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TRANSLATION-EQUIVARIANT MATCHINGS OF COIN FLIPS ON $\mathbb{Z}^d$

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Abstract

Consider independent fair coin flips at each site of the lattice $\mathbb{Z}^d$. A translation-equivariant matching rule is a perfect matching of heads to tails that commutes with translations of $\mathbb{Z}^d$ and is given by a deterministic function of the coin flips. Let Z_Φ be the distance from the origin to its partner, under the translation-equivariant matching rule Φ. Holroyd and Peres (2005) asked, what is the optimal tail behaviour of Z_Φ for translation-equivariant perfect matching rules? We prove that, for every $d \geq 2$, there exists a translation-equivariant perfect matching rule Φ such that $\mathrm{E}\, Z_\Phi^{2/3-\varepsilon} < \infty$ for every $\varepsilon > 0$.

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1. Introduction

Consider the probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, $\mathcal{F}$ is the standard product σ-algebra of $\{0, 1\}^{\mathbb{Z}^d}$, and P is the product measure on $\mathcal{F}$ with parameter $p = \frac{1}{2}$. We call elements of $\mathbb{Z}^d$ *sites*. For $\gamma \in \Omega$, a bijection $\phi\colon \mathbb{Z}^d \to \mathbb{Z}^d$ is a *matching on* γ if every site x with $\gamma(x) = 1$ is mapped to a site y with $\gamma(y) = 0$, and vice versa, and if the composition $\phi \circ \phi$ is the identity mapping on $\mathbb{Z}^d$. For a site z, we define the translation θ^z on $\mathbb{Z}^d$ and Ω as follows: we set $\theta^z x := x + z$ for all $x \in \mathbb{Z}^d$ and, for all $\gamma \in \Omega$, we set $\theta^z\gamma(x) := \gamma(x - z)$ for all $x \in \mathbb{Z}^d$. A measurable mapping $\Phi\colon \{0, 1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d \to \mathbb{Z}^d$ is a *matching rule* if $\Phi(\gamma, \cdot)$ is a matching on γ for P-almost all γ. We say that Φ is *translation equivariant* if it commutes with translations; that is, $\Phi(\theta^z\gamma, \cdot) = \theta^z\Phi(\gamma, \cdot)$ for P-almost all γ.

Let $\|\cdot\|$ be the l^∞ norm on $\mathbb{Z}^d$. We define $Z = Z_\Phi(\gamma) := \|\Phi(\gamma, \mathcal{O})\|$ to be the distance from the origin $\mathcal{O} = (0, \dots, 0) \in \mathbb{Z}^d$ to its partner. We will construct a translation-equivariant matching rule Φ and obtain upper bounds on $\mathrm{P}(Z > r)$.

Theorem 1. *For all $d \geq 1$, there exists a translation-equivariant matching rule Φ such that, for all $r > 0$, we have*

$$\mathrm{P}(Z\phi > r) \leq c(\log r)^4 r^{-\beta}$$

for some $c = c(d) < \infty$, where $\beta = \beta(d) = 2/(1 + 4/d)$.

Prior to this result, the best known decay appears to have been the following.

Theorem 2. ([8] and [10].) *For $d \geq 1$, there exists a translation-equivariant matching rule Φ such that, for all $r > 0$, we have* $\mathrm{P}(Z_\Phi > r) \leq cr^{-1/2}$ *for some $c = c(d) < \infty$.*

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Theorem 2 can be deduced from a simple construction due to Meshalkin [10]. Meshalkin's matching was originally used to construct isomorphisms of Bernoulli schemes [10]; it is the following construction. In $d = 1$ we define a translation-equivariant matching rule inductively, by first matching a 0 to a 1 whenever a 0 is immediately to the left of a 1, i.e.

$$\ldots \underline{\mathbf{01}}10\underline{\mathbf{01}}111100\underline{\mathbf{01}}\ldots.$$

In the next stage we remove the matched pairs, and then follow the same procedure. It is straightforward to check that bounding $\mathrm{P}(Z > r)$ amounts to bounding $R = \inf\{m \geq 1 \colon S_m = 0\}$, where S_m denotes a simple symmetric random walk.

We may deduce the $d \geq 2$ case of Theorem 2 from the following observation. By applying a translation-equivariant matching rule Φ_{d-1} on $\mathbb{Z}^{d-1}$ to each $(d-1)$-dimensional *plane*, given by $\{z\} \times \mathbb{Z}^{d-1}$ for each $z \in \mathbb{Z}$, we obtain a translation-equivariant matching rule Φ_d on $\mathbb{Z}^d = \mathbb{Z} \times \mathbb{Z}^{d-1}$, where $\mathrm{P}(Z_{\Phi_{d-1}} > r) = \mathrm{P}(Z_{\Phi_d} > r)$ for all $r > 0$.

Theorem 1 provides faster decay than that provided by Theorem 2 for all $d > 1$. After this paper was written, Timár [12] proved the following stronger result.

Theorem 3. ([12].) *For any $d \geq 1$, there exists a translation-equivariant matching rule Φ such that, for all $r > 0$, we have $\mathrm{P}(Z_\Phi > r) \leq cr^{-d/2}$ for some $c = c(d) < \infty$.*

Some of the methods of this paper also appear in [12]. New ideas are introduced in [12] and the methods of [12] are much more sophisticated.

For $d = 1, 2$, the bounds obtained in Theorem 3 are essentially best possible.

Theorem 4. ([6] and [8].) *If $d = 1, 2$ then, for any translation-equivariant matching rule Φ, we have $\mathrm{E}\, Z_\Phi^{d/2} = \infty$.*

Hence, for $d = 1, 2$, there does not exist a translation-equivariant matching rule Φ where $\mathrm{P}(Z_\Phi > r) \leq cr^{-\rho}$ for some constants $\rho > d/2$ and $c = c(d) < \infty$.

Note that the result of Theorem 4 is only valid for $d = 1, 2$. In fact, for $d \geq 3$, Timár [12] showed that it is possible to find a translation-equivariant matching rule with even faster decay than that given by Theorem 3.

