# COMPACT SUBSETS IN FUNCTION SPACES

S.K. Kaul

(received January 20, 1969)

1. We wish to study the problem of finding conditions under which a family of maps from one space into another, with a suitable topology, is compact. Some of the results obtained in this direction are in [1; 2; 3]. We propose to give conditions, to be called uniformly regular and regular (the terminology is motivated by [4]), under which "Ascoli" theorems can be proved. These notions turn out to be equivalent to even continuity of Kelley [1, page 235] under such conditions that all the theorems in the section on even continuity in it still hold when in their statements even continuity is replaced by either uniform regularity or regularity (see Theorem A below).

Let $X, Y$ be topological spaces. We denote by $(X, Y)$ the set of all continuous functions from $X$ to $Y$. This set with the compact open topology will be denoted by $C(X, Y)$ and with the pointwise topology by $P(X, Y)$ (see [1, Ch. 7] for terminology).

(1.1). $F \subset (X, Y)$ is said to be uniformly regular if for any open covering $V$ of $Y$ there exists an open covering $U$ of $X$ such that $U$ refines $f^{-1}[V]$ $(U > f^{-1}[V])$ for each $f$ in $F$, where $f^{-1}[V] = \{f^{-1}[v] : v \in V\}$.

(1.2). $F \subset (X, Y)$ is said to be regular at $x \in X$ if for any open set $V$ in $Y$ and $G \subset F$ such that $\overline{G(x)} \subset V$, where $G(x) = \{g(x) : g \in G\}$, there exists an open set $U$ containing $x$ such that $g[U] \subset V$ for each $g$ in $G$. $F$ is said to be regular if it is regular at each point of $X$.

We recall also the definition of even continuity from [1].

(1.3). $F \subset (X, Y)$ is evenly continuous if for each $x$ in $X$, each $y$ in $Y$ and each open set $V$ containing $y$, there exists a neighbourhood $W$ of $y$ and $U$ of $x$ such that $f[U] \subset V$ whenever $f(x) \in W$.

The main results are the following:

THEOREM A. *Suppose $Y$ is a regular space, $F \subset (X, Y)$ and $\overline{F(x)}$ is compact for each $x$ in $X$. Then the following are equivalent:*

Canad. Math. Bull. vol. 12, no. 4, 1969



https://doi.org/10.4153/CMB-1969-057-9 Published online by Cambridge University Press

(a) $F$ *is uniformly regular;*

(b) $F$ *is regular;*

(c) $F$ *is evenly continuous.*

THEOREM B. *Suppose* $Y$ *is regular and Hausdorff and* $X$ *is separable. If* $F \subset C(X, Y)$ *is regular and closed and* $\overline{F(x)}$ *is compact for each* $x$ *in* $X$, *then* $F$ *is sequentially compact.*

We also prove Theorem 20 [1, page 236] with only assuming regular (see Corollary (2.1) below).

2. The following lemmas lead to the proof of Theorem A.

LEMMA (2.1). *If* $F \subset (X, Y)$ *is uniformly regular then it is regular.*

*Proof.* Let $x \varepsilon X$ and $v$ be an open set in $Y$. Let $G \subset F$ be such that $\overline{G(x)} \subset v$. Let $V = \{v, Y - \overline{G(x)}\}$. Given the open covering $V$ of $Y$, there exists, since $F$ is uniformly regular, an open covering $U$ of $X$ such that $U > f^{-1}[V]$ for each $f$ in $F$. Let $u$ be a member of $U$ containing $x$. Since $g(x) \notin Y - \overline{G(x)}$ for any $g \varepsilon G$, $g[u] \not\subset Y - G(x)$. Hence for $g \varepsilon G$, $g[u] \subset v$. Since $x$ and $u$ were arbitrary, this proves the lemma.

LEMMA (2.2). *Let* $Y$ *be a regular (or Hausdorff) space and* $V$ *be an open covering of* $Y$. *If* $F \subset (X, Y)$ *is regular at* $p \varepsilon X$, *and* $\overline{F(p)}$ *is compact, then there exists an open set* $u$ *containing* $p$ *such that* $U = \{u\} > f^{-1}[V]$ *for all* $f$ *in* $F$.

