step extensions. Suppose $T = S \lhd_s B$ for an S-terminal node s and a bush B. Since S is decided, we have $m := \mathrm{spl}_B(s) = \overline{\mathrm{spl}}_S(s) = \mathrm{spl}_U(s)$. The last equality uses the assumption that $s \in U$ is not terminal. It is enough to define the extension $f : T \to U'_M$ extending id_S by putting $f(b_i) \in C_i$ for every $i < k$, where $(b_i)_{i<m}$ is the $\leqslant^{\mathrm{lex}}$-increasing enumeration of the top level of B and $(C_i)_{i<m}$ is the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $\mathrm{Spl}_U(s)$. Next, suppose $T = S \rhd_s B^s$ for a node $s \in S$ and a pointed bush B^s. Let r be the root of B, let $(b_i)_{i<m}$ be the $\leqslant^{\mathrm{lex}}$-increasing enumeration of the top level of B and let $k < m$ be the index with $b_k = s$. By the assumption that U has no root and by the characterizing property, there is a node $t < s$ with $\mathrm{spl}_U(t, s) = (k, m)$. Hence, we may take a $\leqslant^{\mathrm{lex}}$-increasing enumeration $(C_i)_{i<m}$ of $\mathrm{Spl}_U(t)$ with $s \in C_k$. Putting $f(b_i) \in C_i$ for $i < m$ with $f(b_k) = f(s) = s$ defines a desired extension $f : T \to U$ of id_S.

Alternatively, we could show that a countable lexicographic tree U satisfying the conditions admits a level structure such that the expansion satisfies the conditions in Theorem 4.48. The idea is to fix a family $\{B_n\}_{n\in\omega}$ of branches covering U, let B_0 correspond to the set of levels, and construct suitable order-isomorphisms $B_n \to B_0$. We have $B_0 \cap B_1 = (\leftarrow, b]$ for some $b \in U$. We choose an order-isomorphism $g : (b, \to)_{B_n} \to (b, \to)_{B_0}$ such that every combination $(m, m') \in M^2$ appears as the value of $(\mathrm{spl}_U(t), \mathrm{spl}_U(g(t))$ for densely many nodes $t \in (b, \to)$. And we continue similarly with other branches B_n. □

We now begin our discussion on dendrites and their connection to the generic leveless tree U'_M. Recall that a *dendrite* (see, e.g., [20, §X]) is a locally connected metrizable continuum D such that every two points $x, y \in D$ are connected by a unique arc denoted by $[x, y]$ or equivalently there are no (nontrivial) closed curves in D. Also, for every three points $x, y, z \in D$ there is a unique point in the intersection $[x, y] \cap [y, z] \cap [z, x]$ denoted by $\wedge(x, y, z)$ and called a *ternary meet* or a *median*.

Every point $x \in D$ has the *order* $\mathrm{ord}(x) \in \{1, 2, 3, \ldots, \omega\}$ equal to the number of connected components of $D \setminus \{x\}$. Points of order 1 are called *end points*, points of order 2 are called *ordinary points*, and points of order $\geqslant 3$ are called *branch points*. We denote the set of all branch points, which is countable, by $\mathrm{Br}(D)$.

Finally, for every $M \subseteq \{3, 4, 5, \ldots, \omega\}$ there is a unique (up to homeomorphism) dendrite W_M called *(generalized) Ważewski dendrite* such that the order of every branch point is in M and for every $m \in M$

and every arc $A \subseteq W_M$ the set of branch points of order m is dense in A. Generalized Ważewski dendrites were introduced in [3, §6]. The universal Ważewski dendrite $W_{\{\omega\}}$ originates from [25].

Construction 4.58 (Rooting a dendrite). Let D be a dendrite, and let $r \in D$. There is a natural way to turn D into a tree by rooting it at r. We define the tree order by $x \leqslant y \;:\Longleftrightarrow\; [r,x] \subseteq [r,y]$. This is a well-defined order that has meets: $x \wedge y = \wedge(r,x,y)$. Clearly, $[r,z] \subseteq [r,x] \cap [r,y]$ if and only if $[r,z] \subseteq [r, \wedge(r,x,y)]$.

Moreover, for every $x \in D$, $\mathrm{Spl}(x)$ consists of the connected components of $D \setminus \{x\}$ omitting the one containing r unless $x = r$, and hence $\mathrm{spl}(x) = \mathrm{ord}(x) - 1$ if $x \neq r$, and $\mathrm{spl}(r) = \mathrm{ord}(r)$. This is because the components C of $D \setminus \{x\}$ as well as the corresponding sets $C \cup \{x\}$ are arcwise connected, so for $y \in C$ and $z \in C'$, the arc $[y,z]$ goes through x if $C \neq C'$, and stays in C if $C = C'$. Hence, $y < x$ for $y \in C$ if C is the component containing r, and $y \wedge z = x$ if $y \in C$ and $z \in C'$ and $C \neq C'$ are components not containing r, and $y \wedge z > x$ for $y, z \in C \not\ni r$.

Construction 4.59. For a nonempty set $M \subseteq \mathbb{N}^+$ we build a $\mathsf{LexSplTree}$-isomorphic copy of U'_M from the Ważewski dendrite $W_{M'}$, where $M' = \{m+1 : 1 \neq m \in M\} \subseteq \{3,4,5,\dots\}$.