Theorem 5. ([12].) *For any $d \geq 3$ and any $\varepsilon > 0$, there exists a translation-equivariant matching rule Φ such that, for all $r > 0$, we have*

$$\mathrm{P}(Z > r) \leq C \exp(-cr^{d-2-\varepsilon})$$

for some constants $0 < c, C < \infty$.

Variants of matching in continuum settings have also been studied; see [4], [5], [9], [11], and the references therein.

The proof of Theorem 1 will proceed in two steps. We will construct a translation-equivariant matching and then determine bounds for it. To construct a translation-equivariant matching, we will define, in a measurable translation-equivariant way, a sequence P_n of successively coarser partitions of $\mathbb{Z}^d$. Following [7], we call P_n a *clumping rule*. The members of P_n are called *clumps* or *n-clumps*, and we call the clumping rule *locally finite* if all the clumps are bounded. A *component* of a clumping rule is a limit of some increasing (with respect to set inclusion) sequence of clumps. A clumping rule is *connected* if it has only one component. Adapting the construction in [7] we will construct a locally finite connected clumping rule. From a locally finite connected clumping rule, it is easy to obtain a translation-equivariant matching rule Φ;

this is because a translation-equivariant matching rule can be defined by first matching as many sites as possible within each 1-clump and then iteratively matching as many unmatched sites as possible in each n-clump for $n = 2, 3, \dots$. We will obtain, with the central limit theorem and a version of the mass transport principle, a preliminary result which implies that, for $d \geq 3$ and all $\varepsilon > 0$, we have $\mathrm{P}(Z_\Phi > r) \leq cr^{-3/5-\varepsilon}$ for some constant $c = c(d, \varepsilon) < \infty$. The preliminary result will not provide faster decay than that given by Theorem 2 in the cases $d = 1, 2$. Upon closer analysis of the geometry of the clumps, we will show that clumps that are *long and thin* happen with small probability; this analysis is the basis of the proof of Theorem 1.

The outline of the paper is as follows. In Section 2 we discuss clumping rules and matchings from clumping rules. In Section 3 we outline the construction of a clumping rule and collect some useful bounds. In Section 4 we introduce a version of the mass transport principle that will be useful in the proof of Theorem 1. In Section 5 we prove Theorem 1. We conclude the paper with some related open problems.

2. Clumps

Let $\mathcal{P}_F(\mathbb{Z}^d)$ denote all finite subsets of $\mathbb{Z}^d$. For $A \subset \mathbb{Z}^d$, define translations of A via $\theta^z A := \{\theta^z x : x \in A\}$. Formally, a *locally finite connected clumping rule* is a measurable mapping $\mathcal{C} : \{0, 1\}^{\mathbb{Z}^d} \times \mathbb{N} \times \mathbb{Z}^d \to \mathcal{P}_F(\mathbb{Z}^d)$ with the following properties. For all $\gamma \in \{0, 1\}^{\mathbb{Z}^d}$, all $n \in \mathbb{N}$, and all $x, y, z \in \mathbb{Z}^d$, we have

(i) $x \in \mathcal{C}(\gamma, n, x)$,

(ii) $\mathcal{C}(\gamma, n, x) \cap \mathcal{C}(\gamma, n, y) \neq \varnothing$ implies that $\mathcal{C}(\gamma, n, x) = \mathcal{C}(\gamma, n, y)$,

(iii) $\mathcal{C}(\gamma, n, x) \subset \mathcal{C}(\gamma, n + 1, x)$,

(iv) $\mathcal{C}(\theta^z \gamma, n, \theta^z x) = \theta^z \mathcal{C}(\gamma, n, x)$,

(v) $\bigcup_n \mathcal{C}(\gamma, n, \mathcal{O}) = \mathbb{Z}^d$.

Properties (i) and (ii) assure us that, for each $n \in \mathbb{N}$, the map $\mathcal{C}(\cdot, n, \cdot)$ is a partition. Property (iii) makes the partition successively coarser, (iv) is translation equivariance, and (v) is connectedness.

Proposition 1. *There exists a locally finite connected clumping rule almost surely.*

The proof of Proposition 1 will be given in the next section.

2.1. Matchings from clumpings

From a locally finite connected clumping rule we can construct a translation-equivariant matching rule in a countable number of stages. In the first stage, within each of the 1-clumps we match every possible site. Given that the $(n - 1)$th stage is completed, within each of the n-clumps we match every site we can, ignoring the sites that were previously matched. In order to ensure that the resulting matching is translation-equivariant, use, for example, a lexicographic ordering on $\mathbb{Z}^d$, to determine the maximal partial matching on the clumps. Ergodicity, connectedness, and the fact that $p = \frac{1}{2}$ give us that every site will be matched at some stage. Note that for our purposes we do not *need* to make this argument as we obtain upper bounds on $\mathrm{P}(Z > r)$, which easily imply that $\mathrm{P}(Z > r) \to 0$ as $r \to \infty$ (see Theorem 1 or Proposition 4, below).

In the next section we construct an explicit locally finite connected clumping rule $\mathcal{C}$.

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3. Seeds, cutters, and blobs

Our construction of the clumping rule $\mathcal{C}$ is adapted from [7]. In [7] and [13] clumpings are used to obtain factor graphs of point processes. See also [2] for background.

3.1. Basic set-up

Let $\|\cdot\|$ denote the l^∞ norm on $\mathbb{Z}^d$. Let $S(x, r) := \{y \in \mathbb{R}^d : \|x - y\| \le r\}$. Thus, $S(x, r)$ is the cube of side length $2r$ centered at x. We also write $S(\mathcal{O}, r) = S(r)$. We let $\{e_m\}_{m=1}^d$ be the standard unit basis vectors in $\mathbb{R}^d$.