*Proof.* Clearly $\overline{F(p)}$ is a regular closed space whether we assume $Y$ to be regular or Hausdorff. Thus there exists a covering $W$ of $\overline{F(p)}$ by sets $w$ open in the subspace $\overline{F(p)}$ such that $\overline{W} = \{\overline{w} : w \varepsilon W\} > V$. Let $\{w_1, \ldots, w_n\}$ be a finite subset of $W$ that covers $\overline{F(p)}$. Let $F_i = \{f \varepsilon F : f(p) \varepsilon w_i\}$, $1 \leq i \leq n$. Then since $\{w_1, \ldots, w_n\}$ covers $\overline{F(p)}$, $\bigcup_{i=1}^{n} F_i = F$. Since $\overline{W} > V$ there exists $v_i \varepsilon V$ such that $\overline{w}_i \subset v_i$ and consequently $\overline{F_i(p)} \subset \overline{w}_i \subset v_i$, $1 \leq i \leq n$. Since $F$ is regular at $p$, there exists for each $i$ an open set $u_i$ containing $p$ such that $f[u_i] \subset v_i$ for $f \varepsilon F_i$. Let $u = \bigcap_{i=1}^{n} u_i$. Then for each $f \varepsilon F$, $f[u] \subset v_i$ for some $i$, $1 \leq i \leq n$. Hence the result that $U = \{u\} > f^{-1}[V]$ for each $f$ in $F$.



https://doi.org/10.4153/CMB-1969-057-9 Published online by Cambridge University Press

LEMMA (2.3). Suppose $Y$ is regular (or Hausdorff), $F \subset (X, Y)$ and $\overline{F(x)}$ is compact for each $x$ in $X$. If $F$ is regular then $F$ is uniformly regular.

Proof. Let $V$ be any given open covering of $Y$. For each $x \in X$ and $V$ find an open set $u(x)$ as in Lemma (2.2). Then the open covering $U = \{u(x) : x \in X\}$ thus obtained has the required property.

LEMMA (2.4). Suppose $F \subset (X, Y)$ and $\overline{F(p)}$ is compact for $p$ in $X$. If $F$ is evenly continuous then $F$ is regular at $p$.

Proof. Let $v$ be an open set in $Y$ and $G \subset F$ such that $\overline{G(p)} \subset v$. Given $p \in X$, $y \in \overline{G(p)}$ and $v = v(y)$, there exists a neighbourhood $u(y)$ of $p$ and $w(y) \subset v(y)$ of $y$ such that if for $f \in G$, $f(p) \in w(y)$ then $f[u(y)] \subset v(y)$. Carrying out this construction for each $y$ in $\overline{G(p)}$ we get an open covering $W = \{w(y) : y \in \overline{G(p)}\}$ of $G(p)$ and corresponding to it a family $U = \{u(y)\}$ of neighbourhoods of $p$. Let $u$ be the intersection of members of $U$ corresponding to a finite subset of $W$ covering $\overline{G(p)}$. Clearly then for any $g \in G$, $g[u] \subset v$. Since $v$ was arbitrary, $F$ is regular at $p$. This completes the proof.

LEMMA (2.5). Suppose $Y$ is regular. If $F \subset (X, Y)$ is regular, then it is evenly continuous.

Proof. Let $x \in X$, $y \in Y$ and let $u$, a neighbourhood of $y$, be given. Let $w$ be a neighbourhood of $y$ such that $\overline{w} \subset u$. Let $G = \{g \in F : g(x) \in w\}$. Then $\overline{G(x)} \subset \overline{w} \subset u$, and by regularity of $F$, there exists a neighbourhood $v$ of $x$ such that $g[v] \subset u$ for any $g \in G$. Thus if $g \in F$, and $g(x) \in w$, then $g \in G$, and consequently $g[v] \subset u$. This implies that $F$ is evenly continuous.

Proof of Theorem A. Lemmas (2.1) and (2.3) imply that (a) and (b) are equivalent; Lemmas (2.4) and (2.5) imply that (b) and (c) are equivalent. Hence (a), (b), and (c) are equivalent. This completes the proof.

Let $e : (X, Y) \times X \to Y$ be defined by $e(f, x) = f(x)$ for $f \in (X, Y)$ and $x \in X$. If for a topology on $F \subset (X, Y)$, $e$ restricted to $F$ is continuous, then we call it a jointly continuous topology on $F$.

THEOREM (2.1). If $F \subset (X, Y)$ is compact relative to a jointly continuous topology on $F$, then $F$ is regular.

