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A NOTE ON STEADY FLOW INTO A SUBMERGED POINT SINK

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Abstract

The steady, axisymmetric flow induced by a point sink (or source) submerged in an unbounded inviscid fluid is computed. The resulting deformation of the free surface is obtained, and a limit of steady solutions is found that is quite different to those obtained in past work. More accurate solutions indicate that the old limiting flow rate was too high and, in fact, the breakdown of steady solutions at a lower flow rate is characterized by the appearance of spurious wavelets at the free surface.

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1. Introduction

The flow of ideal fluid into a line or point sink has been used for many years as a proxy for the withdrawal of water from reservoirs. The problem is of importance for management of reservoirs and control of water quality in drinking supply [12], and the solutions have been shown to provide a reasonable representation of the real situation in many cases, but have led to some confusion in others; see for example [8, 11, 13, 14, 23].

Some of this work concentrated on unsteady flows [17–19, 22, 24], but, throughout the twentieth century, most work was done by seeking steady flow solutions that provided some estimates of the various critical parameters in such flows [2, 3, 7, 10, 16, 20, 21], to mention but a few. More recently, some cases of supercritical steady flows were obtained for a line sink [4, 5, 9] and for a point sink [6]. The main reason

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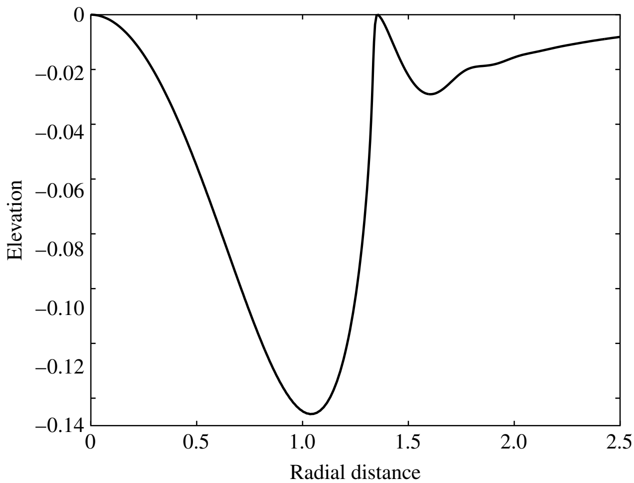

FIGURE 1. Limiting shape of the free surface $F = 6.35$ recomputed using the values in [3]. A uniform grid spacing of $\Delta s = 0.04$ and a truncation point of $tr = 6.0$ were used to obtain this figure.

for studying the steady flow case was the lack of computing power and the lack of stable numerical methods for dealing with free surface hydrodynamical problems – in particular, the formation of a singularity on the free surface after a finite critical time that often occurs in unsteady inviscid flow (see the article by Moore [15]). More complete historical information on this problem can be found in Stokes et al. [17, 18], which discusses all of the relevant line sink and point sink results.

In this note, we revisit one of the fundamental solutions obtained during this period [3]. We consider the simplest withdrawal flow with a three-dimensional aspect; a single point sink submerged beneath a fluid surface where the fluid is unbounded below. In this configuration, there is only one dimensionless parameter that combines the flux into the outlet, Q_S, the depth of the point sink, h_S, and the gravitational acceleration, g, into a quantity known as the Froude number,

$$F = \frac{Q_S}{g h_S^5}. \tag{1.1}$$

Increasing the value of the Froude number corresponds to increasing the flow rate or moving the sink closer to the surface. Forbes and Hocking [3] found that as the Froude number was increased, a circular wave formed on the surface surrounding the sink, and that eventually this rose up to a stagnation ring leading to the breakdown of the solution. Also, they found the limiting Froude number to be $F \approx 6.4$. A depiction of this solution is given in Figure 1.

This solution was obtained using an integral equation formulation with an even grid spacing, $\Delta s = 0.05$, in the arc length, s, along the free surface and truncation of the infinite integral at $tr = 6.0$. It seems surprising now that these results were limited by the capacity of the computer – smaller grid spacing and larger computational windows

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made the length of the computations prohibitive on computers of the day. Tests were performed to compare the accuracy of the solution by decreasing the grid spacing and moving the truncation point further out. Given the limited capabilities, these solutions appeared to have converged, and there was no reason to think that the solutions would change further if longer simulations could be performed.