For each $k \in \mathbb{N}$, we say that a site $x \in \mathbb{Z}^d$ is a *k-seed* if $\gamma(x) = 1$ and $\gamma(y) = 0$ for all $y \in \{x + ne_1 : 1 \le n \le k-1\}$. Whenever x is a k-seed we call the set $\{x + ne_1 : 0 \le n \le k-1\}$ its *shell*. For example, a 4-shell has the form **1**000. Note that the probability of a k-seed occurring at a particular point is exactly 2^{-k}. Two seeds are said to be *overlapping* if their shells intersect. Note that two seeds x and y overlap if and only if $x = y$. This property will be useful later (see Section 5.2). We define

$$r_k = (2^k k^2)^{1/d} + \tfrac{1}{2}. \tag{1}$$

The reason for the choice of r_k will be evident shortly. Define the vector

$$s_k := \lfloor 100 r_k \rfloor e_1.$$

A *k-cutter* is a subset of $\mathbb{R}^d$ of the form $\{y \in \mathbb{R}^d : \|y - x\| = r_k\}$, where $x - s_k$ is a k-seed. We introduce a shift s_k for technical reasons which will surface later. We define $W_k \subset \mathbb{R}^d$ to be the union of all the k-cutters. Note that we have chosen r_k so that $r_k \notin \mathbb{N}$. Thus, we have $W_k \cap \mathbb{Z}^d = \varnothing$ for all $k \in \mathbb{N}$. A *k-blob* is a connected component of $\mathbb{R}^d \setminus \bigcup_{j>k} W_j$. Hence, the sequence of k-blobs defines a successively coarser partition of $\mathbb{R}^d$ (ignoring the elements of $\bigcup_k W_k$.) The k-blobs induce a clumping rule $\mathcal{C}$ when we intersect the k-blobs with $\mathbb{Z}^d$. Note the technical distinction between blobs and clumps.

It is obvious that the induced clumping rule $\mathcal{C}$ is translation equivariant; it remains to show that it is locally finite and connected. It suffices to show that all the blobs are bounded and that, for every $x \in \mathbb{R}^d$, there is a k-blob that contains both x and the origin.

3.2. Estimates

In this section we obtain some estimates that will show that the clumping rule $\mathcal{C}$ defined in the previous section is indeed locally finite and connected. The following events will be important in our analysis. Let

$$E_k(x) := \{x \text{ is enclosed by some } k\text{-cutter}\}; \tag{2}$$

that is, $E_k(x)$ occurs if and only if, for some site x_0, $x_0 - s_k$ is a k-seed and

$$x \in \{y \in \mathbb{Z}^d : \|y - x_0\| \le r_k\}.$$

Also, let $E_k = E_k(\mathcal{O})$. Let

$$U_k(s) := \{S(s) \text{ intersects some } k\text{-cutter}\};$$

that is, $U_k(s)$ occurs if and only if, for some site x_0, $x_0 - s_k$ is a k-seed and

$$\{y \in \mathbb{R}^d : \|y - x_0\| = r_k\} \cap S(s) \ne \varnothing.$$

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Also, let

$$C_k(s) := \bigcup_{j \geq k} U_k(s). \tag{3}$$

From an analysis of these events we will deduce that the clumping rule $\mathcal{C}$ is both locally finite and connected. Moreover, we will see that the tail behavior of Z_Φ (where Φ is a translation-equivariant matching rule obtained from the clumping rule $\mathcal{C}$) also depends on these events.

Lemma 1. (Enclosure bounds.) *For all $k > c_1$, for some $c_1 = c_1(d) < \infty$, we have*

$$\mathrm{P}(E_k^{\mathrm{c}}) \leq \mathrm{e}^{-k}.$$

Proof. Note that

$$E_k = \{S(-s_k, r_k - 1) \text{ contains some } k\text{-seed}\}. \tag{4}$$

Let p_k be the maximum possible number of k-seeds inside $S(-s_k, r_k - 1)$. Recall that no two (distinct) k-seeds overlap and that the probability that a k-seed occurs at a particular point is 2^{-k}. Hence, $\mathrm{P}(E_k) \geq 1 - (1 - 2^{-k})^{p_k} \geq 1 - \mathrm{e}^{-2^{-k} p_k}$. By our choice of r_k in (1) and since $(r_k^d / k) \leq p_k \leq \lceil 2r_k / k \rceil (2r_k)^{d-1}$ for all $k \geq c_1$, for some $c_1 = c_1(d) < \infty$, we have $\mathrm{P}(E_k) \geq 1 - \mathrm{e}^{-k}$.

Corollary 1. *All k-blobs are bounded almost surely.*

Proof. It suffices to show that all k-blobs that contain $\mathcal{O}$ are bounded. By Lemma 1 we have $\mathrm{P}(E_k) \to 1$ as $k \to \infty$, so that E_k occurs for infinitely many k almost surely. Hence, all blobs which contain $\mathcal{O}$ are bounded.

Lemma 2. (Cutter bounds.) *For all $k \geq 1$ and all $s > 0$, we have* $\mathrm{P}(C_k(s)) \leq c_3 s (k^2 / r_k)$ *for some $c_3 = c_3(d) > 0$.*

Proof. Observe that

$$U_k(s) = \{S(-s_k, r_k + s) \setminus S(-s_k, r_k - s) \text{ contains some } k\text{-seed}\}. \tag{5}$$

Clearly, $\mathrm{P}(U_k(s)) \leq N_k(s) 2^{-k}$, where $N_k(s)$ is the number of lattice points in $S(-s_k, r_k + s) \setminus S(-s_k, r_k - s)$. We have

$$N_k(s) = |S(r_k + s)| - |S(r_k - s)| \leq c_2 r_k^{d-1} s$$

for some $c_2 = c_2(d) > 0$. So we obtain $\mathrm{P}(U_k(s)) \leq c_2 s r_k^{d-1} 2^{-k}$. Thus, recalling our choice of r_k in (1), we have

$$\mathrm{P}(C_k(s)) \leq \sum_{j \geq k} \mathrm{P}(U_j(s)) \leq c_2 s \sum_{j \geq k} 2^{-k} r_k^{d-1} \leq c_3 s \left(\frac{k^2}{r_k} \right). \tag{6}$$

Corollary 2. *The clumping rule $\mathcal{C}$ is connected almost surely.*

Proof. Let $s > 0$. By the Borel–Cantelli lemma and (6), we know that $U_k(s)$ occurs infinitely often with probability 0. Thus, any point within distance s of $\mathcal{O}$ will eventually share a blob with it.

Proof of Proposition 1. The proof follows by applying Corollaries 1 and 2.

