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Inductive and projective limits of smooth topological vector spaces

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In *J. Math. Mech.* 15 (1966), 877-898, Bonic and Frampton have laid the foundation for a general theory of smoothness of Banach spaces. In this paper, we shall study one aspect of the smoothness of topological vector spaces, namely, the relationship between smoothness and inductive and projective limits of topological vector spaces. As a consequence, we obtain smoothness results for nuclear spaces and some Montel spaces.

1. Preliminaries

We begin with the various definitions of differentiability. These definitions in topological vector spaces are due to Averbukh and Smolyanov [1], [2]. Let *TVS* denote the class of all topological vector spaces over the real field R. Let $L_1(E, F) = L(E, F)$ denote the set of all continuous linear maps from E into F, where $E, F \in TVS$. We define by induction $L_p(E, F) = L\big(E, L_{p-1}(E, F)\big)$. Each $L_p(E, F)$ is given the topology of uniform convergence on bounded subsets of E.

DEFINITION 1. Let $E, F \in TVS$, A be an open subset of E and $f : A \to F$. Let σ be a class of subsets of E such that every single point set belongs to σ. Then we say f is *σ-differentiable at* $x \in A$, if there exists $u \in L(E, F)$ such that for each $S \in \sigma$ and for each 0-neighbourhood U in F, there exists $\delta > 0$ such that $f(x+th) - f(x) - u.th \in tU$, whenever $h \in S$ and $|t| \le \delta$.

In this case, the mapping u is determined uniquely and is denoted by

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$f'(x)$. We say $f'(x)$ is the *σ-derivative of f at x* .

If $f'(x)$ exists for each $x \in A$, then we may define a map $f' : A \to L(E, F)$ by $x \to f'(x)$. We say f' is the *σ-derivative of f* . Higher order σ-derivatives are then defined in the obvious way [1, p. 227].

Given $E, F \in TVS$ and open sets U in E and V in F , denote by $C^0(U, V)$, the set of all continuous functions from U into V . Denote by $C_\sigma^k(U, V)$ $(k = 1, 2, \ldots)$, the set of all continuous functions from U into V , whose σ-derivatives of all orders $\leq k$ exist and are continuous, and by $C_\sigma^\infty(U, V)$, the set of all continuous functions from U into V , whose σ-derivatives of all orders exist and are continuous. Denote by $D_\sigma^k(U, V)$ $(k = 1, 2, \ldots)$, the set of all functions from U into V , whose σ-derivatives of all orders $\leq k$ exist, and by $D_\sigma^\infty(U, V)$, the set of all functions from U into V , whose σ-derivatives of all orders exist. C^0 will denote all continuous functions between open subsets of topological vector spaces. Analogous meanings are attached to the symbols D_σ^k and C_σ^k $(k = 1, 2, \ldots, \infty)$. An example in [2, p. 107] shows that D_σ^∞ is different from C_σ^∞ , even in the special cases below.

The three most important cases of σ-differentiability are as follows:

(I) If σ is the class of all finite subsets of E , then $f \in D_\sigma^1$ is said to be *Gâteaux differentiable*. The classes D_σ^k and C_σ^k $(k = 1, 2, \ldots, \infty)$ are then denoted by D_G^k and C_G^k .

(II) If σ is the class of all sequentially compact subsets of E , then $f \in D_\sigma^1$ is said to be *Hadamard differentiable*. The classes D_σ^k and C_σ^k $(k = 1, 2, \ldots, \infty)$ are then denoted by D_H^k and C_H^k .

(III) If σ is the class of all bounded subsets of E , then $f \in D_\sigma^1$

is said to be *Fréchet differentiable*. The classes D^k_σ and C^k_σ $(k = 1, 2, \ldots, \infty)$ are then denoted by D^k_F and C^k_F.

The definitions, given in [3], of an S-category and an S-smooth Banach space can be generalized to topological vector spaces with only minor modifications.