Proof. Let $x \in X$, $v$ be an open set in $Y$, and $G \subset F$ be such that $\overline{G(x)} \subset v$. Let $w = Y - \overline{G(x)}$. Then $\{v, w\}$ is an open covering of $Y$. For $f$ in $F$ if $f(x) \in v$ then we find open sets $o(f)$ and $u(f, x)$ containing $f$ and $x$ respectively such that $e(o(f) \times u(f, x)) \subset v$. In case $f \in F$ and $f(x) \notin v$, then of course $f(x) \in w$; again in this case we find open sets as above so that $e(o(f), u(f, x)) \subset w$. From the open covering $\{o(f) : f \in F\}$ thus obtained we obtain a finite open covering $\{o(f_i) : i \in I\}$ of $F$, where $I$ is a finite set. Let $J$ be the subset of I such that $\{o(f_i) : i \in J\}$ covers G, and also each $o(f_i)$, $i \in J$, contains at least one member of G. Now, since $o(f_i)$ for $i \in J$, contains a member g of G and $g(x) \in v$, by the above construction $e(o(f_i) \times u(f_i, x)) \subset v$. Let $u = \bigcap_{i \in J} u(f_i, x)$. Then for $g \in G$, $g[u] \subset v$. Hence F is regular at x. Since x was arbitrary, F is regular and the proof is complete.

From Theorem A and Theorem (2.1), we get immediately:

COROLLARY (2.1). If $F \subset (X, Y)$ is compact relative to a jointly continuous topology on F, and Y is regular (or Hausdorff), then F is evenly continuous.

3. LEMMA (3.1). Let Y be Hausdorff, and $\{f_n\}$ be a sequence in (X, Y) regular at p in X. Then for any open set v containing $q = \lim_{n \to \infty} f_n(p)$ (assuming that the limit exists) there exists an open set u and a positive integer N such that $f_n[u] \subset v$ for $n \geq N$.

Proof. Since $\lim_{n \to \infty} f_n(p) = q \in v$, there exists a positive integer N such that $f_n(p) \in v$ for $n \geq N$. Hence $\{f_n(p) : n \geq N\}^- \subset v$. From regularity of $\{f_n\}$ it follows that there exists an open set u containing x such that $f_n[u] \subset v$ for $n \geq N$.

THEOREM (3.1). Suppose $\{f_n\} \subset (X, Y)$ is regular, and $\{f_n(x)\}$ converges for each x in X. Let $f : X \to Y$ be defined by $f(x) = \lim_{n \to \infty} f_n(x)$ for $x \in X$. If Y is Hausdorff and regular, then f is continuous. Furthermore, f is a limit point of $\{f_n\}$ in C(X, Y).

Proof. Let $x \in X$ and v be an open set containing f(x). Since Y is regular there exists an open set w containing f(x) such that $\overline{w} \subset v$. From Lemma (3.1) there exists an open set u containing x and a positive integer N such that $f_n[u] \subset w$ for $n \geq N$. Hence $f[u] \subset \overline{w} \subset v$. Thus f is continuous at x. Since x is arbitrary f is continuous on X.

To show that f is a limit point of $\{f_n\}$ in C(X, Y), it is enough to show that any sub-basic open set in C(X, Y) containing f contains all but a finite number of $f_n$. Let $M(k, v) = \{h \in (X, Y) : h(k) \subset v\}$, where k is compact in X and v is open in Y, be a sub-basic open set containing f. For each x in k, since $\lim_{n \to \infty} f_n(x) = f(x) \in v$,



https://doi.org/10.4153/CMB-1969-057-9 Published online by Cambridge University Press

there exists a positive integer $N(x)$ such that, for $n \geq N(x)$, $f_n(x) \in v$. Since $Y$ is Hausdorff, $\{f_n(x) : n \geq N(x)\}^- \subset v$. From regularity of $\{f_n\}$, there exists an open set $u(x)$ containing $x$, such that $f_n[u(x)] \subset v$ for $n \geq N(x)$. The family $\{u(x) : x \in k\}$ yields a finite open cover $\{u(x_1), \ldots, u(x_n)\}$ of $k$. If $N = \max\{N(x_1), \ldots, N(x_n)\}$, then for $n \geq N$, $f_n(k) \subset v$. This completes the proof.