Now we obtain a translation-equivariant matching rule Φ from our locally finite connected clumping rule $\mathcal{C}$, via the procedure outlined in Section 2. We will use Lemmas 1 and 2 and the central limit theorem to obtain bounds on Z_Φ.

4. Mass transport

We will require a version of the mass transport principle in order to facilitate calculations. See [1] and [3] for background. Our main application of the mass transport principle will be to prove a modified version of Lemma 3, below, which states that each site has an equal chance of not being matched within its k-clump. Similar ideas also appear in [9].

Let $\mathcal{C}$ be the clumping rule defined in Section 3, and let Φ be the corresponding translation-equivariant matching rule obtained from $\mathcal{C}$. We say that a site is *k-bad* if it is not matched in its k-clump. Let $L_k(x)$ be the k-clump containing the site x, and let $L_k(\mathcal{O}) = L_k$ be the k-clump containing the origin. Let $\#[L_k]$ be the cardinality of L_k. Consider the sum $\zeta := \sum_{x \in L_k}(2\gamma(x) - 1)$, so that $|\zeta|$ is the number of k-bad sites in L_k.

Lemma 3. *For all $k \geq 1$, the probability that the origin is k-bad is exactly*

$$\mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x \in L_k}(2\gamma(x) - 1)\bigg|\bigg).$$

We define a *mass transport* to be a nonnegative measurable function

$$T : \mathbb{Z}^d \times \mathbb{Z}^d \times \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$$

which is translation equivariant; that is, for all $x, y \in \mathbb{Z}^d$, all $\gamma \in \{0, 1\}^{\mathbb{Z}^d}$, and all translations θ of $\mathbb{Z}^d$, we have $T(\theta x, \theta y, \theta\gamma) = T(x, y, \gamma)$. For $A, B \subset \mathbb{Z}^d$, we let $T(A, B, \gamma) := \sum_{x \in A, y \in B} T(x, y, \gamma)$. We think of $T(A, B, \gamma)$ as the mass transferred from A to B under $\gamma \in \Omega$. We will use the following version of the mass transport principle.

Lemma 4. (Mass transport principle.) *For any mass transport, $T : \mathbb{Z}^d \times \mathbb{Z}^d \times \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$, we have* $\mathrm{E}\, T(\mathcal{O}, \mathbb{Z}^d, \cdot) = \mathrm{E}\, T(\mathbb{Z}^d, \mathcal{O}, \cdot)$.

Proof. We have

$$\begin{aligned}
\mathrm{E}\, T(\mathcal{O}, \mathbb{Z}^d, \cdot) &= \sum_{y \in \mathbb{Z}^d} \int T(\mathcal{O}, y, \gamma)\, \mathrm{dP}(\gamma) \\
&= \sum_{y \in \mathbb{Z}^d} \int T(-y, \mathcal{O}, \theta^{-y}\gamma)\, \mathrm{dP}(\gamma) \\
&= \sum_{y \in \mathbb{Z}^d} \int T(-y, \mathcal{O}, \gamma)\, \mathrm{dP}(\gamma) \\
&= \mathrm{E}\, T(\mathbb{Z}^d, \mathcal{O}, \cdot).
\end{aligned}$$

The first and last equalities follow from Fubini's theorem. The second equality follows from the translation equivariance of T and the third equality follows from the translation invariance of P.

To illustrate the versatility of the mass transport principle (Lemma 4), we prove the following (unsurprising) fact.

Proposition 2. *Let $\mathcal{F}$ be the standard product σ-algebra of $\{0, 1\}^{\mathbb{Z}^d}$, and let P_p be the product measure on $\mathcal{F}$ with parameter p. If $p \neq \frac{1}{2}$ then there does not exists a translation-equivariant matching rule.*

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Proof. Let Φ be a translation-equivariant matching rule. Consider the mass transport M defined as follows. Let x be a site, and let $\gamma \in \Omega$. If $\gamma(x) = 1$ then $M(x, x, \gamma) = 1$; that is, x sends one unit of mass to itself. Otherwise, $M(x, y, \gamma) = 1$, where y is a site with $\Phi(x, \gamma) = y$ and $\gamma(y) = 1$; that is, x sends out a unit of mass to the site y that it is matched to under $\Phi(\cdot, \gamma)$. Since Φ is translation equivariant, this defines a mass transport P_p-almost surely. Let E_p be the expected value operator with respect to the measure P_p. Now, since every site sends out exactly one unit of mass, we have $\mathrm{E}_p\, M(\mathcal{O}, \mathbb{Z}^d, \cdot) = 1$. Also, by considering the cases $\gamma(\mathcal{O}) = 1$ or $\gamma(\mathcal{O}) = 0$, we also have $\mathrm{E}_p\, M(\mathbb{Z}^d, \mathcal{O}, \cdot) = 2p$. Hence, we have, by the mass transport principle, $p = \frac{1}{2}$.

Proof of Lemma 3. For each $k \geq 1$, we define a mass transport T_k by saying that if a site x is k-bad then it sends out one unit of mass uniformly to every site in its k-clump $L_k(x)$, while x sends out no mass if the site is not bad. To be precise,

$$T_k(x, y, \gamma) := \left(\frac{1}{\#[L_k(x)]}\right)\mathbf{1}_{[x \text{ is } k\text{-bad}]}(\gamma)\mathbf{1}_{[y \in L_k(x)]}(\gamma).$$

It is easy to see that

$$\mathrm{E}\, T_k(\mathbb{Z}^d, \mathcal{O}, \cdot) = \mathrm{E}\left(\frac{1}{\#[L_k]}\sum_{x \in L_k} \mathbf{1}_{[x \text{ is } k\text{-bad}]}\right).$$

On the other hand, we have $\mathrm{E}\, T_k(\mathcal{O}, \mathbb{Z}^d, \cdot) = \mathrm{P}\{\mathcal{O} \text{ is } k\text{-bad}\}$. Thus, an application of the mass transport principle completes the proof.

Now we are in a position to obtain bounds on $\mathrm{P}(Z > r)$. We will see that the mass transport principle with information about the size of L_k and its diameter gives us an estimate with an application of the central limit theorem.