DEFINITION 2. An *S-category* is a category S, whose objects are all open subsets of all topological vector spaces. For any pair of objects U and V, the morphisms $S(U, V)$ are functions from U into V with the usual composition as their product. We suppose also that the following conditions are satisfied:

S1. $C^\infty_F(U, V) \subset S(U, V) \subset C^0(U, V)$, for all objects U and V;

S2. if $f \in S(U, V)$ and W is an open subset of V containing $f(U)$, then $f \in S(U, W)$;

S3. if $f \in C^0(U, V)$ and for each $x \in U$, there is an open set W with $x \in W \subset U$ such that $f|W \in S(W, V)$, then $f \in S(U, V)$;

S4. if $f_1 \in S(U_1, V_1)$ and $f_2 \in S(U_2, V_2)$, then $f_1 \times f_2 \in S(U_1 \times U_2, V_1 \times V_2)$.

The most important examples of S-categories are C^0, C^k_H, C^k_F, D^k_H and D^k_F $(k = 1, 2, \ldots, \infty)$. The classes C^k_G and D^k_G $(k = 1, 2, \ldots, \infty)$ are not S-categories, since the product of Gâteaux differentiable maps is not necessarily Gâteaux differentiable. Let $\operatorname{supp} f$ denote the support of a real valued function f.

DEFINITION 3. Let $E \in TVS$ and S be an S-category. E is said to be *S-smooth* if given any 0-neighbourhood V in E, there exists a non-trivial $f \in S(E, R)$ such that $\operatorname{supp} f \subset V$.

It is easy to see that E is S-smooth if and only if given $a \in E$ and a neighbourhood V of a, there exists $f \in S(E, R)$ such that $f(a) > 0$, $f(x) \geq 0$ (for each $x \in E$) and $\operatorname{supp} f \subset V$. In case E is a Banach space, the definition given here coincides with the one given by Bonic and Frampton. Let LCS denote the class of all (Hausdorff) locally convex topological vector spaces over R . For each continuous seminorm p on $E \in LCS$, let $N_p = \{x \in E \mid p(x) = 0\}$. N_p is a closed subset of E .

DEFINITION 4. Suppose $E \in LCS$ and S is an S-category. E is said to be *strongly* S*-smooth* if there exists a collection $P(E)$ of continuous seminorms on E which generate the topology on E and satisfy $p \in S\left(E \backslash N_p, R\right)$, for each $p \in P(E)$.

The following propositions give some of the simplest properties of S-smoothness and strong S-smoothness.

PROPOSITION 1. *$E \in TVS$ is S-smooth if and only if the topology on E is the same as the weak topology generated by the functions in $S(E, R)$.*

Proof. The proof is similar to [3, p. 880] and hence is omitted.

PROPOSITION 2. *Let $E \in LCS$. If E is strongly S-smooth, then E is S-smooth.*

Proof. Let V be a 0-neighbourhood in E . Then there exist continuous seminorms $p_1, \ldots, p_n$ and $\varepsilon > 0$ such that $\left\{x \,\middle|\, \sup_{i=1,\ldots,n} p_i(x) \le \varepsilon\right\} \subset V$ and $p_i \in S\left(E \backslash N_{p_i}, R\right)$, for each i . Now choose $\varphi : R \to R$ such that $\varphi \in C_F^\infty(R, R)$, $\varphi = 1$ in some open 0-neighbourhood U , and $\operatorname{supp}\varphi \subset \{t \mid |t| \le \varepsilon\}$. Pick α and β such that $0 < \alpha < 1 < \beta$. Then choose $\psi : R^n \to R$ such that $\psi \in C_F^\infty(R^n, R)$, $\psi\left(t_1, \ldots, t_n\right) > 0$, if $t_i \in (\alpha, \beta)$ $(i = 1, \ldots, n)$, and $\psi = 0$, otherwise. Define $f : E \to R$ by

$$f = \psi \circ \left(\left[\varphi \circ p_1\right] \times \ldots \times \left[\varphi \circ p_n\right]\right) \circ d ,$$

where d is the diagonal map $d : E \to E^n$, $d(x) = (x, \ldots, x)$.