Given the vastly greater computer power currently available, we have revisited this problem and obtained some surprising results. Decreasing the grid spacing over the same computational window makes almost no difference to the solutions (using the earlier formulation [3]), but significantly increasing the computational window makes a dramatic difference that leads to completely different conclusions to the earlier work. Given this fact, a second numerical scheme is developed to verify and compare with the original scheme.

Therefore, the purpose of this note is to *correct* the conclusions of the earlier paper [3] in the light of this. In some ways, these conclusions are disappointing and far less interesting, but they are consistent with the unsteady results that have been considered in more recent work.

2. Problem formulation

While the complete formulation was given in the earlier paper [3], we will summarize it here for completeness. We consider the steady, irrotational, axisymmetric flow of an inviscid, incompressible fluid beneath a free surface. The flow is driven by flux into a point sink of strength Q_S in $\mathrm{m}^3\ \mathrm{s}^{-1}$ situated at a depth h_S in metres, m, beneath the undisturbed level of the free surface. Under these assumptions, the problem can be formulated in terms of a velocity potential $\phi(r, z)$, where r is a radial coordinate centred on the location of the point sink and z is the vertical coordinate with $z = 0$ corresponding to the level of the undisturbed free surface which corresponds to the stagnation level. The radial and vertical components of velocity are given by $u = \Phi_r$ and $v = \Phi_z$, respectively. The sink sits at a depth of $z = -h_S$, and the fluid is not bounded below.

Nondimensionalizing the potential and length with respect to Q_S/h_S and h_S respectively, where the quantity Q_S is the total flux from the full point sink, the problem is to solve

$$\nabla^2\Phi = 0, \quad z < \eta(r), \quad (r, z) \neq (0, -1),$$

subject to

$$\tfrac{1}{2}(\Phi_r^2 + \Phi_z^2) + F^{-2}\eta = 0 \quad \text{on } z = \eta(r) \tag{2.1}$$

and

$$\Phi_r\eta_r - \Phi_z = 0 \quad \text{on } z = \eta(r). \tag{2.2}$$

The sink is now located one unit beneath the free surface and has total withdrawal flux of 4π. These equations include the main parameter that control this flow, the Froude number (1.1).

In the limit as we approach the point sink at $(r, z) = (0, -1)$, the velocity potential should take the form

$$\Phi_S \to \frac{1}{\sqrt{r^2 + (z+1)^2}}.$$

A change of sign reverses the flow direction from a sink flow to a source flow. However, in the case of steady flow, the quadratic nature of the velocity term in the dynamic condition (2.1) means that solutions generated apply for both source and sink flows.

3. The numerical method

To consider the full nonlinear steady flow problem, we implement a numerical scheme identical to that of Forbes and Hocking's [3] and another, different scheme for the sake of comparison. In both cases, we use the same integral equation form, but use completely different numerical methods. The flow is assumed to be axisymmetric, and an integral equation is derived for the elevation and velocity potential on the free surface.

3.1. Formulation We derive an integral equation for the unknown harmonic function, $\Phi(r, z)$, and surface elevation, $z = \eta(r)$. Let Q be a fixed point on the free surface with coordinates $(r, \theta, \eta(r))$ and $P(\gamma, \beta, \mu)$ be another point which is free to move over the same surface. Since Φ is a harmonic function over the full region except at the sink itself, we can define another function $\Psi = 1/R_{PQ}$ which is also analytic, except when P and Q are the same point, that is,

$$\Psi = \frac{1}{R_{PQ}} = \frac{1}{[r^2 + \gamma^2 - 2r\gamma\cos(\beta - \theta) + (z - \mu)^2]^{1/2}}.$$

Invoking Green's second identity,

$$\iint_{\partial V} \left[\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}\right] dS = 0,$$

where n denotes the outward normal direction, and ∂V consists of the surface of the free surface S_T with the point Q carefully excluded by a small hemispherical surface, S_Q, and a small sphere about the sink, S_ϵ.