5. Proof of Theorem 1

5.1. First estimates

Let Φ be the translation-equivariant matching rule we obtain from the clumping rule $\mathcal{C}$ defined in Section 3. Recall that $Z = Z_\Phi$ is the distance from the origin to its partner under Φ. We will obtain bounds on $\mathrm{P}(Z > r)$ by choosing a sequence of events D_k and a $K = K(r)$ so that $\{Z > r\} \cap D_K \subset \{\mathcal{O} \text{ is } K\text{-bad}\}$. The events D_k will be chosen in a such way that we can obtain upper bounds on $\mathrm{P}\{\mathcal{O} \text{ is } K\text{-bad}\}$ and $\mathrm{P}(D_K^{\mathrm{c}})$.

Let $\alpha \in (0, 1)$. In fact, we will end up choosing $\alpha = \alpha(d) = 1/(1 + d/4)$. Recall that the events E_k and $C_k(s)$ were defined earlier in Section 3.2; see (2) and (3). Let B_k be the k-blob containing the origin. The following relations describe the geometry of B_k, when E_k or $C_k(r^\alpha)$ occur. We have

$$E_k \subset \{\text{there exists } x \text{ so that } B_k \subset S(x, r_k) \subset S(2r_k)\} \tag{7}$$

and

$$C_k(r^\alpha)^{\mathrm{c}} \subset \{S(r^\alpha) \subset B_k\}. \tag{8}$$

We consider the following decomposition:

$$\{Z > r\} = ((E_k \cap C_k(r^\alpha)^{\mathrm{c}}) \cap \{Z > r\}) \cup ((E_k^{\mathrm{c}} \cup C_k(r^\alpha)) \cap \{Z > r\}). \tag{9}$$

See Figure 1 for a realization of the event $E_k \cap C_k(r^\alpha)^{\mathrm{c}}$.

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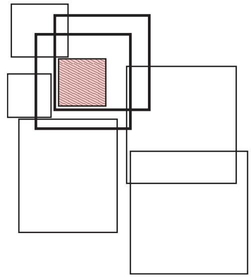

FIGURE 1: An illustration of the event $E_k \cap C_k(r^\alpha)^c$. The thick cutters represent k-cutters enclosing the origin. This corresponds to the event E_k. The shaded region represents the no cutter zone of radius r^α about the origin. This corresponds to the event $C_k(r^\alpha)^c$.

The role of parameter α can be explained heuristically as follows. If the parameter α is small then $C_k(r^\alpha)^c$ occurs with high probability, but then B_k could possibly be very small. If α is close to 1 then B_k would almost contain a cube of length $2r$, but then $C_k(r^\alpha)^c$ occurs with low probability. We will choose α to optimize over these alternatives.

Let $K = K(r)$ be defined to be the unique integer K such that

$$2r_K < r_{K+1} < r \le r_{K+2}. \tag{10}$$

Note that, for some $c_4 = c_4(d) > 0$, we have, for all $k \ge c_4$, $r_k = (2^k k^2)^{1/d} + \frac{1}{2} \le (\mathrm{e}^k/2^{-k})^{1/d}$. Hence, applying (1) with (10) for all sufficiently large r we have

$$\left(\frac{\log 2}{1+\log 2}\right)(K+1) \le \frac{d \log r}{1+\log 2} \le K+2. \tag{11}$$

Proposition 3. (Decay of the first term in (9) via the central limit theorem.) *For all $r > 0$ and the unique integer $K = K(r)$ such that $r_{K+1} < r \le r_{K+2}$, we have*

$$\mathrm{P}((E_K \cap C_K(r^\alpha)^c) \cap \{Z > r\}) \le \frac{c_5}{(r^\alpha)^{d/2}}$$

for some $c_5 = c_5(d) > 0$.

Remark. Note that the decay in Proposition 3 is the decay that appears in Theorem 3, if we set $\alpha = 1$.

Before we begin the proof of Proposition 3, we collect some easy, but important, observations. By (7) and (10), we have

$$(E_K \cap C_K(r^\alpha)^c) \cap \{Z > r\} \subset \{\mathcal{O} \text{ is } K\text{-bad}\}.$$

So from (8) we have

$$((E_K \cap C_K(r^\alpha)^c) \cap \{Z > r\}) \subset (\{\mathcal{O} \text{ is } K\text{-bad}\} \cap E_K \cap \{\#[L_K] \ge r^{\alpha d}\}). \tag{12}$$

Recall that L_k is the k-clump containing $\mathcal{O}$. To analyze the right-hand side of (12), we will use the following version of Lemma 3.

Lemma 5. *For all $k \geq 1$, we have*

$$\begin{aligned}
&\mathrm{P}(\{\mathcal{O} \text{ is } k\text{-bad}\} \cap E_k \cap \{\#[L_k \geq r^{\alpha d}]\}) \\
&\quad = \mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x\in L_k}(2\gamma(x)-1)\bigg|;\, E_k \cap \{\#[L_k] \geq r^{\alpha d}\}\bigg).
\end{aligned} \tag{13}$$

Proof. We use the mass transport principle. Recall T_k as defined in the proof of Lemma 3:

$$T_k(x, y, \gamma) := \left(\frac{1}{\#[L_k(x)]}\right)\mathbf{1}_{[x \text{ is } k\text{-bad}]}(\gamma)\mathbf{1}_{[y\in L_k(x)]}(\gamma).$$

We define another mass transport, namely

$$\hat{T}_k(x, y, \gamma) := T_k(x, y, \gamma)\mathbf{1}_{[E_k(x)\cap\{\#[L_k(x)]\geq r^{\alpha d}\}]}(\gamma).$$

Note that, on the event $\{y \in L_k(x)\}$, the event $E_k(x)$ occurs if and only if the event $E_k(y)$ occurs and $\#[L_k(x)] = \#[L_k(y)]$. Hence, we obtain

$$\begin{aligned}
\mathrm{E}\,\hat{T}_k(\mathbb{Z}^d, \mathcal{O}, \cdot) &= \mathrm{E}\sum_{y\in\mathbb{Z}^d} T_k(y, \mathcal{O}, \cdot)\mathbf{1}_{[E_k(y)\cap\{\#[L_k(y)]\geq r^{\alpha d}\}]} \\
&= \mathrm{E}\sum_{y\in\mathbb{Z}^d} T_k(y, \mathcal{O}, \cdot)\mathbf{1}_{[E_k\cap\{\#[L_k]\geq r^{\alpha d}\}]} \\
&= \mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x\in L_k}(2\gamma(x)-1)\bigg|;\, E_k \cap \{\#[L_k] \geq r^{\alpha d}\}\bigg).
\end{aligned}$$

On the other hand, we have

$$\mathrm{E}\,\hat{T}_k(\mathcal{O}, \mathbb{Z}^d, \cdot) = \mathrm{P}(\{\mathcal{O} \text{ is } k\text{-bad}\} \cap E_k \cap \{\#[L_k] \geq r^{\alpha d}\}).$$

Thus, an application of the mass transport principle (Lemma 4) completes the proof.