First we show that $f \in S(E, R)$. Clearly $f \in C^0(E, R)$. Now let $x_0 \in E$. Then there are two cases:

(I) $x_0 \notin \bigcup_{i=1}^{n} N_{p_i}$.

Put $W = E - \bigcup_{i=1}^{n} N_{p_i}$. Then $x_0 \in W$, W is open and $f|W \in S(W, R)$.

(II) $x_0 \in \bigcup_{i=1}^{n} N_{p_i}$.

By changing subscripts if necessary, suppose $x_0 \in N_{p_1}, \ldots, N_{p_m}$ and $x_0 \notin N_{p_{m+1}}, \ldots, N_{p_n}$. If $m < n$, since $\bigcup_{i=m+1}^{n} N_{p_i}$ is closed, there exists open W_1 such that $x_0 \in W_1$ and $W_1 \cap \bigcup_{i=m+1}^{n} N_{p_i} = \emptyset$. Put $W = W_1 \cap \bigcap_{i=1}^{m} p_i^{-1}(U)$. If $m = n$, put $W = \bigcap_{i=1}^{n} p_i^{-1}(U)$. Now, for $i = 1, \ldots, m$, $(\varphi \circ p_i)|W = 1$, and for $i = m+1, \ldots, n$, $(\varphi \circ p_i)|W \in S(W, R)$. Thus $x_0 \in W$, W is open and $f|W \in S(W, R)$. Hence, by axiom S3, $f \in S(E, R)$.

Now f is non-trivial, since $f(0) = \psi(1, 1, \ldots, 1) > 0$. Finally $\operatorname{supp} f \subset V$. For suppose $f(x) \neq 0$. Hence $(\varphi \circ p_i)(x) \in (\alpha, \beta)$, and so $(\varphi \circ p_i)(x) \neq 0$, for each i . Thus $p_i(x) \in \operatorname{supp}\varphi \subset \{t \mid |t| \leq \varepsilon\}$, for each i . Thus $\operatorname{supp} f \subset \left\{x \,\middle|\, \sup_{i=1,\ldots,n} p_i(x) \leq \varepsilon\right\} \subset V$.

It is not known, even for Banach spaces [3, p. 882], if the converse of Proposition 2 holds. We also remark that the results of Section 3 in [3], on the existence of partitions of unity, can be generalized without difficulty to separable metrizable topological vector spaces [7, p. 36].

There is a simple connection between the theory of smoothness of Bonic and Frampton and the older theory of Day, Klee and others. This connection depends on the following result.

PROPOSITION 3. *Let* $E \in LCS$ *and* p *be a continuous seminorm on* E . *Then* p *is Gâteaux differentiable at* $x \in E$ *if and only if* p *is*

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Hadamard differentiable at x .

Proof. Suppose that p is Gâteaux differentiable at $x \in E$. It suffices to show that, given $h_n \to h$ in E and $t_n \to 0$ in R , then $t_n^{-1}.\left[p\left(x+t_n h_n\right)-p(x)-p'(x).t_n h_n\right] \to 0$, where $p'(x)$ is the Gâteaux derivative at x .

Now

$$\left|t_n^{-1}.\left[p\left(x+t_n h_n\right)-p(x)-p'(x).t_n h_n\right]\right|$$
$$\leq p\left(h_n-h\right) + \left|p'(x).\left(h_n-h\right)\right| + \left|t_n^{-1}.\left[p\left(x+t_n h\right)-p(x)-p'(x).t_n h\right]\right| \to 0 ,$$

by the continuity of p and $p'(x)$, and the existence of the Gâteaux derivative. Thus p is Hadamard differentiable at x .