It is not difficult to show that the contributions from all of these surfaces lead to an integral equation of the form

$$2\pi\Phi(Q) = \frac{1}{(r^2 + (z + h_S)^2)^{1/2}} - \iint_{S_T} \Phi(P) \frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right) dS_P.$$

Following the previous work [3], the surface integral can be specified in terms of the variables of the problem as

$$2\pi\Phi(Q) = \frac{1}{(r^2 + (z+1)^2)^{1/2}} - \int_0^\infty \Phi(P)\mathcal{K}(a, b, c, d)\, d\rho,$$

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in which the kernel function is

$$\mathcal{K}(a,b,c,d) = \gamma \int_0^{2\pi} \frac{a - b\cos(\beta - \theta)}{[c - d\cos(\beta - \theta)]^{3/2}}\, d\beta$$

and the intermediate quantities a–d are defined as

$$a = \gamma\eta_\gamma(P) - (\eta(P) - \eta(Q)), \quad b = r\eta_\gamma(P),$$
$$c = \gamma^2 + r^2 + (\eta(P) - \eta(Q))^2, \quad d = 2r\gamma.$$

Forbes and Hocking [3] reduced this to the form

$$\mathcal{K}(a,b,c,d) = \frac{2}{\sqrt{c+d}}\Big[\eta_\gamma K\Big(\frac{2d}{c+d}\Big) + \Big(\frac{2ar - \eta_\gamma c}{c-d}\Big)E\Big(\frac{2d}{c+d}\Big)\Big],$$

where K and E are complete elliptic integrals of the first and second kinds as defined by Abramowitz and Stegun [1]. At this point, we note that E is well behaved over the interval of interest, but that K has a logarithmic singularity as $P \to Q$ in the integral over the free surface.

This problem was solved using a formulation based on arc length along the surface, so that s is the distance from $\gamma = 0$ to Q, and σ is the distance along the surface to P. The standard formula

$$\Big(\frac{dr}{ds}\Big)^2 + \Big(\frac{d\eta}{ds}\Big)^2 = 1 \tag{3.1}$$

defines the arc length s in terms of r and η. We define a surface potential $\phi(s)$ and, applying the chain rule, we find that along the surface,

$$\frac{\partial\phi}{\partial r} = \Phi_r(r,\eta) + \Phi_z(r,\eta)\frac{d\eta}{dr}.$$

Eliminating Φ_z from the Bernoulli equation (2.1) and the kinematic condition (2.2) and combining leads to a single relation

$$\frac{1}{2}F^2\Big(\frac{d\phi}{ds}\Big)^2 + \eta(s) = 0 \tag{3.2}$$

on the free surface $z = \eta(r)$.

Rewriting the integral equation in terms of arc length,

$$2\pi\phi(s) = \frac{1}{[r^2(s) + (\eta(s)+1)^2]^{1/2}} - \int_0^\infty \phi(\sigma)\mathcal{K}(A,B,C,D)\, d\sigma, \tag{3.3}$$

where

$$A = r(\sigma)\eta'(\sigma) - r'(\sigma)(\eta(\sigma) - \eta(s)), \quad B = r(s)\eta'(\sigma),$$
$$C = r^2(\sigma) + r^2(s) + (\eta(\sigma) - \eta(s))^2, \quad D = 2r(s)r(\sigma).$$

In the previous work [3], the authors noted that

$$\int_0^\infty \mathcal{K}(A,B,C,D)\, d\sigma = 0,$$

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so that

$$2\pi\phi(s) = \frac{1}{[r^2(s) + (\eta(s) + 1)^2]^{1/2}} - \int_0^\infty [\phi(\sigma) - \phi(s)]\mathcal{K}(A, B, C, D)\, d\sigma \qquad (3.4)$$

is equivalent to (3.3). It is this form (3.4) that is used in Method I below, while Method II uses (3.3).

This integral equation is coupled with the condition (3.1), subject to (3.2), to give the complete formulation of the problem. The arc-length formulation allows the method to find multiple-valued or overhanging free surface shapes should they exist.