We pick an end point $r \in W_{M'}$ and root the dendrite at r according to the previous construction. Note that in the resulting tree every branch has the order type of the closed real interval $[0,1]$ and contains densely many branch points of every order from M' as well as densely many ordinary points (there are only countably many branch point in a dendrite). Also, note that the end points of the dendrite correspond to maxima of the tree and to the root.

Let $T \subseteq W_{M'}$ be the countable subset consisting of all branch points and, if $1 \in M$, also of an ordinary point $z \in (x,y)$ for every $x < y \in \mathrm{Br}(W_{M'})$. For every $x, y \in T$, we have $x \wedge y = \wedge(r,x,y)$, which is either one of x, y or a branch point. Since T contains all branch points, it follows that T is closed under meets. Since also the branch points are dense in every arc of the dendrite, we have that $\mathrm{spl}_T(t) = \mathrm{spl}_{W_{M'}}(t)$ for every $t \in T$. Hence, the inclusion $T \subseteq W_{M'}$ is a $\mathsf{SplTree}$-embedding. Moreover, every branch in T is ordered like the rationals since T contains no endpoints, and for every $x < y \in T$ and $m \in M$, there is $z \in (x,y)_T$ with $\mathrm{spl}_T(z) = m$.

To construct the desired $\mathsf{LexSplTree}$-isomorphic copy of U'_M, we endow T with a lexicographic order such that the resulting tree satisfies the characterizing condition of Theorem 4.57. To define a compatible lexicographic order, it is enough to fix a linear order on every $\mathrm{Spl}_T(x)$ for $x \in T$, but to ensure the condition we choose a point $w_{x,y,k,m} \in (x,y)_T$ of splitting degree m for every $x < y \in T$ and $(k,m) \in M^<$ such that the map $(x,y,k,m) \mapsto w_{x,y,k,m}$ is one-to-one, and we make sure that $y \in C_k$ where $(C_i)_{i<m}$ is the $\leqslant^{\mathrm{lex}}$-increasing enumeration of $\mathrm{Spl}_T(w_{x,y,k,m})$.

Finite rooted subtrees obtained from the branch points of dendrites have been extensively studied, often to the great success of computing the universal minimal flow of the Ważewski dendrites; see Duchesne [5] and Kwiatkowska [15]. In fact, a rephrasing of a result by Kwiatkowska gives us the Ramsey property of $\mathfrak{D}'_M$.

Theorem 4.60. *For every nonempty $M \subseteq \mathbb{N}^+$, the category $\mathfrak{D}'_M$ has the Ramsey property. Hence, $\mathfrak{T}'_M$ has the weak Ramsey property and* $\mathrm{Aut}(U'_M)$ *is extremely amenable.*

Proof. The second part follows from our general theory. The first part is a reformulation of [15, Theorem 3.6]. Namely, let $\mathfrak{D}''_M$ be the modified version of the category $\mathfrak{D}'_M$ where the predetermined splitting degree $\overline{\mathrm{spl}}$ may be strictly greater (but still from M) than the actual splitting degree spl. The $\mathfrak{D}''_M$-arrows preserve the relations R_m, $m \in M$, that encode $\overline{\mathrm{spl}}$, so it is the predetermined splitting degree rather than the actual one that is being preserved. Nevertheless, $\mathfrak{D}'_M \subseteq \mathfrak{D}''_M$ is a full cofinal subcategory since we may simply add the missing immediate successors to any nonterminal node s such that $\mathrm{spl}(s) < \overline{\mathrm{spl}}(s)$. Hence, $\mathfrak{D}'_M$ has the Ramsey property if and only if $\mathfrak{D}''_M$ has the Ramsey property.

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The statement of [15, Theorem 3.6] is that a certain category $\mathcal{T}_P^*$ has the Ramsey property. We argue that $\mathfrak{D}''_M$ is equivalent to $\mathcal{T}_P^*$ for $P = M + 1$. The objects of $\mathcal{T}_P^*$ are finite trees, but the unordered graph-theoretic tree structure is encoded by a quaternary relation $D(a, b, c, d)$ holding if the finite paths from a to b and from c to d do not intersect, and the tree order is encoded by a ternary relation $C(a, b, c)$ defined by $D(a, b, c, r)$, where r is a fixed root. By [15, Proposition 3.3], a map between trees preserves the relations C and D if and only if it preserves the tree order and meets. The trees are labeled by unary relations K_p, $p \in P$, giving an upper bound for a splitting degree and directly corresponding to our relations R_m, where $p = m + 1$. Finally, the lexicographic order is encoded by binary relations $G_i(a, b)$, $i \in \mathbb{N}^+$: each node a with a predetermined splitting degree m has 'slots' for immediate successors indexed by $i \in \{1, \ldots, m\}$, and $G_i(a, b)$ holds if b is an actual immediate successor of a occupying the ith slot.

Note that the statement of [15, Theorem 3.6] does not cover the cases when $1 \in M$ (though it allows infinite bound on a splitting degree); however, the proof could be rewritten using our language to directly show that $\mathfrak{D}'_M$ has the Ramsey property for any M. □

The following question remains open as the level structure poses a new challenge. The above, however, seems like a step forward to proving this fact and deducing another structural variant of the classic Milliken's theorem.

Question 4.61. Does the category $\mathfrak{T}_M$ have the weak Ramsey property? Equivalently, does $\mathfrak{D}_M$ have the Ramsey property? Equivalently, is $\mathrm{Aut}(U_M)$ extremely amenable?

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