Now, in Day's terminology, a Banach space E is smooth if its norm is Gâteaux differentiable away from the origin. Hence, by Proposition 3, a Banach space which is smooth (in Day's sense) is strongly D_H^1-smooth, and hence D_H^1-smooth. Day [4, p. 519] proved that every separable Banach space has an equivalent norm which is Gâteaux differentiable away from the origin. Thus, by Proposition 3, every separable Banach space is strongly D_H^1-smooth. Later we extend this result to separable locally convex spaces. Notice, however, that we cannot even hope to show that every separable Banach space is C_H^1-smooth. For if $f \in C_G^1$, then $f \in D_F^1$. But Bonic and Frampton have shown that the separable space l^1 is not D_F^1-smooth [3, p. 882].

2. Results

The main concern of this paper is with inductive and projective limits of S-smooth topological vector spaces. Our first theorem gives a connection between S-smoothness and locally convex kernels. We follow the notation and terminology of Köthe [5, §19].

THEOREM 1. *Let* $E[T] = \underset{\alpha}{\mathrm{K}}\, A_\alpha^{-1}\left(E_\alpha\left[T_\alpha\right]\right)$ *be the locally convex kernel of the locally convex spaces* E_α *, where each* A_α *is a linear map from* E

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into E_α . *Let* S *be an* S*-category.*

(I) *If each* E_α *is* S*-smooth, then* E *is* S*-smooth. The converse does not hold.*

(II) *If each* E_α *is strongly* S*-smooth, then* E *is strongly* S*-smooth. The converse does not hold.*

Proof (I). Let $a \in E$ and U be a neighbourhood of a . Since E is regular, there exists another neighbourhood V of a such that $a \in V \subset \overline{V} \subset U$. By the definition of the kernel topology on E , there exist finitely many neighbourhoods $V_1, \ldots, V_n$ in $E_{\alpha_1}, \ldots, E_{\alpha_n}$ such that $a \in \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}(V_i) \subset V$.

Now, for each $i = 1, \ldots, n$, there exists $f_i \in S\left(E_{\alpha_i}, R\right)$ such that $f_i\left(A_{\alpha_i}(a)\right) > 0$ and $\operatorname{supp} f_i \subset V_i$. Choose α_i, β_i in R such that $0 < \alpha_i < f_i\left(A_{\alpha_i}(a)\right) < \beta_i$. Thus, for each i , we have $A_{\alpha_i}(a) \in f_i^{-1}(\alpha_i, \beta_i) \subset V_i$. Choose $\varphi : R^n \to R$ such that $\varphi \in C_F^\infty(R^n, R)$, $\varphi(t_1, \ldots, t_n) > 0$, if $t_i \in (\alpha_i, \beta_i)$ $(i = 1, 2, \ldots, n)$, and $\varphi = 0$ otherwise. Define $f : E \to R$ by

$$f = \varphi \circ (f_1 \times \ldots \times f_n) \circ \left(A_{\alpha_1} \times \ldots \times A_{\alpha_n}\right) \circ d ,$$

where d is the diagonal map. We show that f has the required properties.

Clearly, $f \geq 0$ and $f(a) > 0$. Also $f \in S(E, R)$ since $\varphi \in C_F^\infty(R^n, R)$, $f_1 \times \ldots \times f_n \in S\left(E_{\alpha_1} \times \ldots \times E_{\alpha_n}, R^n\right)$, $A_{\alpha_1} \times \ldots \times A_{\alpha_n} \in C_F^\infty\left(E^n, E_{\alpha_1} \times \ldots \times E_{\alpha_n}\right)$ and $d \in C_F^\infty(E, E^n)$. Finally we show that $\operatorname{supp} f \subset U$. Suppose $f(x) > 0$. Hence

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$\alpha_i < f_i\left(A_{\alpha_i}(x)\right) < \beta_i$, and so $A_{\alpha_i}(x) \in f_i^{-1}\left(\alpha_i, \beta_i\right) \subset V_i$, for each i . Hence $x \in \bigcap_{i=1}^{n} A_{\alpha_i}^{-1}\left(V_i\right) \subset V$. Thus $\operatorname{supp} f \subset \overline{V} \subset U$, and the proof is finished.