3.2. Computational details – two methods The equations derived in the previous section are highly nonlinear both because of the quadratic dependence on velocity of the surface condition and the fact that the surface shape is unknown. The equations were, therefore, solved numerically using collocation. A grid of points was chosen at arc-length values $s = s_0, s_1, s_2, s_3, \ldots, s_N$. An initial guess for the potential function, $\phi = \phi_0, \phi_1, \ldots, \phi_N$, on the surface was made and used to compute the surface shape $\eta = \eta_0, \eta_1, \eta_2, \ldots, \eta_N$ from the surface condition (3.2). These values were then used to compute the error in the integral equation (3.3). The initial guess was then updated using a damped Newton's method until the error in all equations dropped below 10^{-8}.

Earlier [3], the grid of points was chosen to be uniform, but we modified this in Method I here to have a very gradual geometric expansion of points to much greater distances. Even with the large computational power available today, calculations with a uniform grid become prohibitive for large computational windows. In this way, truncation can be moved out as far as we wish, so long as sufficiently many points are used to ensure accurate solutions.

In the previous work [3] and Method I, the integral was evaluated by extracting the singularity as $\sigma \to s$. The correction for this can be shown to be zero if the integral extends to infinity. Therefore, the implementation involves the modified integral equation (3.4). Once the singular point was removed, all integrals were evaluated using cubic spline integration. The solutions converged very well at a fixed truncation value s_N, accurate to graphical accuracy with $\Delta s = 0.01$ in the central (nonexpanding) region of the grid.

In Method II, a product integration scheme with quadratic segments was used to evaluate the integral in equation (3.3), noting the logarithmic singularity in the K elliptic integral as $\sigma \to s$. The nonsingular part was computed using cubic spline integration. Tests showed no significant difference in the results if linear segments were used in the product integration.

Calculations were done with both methods, and it was found that for Method I the size of the computational window began to have a significant impact while the step size was found to be less important. However, the variation in results as the window size was increased was very small, partly explaining the mistaken conclusions earlier [3]. The conclusion of this is that although the omission of the extra term explained in equation (3.4) is valid, the missing component may be significant if the window is truncated too soon.

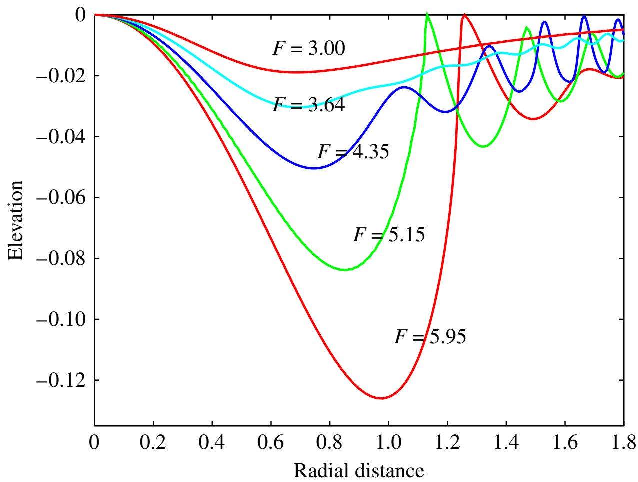

FIGURE 2. Limiting free surface shapes as the grid spacing is reduced.

Method II was the opposite in that a sufficient computational window was found to be approximately $s_N = 10$, but that decreasing step size made a big difference. Thus, using the product integration method directly removes the problem of truncation in Method I, but dealing with the logarithmic singularity directly gives a less accurate method in terms of grid resolution. In order to compute results for very small grid spacing, the Jacobian was not computed fully. It was found that computing the leading diagonal together with 40 or 50 diagonals either side provided sufficient accuracy in the iterative scheme, but made the process run orders of magnitude faster (that is, instead of $N \times N$ calculations to find the Jacobian there were approximately $80 \times N$).

However, and importantly, the effect of making both methods more accurate, whether by increasing the computational window in Method I or by increasing the resolution in Method II, was the same.