Next we will use the central limit theorem to estimate the right-hand side of (13), but first we need to verify that we have the necessary independence. For $k \geq 1$, consider the events

$$H_k(x_1, x_2, \ldots, x_n) := \{\{x_1, \ldots, x_n\} = L_k \cap S(2r_k)\},$$

where $x_i \in \mathbb{Z}^d$ and $\|x_i\| \leq 2r_k$. Let $\mathcal{G}_k := \sigma(\gamma(x)\colon x \in S(2r_k))$. The following lemma is behind why the (large) shift $s_k = \lfloor 100r_k\rfloor e_1$, along the axis e_1, appears in the definition of the k-cutters.

Lemma 6. *For all $k \geq 1$ and all $\|x_i\| \leq 2r_k$, the σ-field $\mathcal{G}_k$ is independent of*

$$\sigma(H_k(x_1, \ldots, x_n), E_k).$$

Proof. Consider a site y with $\|y\| < s_k/3$. The event $\{y \in L_k\}$ is determined by whether there exist j-seeds, with $j \geq k$, to give rise to j-cutters that can separate y from $\mathcal{O}$. However, such j-seeds (and their shells) are at least at distance $s_k/2$ from the origin. So, $\{\gamma(x)\colon \|x\| < s_k/3\}$ is independent of $H_k(x_1, x_2, \ldots, x_n)$ for all $x_i \in \mathbb{Z}^d$ such that $\|x_i\| \leq 2r_k$. Also, recall that E_k from (4) is determined by $\gamma(x)$, where $\|x\| \geq s_k/2$.

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Now the proof of Proposition 3 amounts to a simple calculation, whose result we record in the next lemma.

Lemma 7. *For all $k \geq 1$, we have*

$$\mathrm{P}(\{\mathcal{O} \text{ is } k\text{-bad}\} \cap E_k \cap \{\#[L_k] \geq r^{\alpha d}\}) \leq \frac{c_5}{(r^\alpha)^{d/2}}$$

for some $c_5 > 0$.

Proof of Proposition 3. From (12) and Lemma 7, Proposition 3 follows immediately.

Proof of Lemma 7. Let $k \geq 1$, and recall that by (7) we know that on the event E_k we have $L_k \subset S(2r_k)$. Fix $x_1, \ldots, x_n \in S(2r_k)$, and let $H_k = H_k(x_1, \ldots, x_n)$. We will now compute

$$A := \mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x \in L_k}(2\gamma(x) - 1)\bigg|; E_k \cap H_k(x_1, \ldots, x_n)\bigg),$$

by conditioning on $\mathcal{G}_k$. Let $S_n = \sum_{i=1}^n (2\gamma(x_i) - 1)$. Consider the following calculation:

$$\begin{aligned} A &= \mathrm{E}\bigg(n^{-1}\bigg|\sum_{i=1}^n (2\gamma(x_i) - 1)\bigg|\mathbf{1}_{[E_k \cap H_k]}\bigg) \\ &= \mathrm{E}(\mathrm{E}(n^{-1}|S_n|\mathbf{1}_{[E_k \cap H_n]} \mid \mathcal{G}_k)) && (14) \\ &= \mathrm{E}\bigg(\frac{1}{n}|S_n|\,\mathrm{E}(\mathbf{1}_{[E_k \cap H_k]} \mid \mathcal{G}_k)\bigg) && (15) \\ &= \mathrm{E}\bigg(\frac{1}{n}|S_n|\bigg)\,\mathrm{E}(\mathbf{1}_{[E_k \cap H_k]}) && (16) \\ &\leq \frac{c_5}{\sqrt{n}}\,\mathrm{P}(E_k \cap H_k) && (17) \end{aligned}$$

for some $c_5 > 0$. Equality (14) is obtained by conditioning on the σ-field $\mathcal{G}$. Equality (15) comes from the fact that the $\gamma(x_i)$ are all $\mathcal{G}$ measurable. By Lemma 6 we know that $\mathcal{G}_k$ and $\sigma(H_k, E_k)$ are independent, thus establishing equality (16). Inequality (17) is obtained by applying the central limit theorem.

By summing over all possible $H_k(x_1, \ldots, x_n)$ we obtain

$$\mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x \in L_k}(2\gamma(x) - 1)\bigg|; E_k \cap \{\#[L_k] = n\}\bigg) \leq \frac{c_5}{\sqrt{n}}\,\mathrm{P}\{\#[L_k] = n\}.$$

Furthermore, by summing over all $n \geq r^{\alpha d}$ we see that

$$\mathrm{E}\bigg(\frac{1}{\#[L_k]}\bigg|\sum_{x \in L_k}(2\gamma(x) - 1)\bigg|; E_k \cap \{\#[L_k] \geq r^{\alpha d}\}\bigg) \leq \frac{c_5}{(r^\alpha)^{d/2}}.$$

Thus, an application of Lemma 5 completes the proof.

Now we turn our attention to the second term in (9): $(E_k^{\mathrm{c}} \cup C_k(r^\alpha)) \cap \{Z > r\}$. We will bound this term in two different ways. As a first step, let us just throw away the term $\{Z > r\}$, since this will allow us to obtain a novel result for the case $d \geq 3$ without much more additional effort.