The counterexample to the converse is as follows. The space c_0 is a subspace of l^∞ . However, c_0 is strongly C_F^∞-smooth [3, p. 896], but l^∞ is not even D_F^1-smooth [3, p. 882].

(II). For each index α , let $\left\{p_\beta^\alpha\right\}$ be a collection of continuous seminorms on E_α which generate the topology of E_α and satisfy $p_\beta^\alpha \in S\left(E_\alpha \backslash N_{p_\beta^\alpha}, R\right)$, for each β . Then the kernel topology on E is generated by all seminorms of the form $p_\beta^\alpha \circ A_\alpha$. For some fixed α, β , let $p = p_\beta^\alpha \circ A_\alpha$. Clearly $N_p = \left\{x \in E \mid A_\alpha(x) \in N_{p_\beta^\alpha}\right\}$, and since $A_\alpha \in C_F^\infty\left(E \backslash N_p, E_\alpha \backslash N_{p_\beta^\alpha}\right)$ and $p_\beta^\alpha \in S\left(E_\alpha \backslash N_{p_\beta^\alpha}, R\right)$, we have that $p \in S\left(E \backslash N_p, R\right)$.

The example given in part (I) also serves as a counterexample to the converse of part (II).

COROLLARY 1. *Every nuclear space is strongly* C_F^∞*-smooth.*

Proof. Every nuclear space is topologically isomorphic to a subspace of a topological product of spaces $\{E_\alpha\}$, where each E_α is a subspace of l^2 [6, p. 101]. But l^2 is strongly C_F^∞-smooth.

COROLLARY 2. *Every separable locally convex space is strongly* D_H^1*-smooth.*

Proof. Every locally convex space E is topologically isomorphic to

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a subspace of a topological product of Banach spaces, $\{E_\alpha\}$ [5, p. 208]. If E is separable, then so is each E_α. Hence, by our previous remarks, each E_α will be strongly D_H^1-smooth, and the result now follows from the theorem.

COROLLARY 3. *Every* (FM)*-space is strongly* C_F^1*-smooth.*

Proof. Every (FM)-space E is separable [5, p. 370]. Thus, by Corollary 2, E will be strongly D_F^1-smooth. But if a continuous seminorm p is Fréchet differentiable on some open subset $A \subset E$, then p' is continuous on A [7, p. 39]. Thus E will be strongly C_F^1-smooth.

COROLLARY 4. *Every locally convex space with the weak topology is strongly* C_F^∞*-smooth.*

We now consider the connection between smoothness and inductive limits. First suppose that $E[T] = \sum_\alpha E_\alpha[T_\alpha]$ is a topological inductive limit with the property that T induces the topology T_α on each E_α. Then, by Theorem 1, if E is S-smooth (strongly S-smooth), each E_α is S-smooth (strongly S-smooth). However, in more general situations, we cannot make the same conclusion. By [5, p. 280], every separable Banach space is topologically isomorphic to a suitable quotient space of l^1. So, in particular, R is topologically isomorphic to a quotient space of l^1. But R is strongly C_F^∞-smooth, while l^1 is not D_F^1-smooth.

The converse situation is more interesting. Suppose E is a topological inductive limit of the S-smooth spaces E_α. Then is E S-smooth? We give a positive result in a particular case (Theorem 3). First we prove a result which will be needed in the proof of Theorem 3, and which has some interest itself.