4. Results and conclusions

Solutions identical to those in the earlier paper by the authors [3] were obtained using parameters matching those in that work for Method I. However, increasing accuracy of both methods led to an increase in waviness on the free surface, which caused the maximum Froude number that could be obtained to decrease quite significantly.

Figure 2 shows the limiting free surface shapes as the accuracy was increased. At $\Delta s = 0.03$, the maximum $F = 5.95$ looks very similar to the original solution in [3], except that the stagnation ring is closer to the centre and the deepest dip is shallower. This pattern continues in the case $\Delta s = 0.01$ with limiting $F = 5.15$, in which the surface ring has again moved inward, and the second ripple has become much steeper as has the third. By the time Δs is reduced to $\Delta s = 0.005$, $F = 4.35$, the first ripple is

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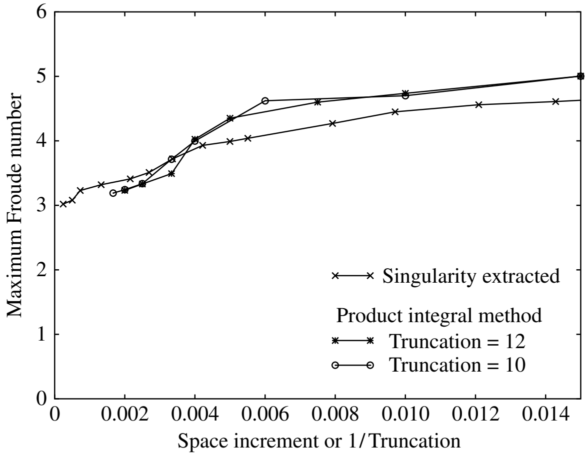

FIGURE 3. Limiting Froude number as the grid spacing is reduced (product integration method) or the truncation point is increased (singularity extraction method) (limiting F is plotted against 1/truncation point). There is no reason why the truncation curve should match the grid spacing curves except as they all approach zero on the horizontal axis.

no longer the highest, and it is the fact that the fourth local maximum has reached the stagnation level that causes the method to fail. This trend continues as the solutions are computed more accurately, so that when $\Delta s = 0.0033$, the limiting $F \approx 3.64$ and none of the inner ripples reaches the stagnation level. Presumably, ripples further out rise up to *destroy* the solutions. In the final surface shape shown, $\Delta s = 0.0029$, $F = 3.0$, there no longer appears to be any significant wave on the surface. It seems likely that more accurate solutions will also converge up to this value of F, as there is no wave on the surface to confound the solutions. Careful examination of the solution with $F = 3.0$ shows tiny ripples on the surface which exist only over a short interval in r on the upslope of the major dip, suggesting that the maximal F solution is slightly below this value, but it was found to be impossible to compute solutions with smaller Δs in a reasonable time. This conclusion appears to be supported by Figure 3, in which limiting F values are shown for increasingly accurate solutions using both Method I, that is, $F_{\max}$ versus 1/truncation and Method II, that is, $F_{\max}$ versus $1/N$. There is no reason why the two curves for the product integral method and the curve for the singularity extraction method should match, except in the limit as they approach zero. Using both methods, a case can be made that the limiting value is just below $F = 3.0$. Furthermore, a spectral method was developed (the details are not given for succinctness) and was found to converge only for values of $F < 3$, again seeming to confirm the above conclusions.

However, it is still possible that for *infinitely accurate* solutions there are infinitesimal waves on the free surface for every value of F and, therefore, no steady solution exists.

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While the solutions with a stagnation ring are the most *attractive* from an interest point of view, they do not seem to be achievable as real steady flows as the accuracy of the two schemes is increased. The only reasonable conclusion is that if significant waves begin to form on the surface, then no steady solution is possible. Thus, the critical value of Froude number drops from $F \approx 6.4$ [3] to $F \approx 3$ and the free surface has no waves or ripples.

The solutions computed over the full range, however, can be thought of as being very close to steady state (with only very small *errors* on the free surface), suggesting that perhaps they exist as *almost-steady* solutions, with small, wavering ripples on an otherwise unchanging surface. Only full unsteady simulations will be able to answer this question. This issue and also the stability of these flows are the subject of further work.

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