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Lemma 8. (Decay of the second term in (9): first bound.) *For all $r > 0$ and the unique integer $K = K(r)$ such that $r_{K+1} < r \leq r_{K+2}$, we have*

$$\mathrm{P}(E_K^{\mathrm{c}} \cup C_K(r^\alpha)) \leq c_6 r^{\alpha-1}(\log r)^2$$

for some $c_6 = c_6(d) > 0$.

Proof. Recall that, from Lemma 1 and Lemma 2, we already have bounds for the events appearing in this term. From (11) we see that

$$\mathrm{P}(E_K^c) \leq c_7 r^{-d/(1+\log 2)} \tag{18}$$

for some $c_7 = c_7(d) > 0$. Note that $d/(1+\log 2) > d/2$. On the other hand, applying (11) to Lemma 2 we obtain

$$\mathrm{P}(C_K(r^\alpha)) \leq c_8 r^{\alpha-1}(\log r)^2$$

for some $c_8 = c_8(d) > 0$.

Proposition 4. (Easy preliminary result.) *For all $d \geq 1$, there exists a translation-equivariant matching rule Φ such that $Z = Z_\Phi(\gamma) = \|\Phi(\gamma, \mathcal{O})\|$ has the tail behavior* $\mathrm{P}(Z > r) \leq c_9 r^{-\beta'}(\log r)^2$*, where $c_9 = c_9(d) > 0$ and $\beta' = \beta'(d) = 1/(1+2/d)$.*

Proof. We can see, from (9), Proposition 3, and Lemma 8, that

$$\mathrm{P}(Z > r) \leq c_5 r^{-\alpha d/2} + c_6 r^{\alpha-1}(\log r)^2.$$

Hence, we are led to minimize the quantity $\max(-\alpha d/2, \alpha-1)$. So we choose (for the purposes of this proposition) $\alpha = \alpha(d) = 1/(1+d/2)$.

Proposition 4 gives, for $d = 2$, a decay of order $(\log r)^2/r^{1/2}$. For $d = 2$, Theorem 2 still provides a better result, but, for $d \geq 3$, Proposition 4 provides faster decay than Theorem 2.

5.2. Long and thin blobs

With a closer analysis of the second term in (9) we will prove the following.

Proposition 5. (Decay of the second term in (9): closer analysis.) *For all $r > 0$ and the unique integer $K = K(r)$ such that $r_{K+1} < r \leq r_{K+2}$, we have*

$$\mathrm{P}((E_K^c \cup C_K(r^\alpha)) \cap \{Z > r\}) \leq c_{10} r^{-\alpha d/2} + c_{11} r^{2(\alpha-1)}(\log r)^4$$

for some $c_{10} = c_{10}(d) > 0$ and $c_{11} = c_{11}(d) > 0$.

Proposition 5 together with Proposition 3 yields a proof of Theorem 1.

Proof of Theorem 1. From the previous results, (9), Proposition 3, and Proposition 5, we have

$$\mathrm{P}(Z_\Phi > r) \leq c_5 r^{-\alpha d/2} + c_{10} r^{-\alpha d/2} + c_{11} r^{2(\alpha-1)}(\log r)^4.$$

Hence, we are led to minimize the quantity $\max(-\alpha d/2, 2(\alpha-1))$. It is easy to verify that we should take $\alpha(d) = 1/(1+d/4)$. Let $\beta(d) := d\alpha(d)/2 = 2/(1+4/d)$. Thus, we obtain

$$\mathrm{P}(Z_\Phi > r) \leq c(\log r)^4 r^{-\beta},$$

where $c = c(d) < \infty$ and $\beta = \beta(d) = 2/(1+4/d)$.

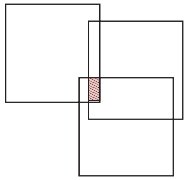

Figure 2: The shaded region represents the k-blob containing the origin. Note that on the event $C_k^2(r^\alpha)$ the k-blob can be quite *small*. For this reason, it seems we will not be able to do any better by including $\{Z > r\}$.

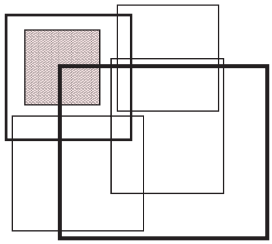

Figure 3: An illustration of the event $C_k^1(r^\alpha) \cap E_k$. The shaded region represents the restricted cutter zone of radius r^α about the origin. The very thick cutter represents the unique $j \geq k$ that intersects $S(r^\alpha)$. This corresponds to the event $C_k^1(r^\alpha)$. The other thick cutter represents a k-cutter enclosing the origin. This corresponds to the event E_k.

We will now work towards a proof of Proposition 5. We will need to examine the geometry of the blobs a bit closer to prove Proposition 5. Again, in light of (18) we do not need to worry about the event $E_k^{\rm c}$. Let us consider the decomposition,

$$C_k(r^\alpha) = (E_k^{\rm c} \cap C_k(r^\alpha)) \cup (E_k \cap C_k(r^\alpha)). \tag{19}$$

The second term puts us in a position akin to the situation of Proposition 3, since we can control the diameter of the k-blob containing the origin. We now examine two situations. One where the k-blob containing $\mathcal{O}$ is possibly very small (see Lemma 9, below, and Figure 2) and another where there are *enough* points inside the k-blob to make good use of the central limit theorem (see Lemma 11, below, and Figure 3).

Let $j \geq k$. Consider again j-seeds on the sets

$$A_j = A_j(r^\alpha) := S(-s_j, r_j + r^\alpha) \setminus S(-s_j, r_j - r^\alpha).$$

Observe that seeds on two levels will not overlap; that is, a seed in A_j will not overlap with a seed in A_m for $j \neq m$. Also, recall that, by our definition of k-seeds, no two (distinct) k-seeds will overlap. Since $C_k(r^\alpha)$ is the event that, for some $j \geq k$, the set A_j contains a j-seed, we will further split up this event. Define

$$\begin{aligned} C_k^1(r^\alpha) := \{&\text{for all } j \geq k, \text{ the set } A_j \text{ contains at most one } j\text{-seed and there is a} \\ &\text{unique } j \geq k \text{ such that } A_j \text{ contains a } j\text{-seed}\}, \\ C_k^2(r^\alpha) := {} & C_k(r^\alpha) \setminus C_k^1(r^\alpha). \end{aligned} \tag{20}$$

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We will throw away the term $\{Z > r\}$ when we bound $\mathrm{P}(C_k^2(r^\alpha) \cap E_k \cap \{Z > r\})$, but we will keep it when we bound $\mathrm{P}(C_k^1(r^\alpha) \cap E_k \cap \{Z > r\})$.