THEOREM 2. *Let* $E[T] = \sum_\alpha E_\alpha[T_\alpha]$ *be a strict inductive limit* [5, p. 222], *with the property that a subset* $B \subset E$ *is bounded if and only if* B *is contained in some* E_α *and is bounded there. Suppose* $F \in LCS$, U

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is an open subset of E *and* $f : E \to F$. *Then* $f \in D_F^k(U, F)$ $(k = 1, 2, \ldots, \infty)$ *if and only if* $f|E_\alpha \in D_F^k(U \cap E_\alpha, F)$, *for each* α .

Proof. The "only if" part is trivial. For the converse, suppose first that $k = \infty$. We show by induction that $f \in D_F^\infty(U, F)$. Let $x \in U$. We define a map $u_1 : E \to F$ as follows. Given $y \in E$, since E is a *strict* inductive limit, there exists α such that $x, y \in E_\alpha$. Then define $u_1(y) = (f|E_\alpha)'(x).y$. By the uniqueness of the Fréchet derivative, the value of $u_1(y)$ is independent of the choice of α , so long as $x, y \in E_\alpha$. Also u_1 is linear, and is continuous, since $u_1|E_\alpha = (f|E_\alpha)'(x)$ is continuous, for each α .

We show that $u_1 = f'(x)$. Let B be a bounded subset of E . Then there exists an α such that $B \subset E_\alpha$ and is bounded there. Also, $x \in E_\beta$, for some β . Now choose γ such that $\gamma \geq \beta$ and $\gamma \geq \alpha$. Then $x \in E_\gamma$ and $B \subset E_\gamma$. Also B is bounded in E_γ , since the topology induced by E_γ on E_α is the original topology T_α on E_α .

Now let V be a 0-neighbourhood in F . Then the existence of $(f|E_\gamma)'(x)$ gives the existence of $\delta > 0$ such that $f(x+th) - f(x) - u_1.th \in tV$, whenever $|t| \leq \delta$ and $h \in B$. That is, $f'(x) = u_1$.

Now suppose $f^{(n)}(x)$ exists. Define $u_{n+1} : E \to L_n(E, F)$ as follows: $u_{n+1}(y) = (f|E_\alpha)^{(n+1)}(x).y$, where $x, y \in E_\alpha$. Then, as before, u_{n+1} is a well-defined, continuous linear map. Further, as before, given a 0-neighbourhood V in $L_n(E, F)$ and given a bounded set B in E , there exists $\delta > 0$ such that $f^{(n)}(x+th) - f^{(n)}(x) - u_{n+1}.th \in tV$, whenever $h \in B$ and $|t| \leq \delta$. That is, $f^{(n+1)}(x) = u_{n+1}$. Thus

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$f \in D_F^\infty(U, F)$.

When k is finite, the proof is similar.

The following theorem is a special case of Corollary 1 of Theorem 1. However, in the special case considered here, it is also possible to give a simple construction of a class of generating C_F^∞-seminorms.

THEOREM 3. *Let E be a vector space (over R) with a countable Hamel basis, and let E have the finest locally convex topology T . Then E is strongly C_F^∞-smooth.*

Proof. Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be a basis for E . For each n , let E_n be the vector subspace spanned by $\{e_1, e_2, \ldots, e_n\}$. Give each E_n the Euclidean topology. Then we may consider E as the strict inductive limit of the sequence $\{E_n\}$. A 0-neighbourhood base V for T consists of all absolutely convex absorbent subsets of E .

Consider the sequences $a = (a_1, \ldots, a_n, \ldots)$ of real numbers, where each $a_i > 0$. Put $U_a = \left\{x \in E \mid x = \sum\limits_{i=1}^n x_i e_i ,\ \sum\limits_{i=1}^n (x_i/a_i)^2 \le 1\right\}$. Then each U_a is absolutely convex and absorbent. Thus each U_a is a 0-neighbourhood in T . We show that $\{U_a\}$ is a 0-neighbourhood base for T .

