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Foreword

SERGEI SOLOVIEV

IRIT, Université Paul Sabatier, 118, route de Narbonne, 31062, Toulouse Cedex – FRANCE
Email: soloviev@irit.fr

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It is interesting to note that the very first papers related to isomorphism of types were written before the notion itself appeared. Their subjects were the study of equality of terms defined on numbers, the isomorphism of objects in certain categories and the invertibility of λ-terms, but not the isomorphism of types 'as such'. One may cite the so-called 'Tarsky High School Algebra Problem': whether all identities between terms built from $+, x, \uparrow$, variables and constants are derivable from basic 'high school equalities', like $(xy)^z = x^z y^z$. The earliest publications related to this problem date from the 1940s, *cf.* Birkhoff (1940) – an extensive bibliography may be found in Burris and Yeats (2002).

The pioneering work by Mariangiola Dezani-Ciancaglini (Dezani-Ciancaglini 1976) on the invertibility of type-free λ-terms has turned out to be one of the principal instruments that has been widely used in the study of type isomorphisms in typed λ-calculus.

The finite model property in simply typed λ-calculus, studied in Statman (1982), has important implications for the study of isomorphisms and retractions, but the main inspiration for the first paper in which retractions in this calculus were studied (Bruce and Longo 1985) came from domain theory. For more recent work, we can mention Padovani (2001) and Regnier and Urzyczyn (2002).

Category theory was one of main inspirations for Soloviev (1981).

The same pattern has been repeated many times: theoretical results appearing before new and quite unexpected applications (for example, the retrieval of data using types as keys with types considered up to isomorphism). This is often a sign that a field of study has great unexplored potential.

A detailed survey of the study of isomorphisms of types is presented in this issue by Roberto Di Cosmo.

Amongst the research papers selected for this special issue, two are about extending the notion of isomorphism to new domains. The paper *Classical isomorphisms of types* by O. Laurent develops the theory of isomorphism for 'classical' λ-calculi, that is, the call-by-name and call-by-value variants of Parigot's $\lambda\mu$-calculus augmented by products and sums. The paper *Isomorphisms of simple inductive types through extensional rewriting* by D. Chemouil considers the notion of isomorphism for typed λ-calculus with an extended notion of *reduction*, and shows that in many important cases, such as an isomorphic change of parameter in parameterised inductive types and the representation of products as inductive types, one may work efficiently with the generalised notion of isomorphism.

Two other papers included in this issue are concerned with important practical issues. In the paper *Type isomorphisms and back-and-forth coercions in type theory* by G. Barthe, the technology for computer-assisted formal proofs is considered. The isomorphisms are used as coercions to deal with the difficulties that often appear because the same mathematical structure (say polynomials) can use different representations in the computer (say a list of monomials, functions, and so on). This is an important problem; and may become increasingly important as systems rely more on computations that require representations tuned for efficiency. In this context, coercions are the representation-changing transformations that leave the structure essentially unchanged. The isomorphisms are a natural source of such coercions. The paper *Efficient algorithms for isomorphisms of simple types* by Y. Gil and Y. Zibin describes efficient algorithms for deciding isomorphisms of simple (non-recursive, first-order) types. The authors improve the state of the art by giving:

1 an $O(n)$ time and $O(n)$ space algorithm for the linear isomorphism problem, which does not include the distributive axiom; and
2 an $O(nlog^2n)$ time and $O(n)$ space algorithm for the first-order isomorphism problem.

The second result, in particular, is especially striking and may have strong practical interest; the best previously known algorithm had quadratic time and space complexity. It is worth mentioning that for many years it was commonly assumed (based on algorithms suggested in early work) that the complexity of the deciding algorithm for the isomorphism of simple types will be at least subexponential.

We also include in this issue a seminal paper by Bernard Lang. This paper, which is about matching up to multiplication and exponentiation, has previously only been available as a privately circulated technical report (written in 1978), and is of great historical interest.

It is convenient as a conclusion to this foreword to outline some directions of future research as we see them now. The notion of isomorphism of types has turned out to be so fruitful that one may expect further extensions of the theory of isomorphisms to new areas and type systems, following the examples of ‘classical’ λ-calculus or certain classes of inductive types, as studied in this issue.

Special attention should be paid to isomorphisms in type systems used in proof assistants. The definition of isomorphism uses both the idea of the transformation of types and equality between transformations. Thus, the extension of the notion of isomorphism may involve new transformations as well as modifications of the notion of equality. Two transformations that are not mutually inverse with respect to a weaker equality may become isomorphisms with respect to a stronger one. It is our opinion, taking into account the profound problems concerning the equality in type theory, that very interesting perspectives may open up when various forms of equality (for example, ones based on extensions of reduction systems) are studied in connection with isomorphism.

On the practical side, further developments involve the exploration of links with coercive subtyping and program reuse. Extensions to the notion of isomorphism will permit a study of the development of information retrieval applications in a much larger context. We have in mind, for example, the use of isomorphisms in the framework of the ‘Semantic Web’ (use of types to characterise the data posted on the Internet, typed XML, and so on). Another important practical application in view is the creation of better structured user interfaces.

The advances in understanding of the complexity questions related to the notion of isomorphism and retraction (one-side invertibility) permit us to look toward other 'hot' practical topics, such as coding, including public-key cryptography. Indeed, the term $t : A \to B$ may play the role of 'public' coding operator for cases when the problem of finding its inverse is sufficiently complex. The advantage of this approach may lie in the fact that it is quite flexible with respect to the level of structures that are coded (for example, trees instead of individual symbols).

Of course the list of future topics suggested here is far from exhaustive. We have every reason to hope that many more new and unexpected applications will arise, as has already happened many times in the case of type isomorphisms.

To conclude this short foreword, the editors of this special issue (Sergei Soloviev and Roberto Di Cosmo) would like to thank the department STIC/CNRS for its partial support of the workshop WIT2002 held in Toulouse in August 2002, where the research activities that have given rise to some of the papers in this issue were first presented, and for its partial support of the editors' work.

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