LEMMA (3.2). <u>Let</u> $A \subset X$ <u>be dense, and</u> $\{f_n\}$ <u>be a sequence in</u> $(X, Y)$ <u>such that</u> $\{f_n(a)\}$ <u>converges for each</u> $a \in A$. <u>If</u> $Y$ <u>is Hausdorff and regular</u>, $\{f_n(x)\}^-$ <u>is compact for each</u> $x$ <u>in</u> $X$, <u>and</u> $\{f_n\}$ <u>is regular, then</u> $\{f_n(x)\}$ <u>converges for each</u> $x$ <u>in</u> $X$.

<u>Proof</u>. Let $x \in X$ and suppose that $\{f_n(x)\}$ does not converge. Since $\{f_n(x)\}^-$ is compact and $Y$ is $T_2$, we find two subsequences $\{g_n\}$ and $\{h_n\}$ of $\{f_n\}$ such that $\{g_n(x)\}$ converges to say $p$, and $\{h_n(x)\}$ converges to say $q$, where $p \neq q$. Since $Y$ is $T_2$, there exist disjoint open sets $v_1$ and $v_2$ containing $p$ and $q$ respectively. From Lemma (3.1) there exist open sets $u_1$ and $u_2$ containing $x$ and positive integers $N_1$ and $N_2$ such that $g_n[u_1] \subset v_1$, for $n \geq N_1$ and $h_n[u_2] \subset v_2$ for $n \geq N_2$. Let $u = u_1 \cap u_2$. Since $A$ is dense in $X$, there exists an $a \in A \cap u$. Clearly $\{f_n(a)\}$ does not converge to any point contrary to the hypothesis. Hence $\{f_n(x)\}$ converges, and the lemma is proved.

THEOREM (3.2). <u>Let</u> $Y$ <u>be Hausdorff and regular</u>, $A \subset X$ <u>be dense, and</u> $\{f_n\} \subset (X, Y)$ <u>be a sequence such that</u> $\{f_n(a)\}$ <u>converges for each</u> $a$ <u>in</u> $A$. <u>Then the function</u> $g$ <u>defined on</u> $A$ <u>by</u> $g(a) = \lim_{n \to \infty} f_n(a)$ <u>for</u> $a$ <u>in</u> $A$, <u>has a unique extension to a continuous function</u> $f : X \to Y$ <u>provided that</u>, (1) $\{f_n(x)\}^-$ <u>is compact for each</u> $x$ <u>in</u> $X$, <u>and</u> (2) $\{f_n\}$ <u>is regular</u>.

<u>Proof</u>. From Lemma (3.2) it follows that $\{f_n(x)\}$ converges for each $x$ in $X$. Hence $f(x) = \lim_{n \to \infty} f_n(x)$ is a well-defined function on $X$ which agrees with $g$ on $A$. From Theorem (3.1) it follows that $f$ is continuous on $X$. Since $f$ is a continuous extension of a continuous function $g$ on a dense subset $A$ in $X$ and $Y$ is Hausdorff, $f$ is unique. This completes the proof.

Proof of Theorem B. Since $X$ is separable, there exists a countable dense subset $A$ in $X$. Since $\{f_n(a)\}^-$ for $\{f_n\} \subset F$ is compact for each $a$ in $A$, by the Cantor diagonal process [5, Theorem 9, page 45] there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(a)\}$ converges for each $a$ in $A$. By Theorem (3.2) the function $g : A \to y$ defined by $g(a) = \lim_{n\to\infty} g_n(a)$ has a unique extension to a continuous function $f : X \to Y$. By Theorem (3.1), $f$ is a limit point of $\{g_n\} \subset F$, and consequently, since $F$ is closed, $f$ lies in it. Thus an arbitrary sequence $\{f_n\}$ of $F$ has a convergent subsequence converging to a point of $F$. This proves the theorem.

Remark. In view of Theorem A, "even continuity" can be replaced by "regularity" in Theorems 22 and 23 [1, page 237].

## REFERENCES.


1. J.L. Kelley, General topology. (Van Nostrand, 1955.)

2. D. Gale, Compact sets of functions and function rings. Proc. Amer. Math. Soc. 1 (1950) 303-308.

3. A. Grothendieck, Critères de compacité dans les espaces fonctionnels généraux. Amer. J. Math. 74 (1952) 168-186.

4. B.V. Kerekjarto, Topologische charakterisierung der linearen Abbildungen. Acta. Litt. ac. Sci. Szeged 6 (1934), 235-262.

5. L. Pontrjagin, Topological groups. (Princeton Univ. Press, 1958.)

6. G.F. Simmons, Introduction to topology and modern analysis. (McGraw-Hill Book Co., 1963.)



University of Saskatchewan
Regina