Let $V \in V$. Since V is absorbent, there exist $\varepsilon_i > 0$ such that $\varepsilon_i e_i \in V$, for $i = 1, 2, \ldots$. Put $a_i = \varepsilon_i . 2^{-i}$ and $a = (a_i)_{i=1}^\infty$. Put

$$A = \left\{\sum_{i \in I} \alpha_i \varepsilon_i e_i \mid \sum_{i \in I} |\alpha_i| \le 1 \text{ , where } I \text{ is some finite set of positive integers}\right\} = \text{absolutely convex cover of } \{\varepsilon_i e_i\}_{i=1}^\infty .$$

Since $A \subset V$, it suffices to show that $U_a \subset A$. Suppose $x = \sum\limits_{i=1}^n x_i e_i \in U_a$. Clearly $|x_i| \le \varepsilon_i . 2^{-i}$, $i = 1, 2, \ldots, n$. Thus we can write $x = \sum_{i=1}^{n} \alpha_i \varepsilon_i e_i$, where $\sum_{i=1}^{n} |\alpha_i| \le \sum_{i=1}^{n} 2^{-i} < 1$. Thus $x \in A$, and so $\{U_\alpha\}$ is a 0-neighbourhood base for T .

Now, let $P(E)$ be the collection of gauges of the sets U_a . Each gauge will in fact be a norm. Then certainly $P(E)$ generates T . Thus we have only to show that $P(E) \subset C_F^\infty(E\backslash\{0\}, R)$. So let p be the gauge of some U_a . A straightforward calculation shows that if $x = \sum_{i=1}^{n} x_i e_i \in E_n$, then $p(x) = \left[\sum_{i=1}^{n} (x_i/a_i)^2\right]^{\frac{1}{2}}$. Thus, for each n , $p|E_n \in D_F^\infty(E_n\backslash\{0\}, R)$. Further, by [5, p. 223], a subset B of E is bounded if and only if it is contained in some E_n and is bounded there. Thus, by Theorem 2, $p \in D_F^\infty(E\backslash\{0\}, R)$.

Now in [2, p. 106] it is proved that if E is the strict inductive limit of the increasing sequence $\{E_n\}_{n=1}^\infty$ of locally convex spaces, where the embeddings $E_n \to E_{n+1}$ are compact, then any map from E to $F \in TVS$ that is Hadamard differentiable in some neighbourhood of $x \in E$ is continuous at x . Thus we have $p \in C_F^\infty(E\backslash\{0\}, R)$, and the theorem is proved.

We remark that Proposition 2 now shows that E is C_F^∞-smooth. This can also be proved directly. With the same notation as above, define $f : E \to R$ as follows:

$$f(x) = \exp\left\{-\left[1 - \sum_{i=1}^{n} \left(x_i . 2^i/\varepsilon_i\right)^2\right]^{-1}\right\} ,$$

whenever

$$\sum_{i=1}^{n} \left(x_i . 2^i/\varepsilon_i\right)^2 < 1 \quad \left(x = \sum_{i=1}^{n} x_i e_i\right) ,$$

and $f(x) = 0$, otherwise.

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Then similar methods show that $f \in C_F^\infty(E, F)$, f is non-trivial and $\operatorname{supp} f \subset V$.

Finally, we give an example to show that if $E[T] = \sum_\alpha A_\alpha\left(F_\alpha\left[T_\alpha\right]\right)$ is a locally convex hull and each F_α is S-smooth (strongly S-smooth), then E is not necessarily S-smooth (strongly S-smooth). Grothendieck has constructed an (FM)-space E which has a quotient space topologically isomorphic to l^1 [5, p. 433]. By Corollary 3 of Theorem 1, E is strongly C_F^1-smooth, but l^1 is not even D_F^1-smooth.

Note added in proof, 7 December 1971. Every Schwartz space can be embedded in a topological product of separable Banach spaces and hence is D_H^1-smooth.

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