Lemma 9. *For all* $r > 0$ *and the unique integer* $K = K(r)$ *such that* $r_{K+1} < r \le r_{K+2}$*, we have* $\mathrm{P}(C_K^2(r^\alpha)) \le c_{13}(r^{\alpha-1}(\log r)^2)^2$ *for some* $c_{13} = c_{13}(d) > 0$.

Proof. For all $j \ge 1$, let $U_j = \{A_j \text{contains a } j\text{-seed}\}$. Thus, from (5) we have $U_j = U_j(r^\alpha)$. Since, for all $j \ge 1$, no two (distinct) j-seeds overlap, we have

$$\mathrm{P}\{A_j \text{ contains more than one } j\text{-seed}\} \le \mathrm{P}(U_j)^2.$$

Similarly, since seeds in A_j and A_m do not overlap for $j \ne m$, we have $\mathrm{P}(U_j \cap U_m) = \mathrm{P}(U_j)\,\mathrm{P}(U_m)$ for all $j \ne m$. Since

$$C_k^2(r^\alpha) \subset \Bigg(\bigcup_{j>m\ge k} U_j \cap U_m\Bigg) \cup \Bigg(\bigcup_{j\ge m}\{A_j \text{ contains more than one } j\text{-seed}\}\Bigg),$$

we have

$$\mathrm{P}(C_k^2(r^\alpha)) \le \sum_{j\ge k}\mathrm{P}(U_j)\sum_{m\ge k}\mathrm{P}(U_m).$$

By (6) and (11), it is easy to see that $\mathrm{P}(C_K^2(r^\alpha)) \le c_{13}(r^{\alpha-1}(\log r)^2)^2$ for some $c_{13} = c_{13}(d) > 0$. Thus, we have an improved term $r^{2(\alpha-1)}$. For a realization of the event $C_k^2(r^\alpha)$, see Figure 2.

We now turn our attention to the event $C_k^1(r^\alpha)$.

Lemma 10. *For all* $r > 0$ *and the unique integer* $K = K(r)$ *such that* $r_{K+1} < r \le r_{K+2}$*, we have* $C_K^1(r^\alpha) \subset \{\#[L_K] \ge c_{14}r^{\alpha d}\}$ *for some constant* $0 < c_{14} = c_{14}(d) < \infty$.

Proof. On the event $C_K^1(r^\alpha)$, there is exactly one j-cutter that has the property that it intersects $S(r^\alpha)$ and $j \ge K$; call this unique cutter C. Observe that if the cutter C was removed, the blob containing the origin would contain all of $S(r^\alpha)$. Note that C has side length at least $2r_K$. It is easy to see that there is a constant $c_{15} < \infty$ independent of k, so that $c_{15}r_k \ge r_{k+2}$. Thus, $c_{15}r_K \ge r$. Therefore, the blob containing the origin must contain a d-cube with side length r^α/c_{15}.

Lemma 11. *For all* $r > 0$ *and the unique integer* $K = K(r)$ *such that* $r_{K+1} < r \le r_{K+2}$*, we have* $\mathrm{P}(C_K^1(r^\alpha) \cap E_K \cap \{Z > r\}) \le c_{12}/(r^\alpha)^{d/2}$ *for some* $c_{12} = c_{12}(d) > 0$.

Proof. Again, from (7) and (10), we have $C_K^1(r^\alpha) \cap E_K \cap \{Z > r\} \subset \{\mathcal{O} \text{ is } K\text{-bad}\}$. So, by Lemma 10, it suffices to show that, for all $k \ge 1$, we have

$$\mathrm{P}(\{\mathcal{O} \text{ is } k\text{-bad}\} \cap E_k \cap \{\#[L_k] \ge c_{14}r^{\alpha d}\}) \le \frac{c_{12}}{(r^\alpha)^{d/2}} \tag{21}$$

for some $c_{12} = c_{12}(d) > 0$. Equation (21) follows from Lemma 7.

Proof of Proposition 5. Using (19) and (20), we have

$$\mathrm{P}((E_K^c \cup C_K(r^\alpha)) \cap \{Z > r\}) \le \mathrm{P}(E_K^c) + \mathrm{P}(C_K^2(r^\alpha)) + \mathrm{P}((C_K^1(r^\alpha)) \cap E_K \cap \{Z > r\}).$$

From (18) and Lemmas 9 and 11, we obtain

$$\mathrm{P}((E_K^c \cup C_K(r^\alpha)) \cap \{Z > r\}) \le c_7 r^{-d/(1+\log 2)} + c_{13}(r^{\alpha-1}(\log r)^2)^2 + \frac{c_{12}}{(r^\alpha)^{d/2}}.$$

6. Open problems

1. What is the optimal tail behavior for translation-equivariant matchings on $\mathbb{Z}^d$ in the case $d \geq 3$? When $d \geq 3$, from [12], for all $\varepsilon > 0$, there exists a translation-equivariant matching rule with exponential tails of order $\exp(-cr^{d-2-\varepsilon})$, where $c > 0$ is some constant. Does there exist a translation-equivariant matching rule with tails of order $\exp(-cr^d)$? The original problem is from [8], which also contains a few other related open problems.

2. We say that a translation-equivariant matching rule is *oriented* if it satisfies the additional restriction that if a site x is matched to a site y that contains a 1, then $y_i \geq x_i$ for all $i \leq d$. Observe that in Meshalkin's matching, a 0 is always matched to a 1 that is to the right of it. Note that it is not obvious that the method employed in this paper can be modified to work in an oriented setting. In one dimension, the restriction of orientation does not make a difference; one might think it should not for higher dimensions as well. What is the optimal tail behavior for matchings in $\mathbb{Z}^d$ with the restriction that we consider orientation as well?

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