Bull. Aust. Math. Soc. **83** (2011), 177–188
doi:10.1017/S0004972710001942

ON THE STABILITY OF STATIONARY LINE AND GRIM REAPER IN PLANAR CURVATURE FLOW

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(Received 7 June 2009)

Abstract

The asymptotic stability of two types of invariant solutions under a curvature flow in the whole plane is studied. First, by extending the work of others, we prove that the stationary line with nonzero slope will attract the graphical curves which surround it. Then a similar property is obtained for the grim reaper.

2010 *Mathematics subject classification*: primary 35K55; secondary 35B35, 35K15, 53C44.
Keywords and phrases: curvature flow, asymptotic behavior, stability.

1. Introduction

In this paper we study a curvature flow in the whole plane. Specifically, the asymptotic stability of the stationary solution and the translating solution is studied for the Cauchy problem

$$u_t = \frac{u_{xx}}{1+u_x^2}, \qquad x \in I,\ t > 0, \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x), \qquad x \in I, \tag{1.2}$$

where I may be $\mathbb{R}$ or an open interval of $\mathbb{R}$, and $u_0(x)$ belongs to $C^{2+\alpha}_{\mathrm{loc}}(I)$ for some $\alpha \in (0, 1)$.

Equation (1.1) is the simplest evolution equation reduced from a curvature flow arising in applied areas such as phase transitions and image processing (see [12, 15, 17]). Denote by $\gamma(t) \subset \mathbb{R}^2$ an interface that separates two different chemical or physical states. One of the mathematical models that describe the motion of the interface is characterized by the curvature flow

$$\frac{\partial \gamma}{\partial t} = -\kappa \vec{\nu}, \tag{1.3}$$

where $\vec{\nu}$ is a continuous choice of outer unit normals and κ is the curvature of the curve with respect to $\vec{\nu}$. The flow (1.3) is usually called a shortening flow (see [7, 8] for instance). Let the initial curve $\gamma(0)$ be given by the graph $(x, u_0(x))$ on the whole

This work is supported by the Grant for New PhD Teacher Program of Southeast University.

x-axis. Then under the flow (1.3), it is well known that the evolving curve $\gamma(t)$ can be expressed in the form of a graph $(x, u(x, t))$ and $u(x, t)$ satisfies problem (1.1)–(1.2) with $I = \mathbb{R}$. For more references about the flow (1.3), one can refer to the monograph [5].

A pioneering work on the flow (1.3) of entire graphs in higher dimensions was contributed by Ecker and Huisken [6], who found a class of initial interfaces under which the flow converges to an expanding self-similar solution. Then, under more general initial conditions, Stavrou [16] obtained the same convergence result. Ishimura improved their results in [10] in the planar setting. In fact, all of their work can be regarded as relating to the stability of expanding self-similar solutions to the flow (1.3).

Our aim in this paper is to study the stability of other invariant solutions for the flow (1.3), such as a stationary line, or a translating solution. In the whole plane, every line is a stationary solution to the flow. Recently, Nara and Taniguchi [13] considered the stability of the stationary line $u(x, t) \equiv 0$. Their result is as follows.

PROPOSITION 1.1. *Let $\alpha \in (0, 1)$. Assume that initial data u_0 satisfies $\|u_0\|_{C^{2+\alpha}} \le C_0$ for some constant $C_0 > 0$ and*

$$\lim_{R\to+\infty} \sup_{x_0\in\mathbb{R}} \frac{1}{2R}\left|\int_{x_0-R}^{x_0+R} u_0(x)\,dx\right| = 0. \tag{1.4}$$

Then there exists a classical global-in-time solution $u(x, t)$ to problem (1.1)–(1.2). Moreover, the solution $u(x, t)$ satisfies

$$\sup_{x\in\mathbb{R}} |u(x, t)| \le C_1(1 + t)^{-1/2}, \quad t > 0, \tag{1.5}$$

for a constant C_1 depending only on u_0.

Motivated by Nara and Taniguchi's work [13], we wish to study the asymptotic stability of lines $y = kx$ with $k \ne 0$. Our result reads as follows.

THEOREM 1.2. *Assume that u_0 satisfies*

$$\|u_0 - kx\|_{C^{2+\alpha}} \le C_0, \quad 0 < \alpha < 1, \tag{1.6}$$

and

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}} \frac{1}{2R}\left|\int_{x_0-R}^{x_0+R} (u_0(x) - kx)\,dx\right| = 0, \tag{1.7}$$

where k is a nonzero constant. Then there exists a classical global-in-time solution $u(x, t)$ to problem (1.1)–(1.2) and the solution satisfies

$$\sup_{x\in\mathbb{R}} |u(x, t) - kx| \le C_2(1 + t)^{-1/2}, \quad t > 0, \tag{1.8}$$

where the constant C_2 depends only on u_0.

REMARK 1.3. As far as the uniqueness of the solution is concerned, one can refer to the remark after Proposition 2.1. Under the assumption (1.6), the condition (1.7)

https://doi.org/10.1017/S0004972710001942 Published online by Cambridge University Press

is also necessary to guarantee that the solution converges to the line $y = kx$. This is explained in Section 4.

Here, we point out that our result can be regarded as an interesting generalization of Nara and Taniguchi's work [13], since the initial curve $(x, u_0(x))$ in Theorem 1.2 could fail to be graphical on the line $y = kx$. Because of this, the method used in [13] cannot be applied directly to our problem. Fortunately, a decay estimate is obtained to guarantee that the curve $\gamma(t) = (x, u(x, t))$ becomes a graph on the line $y = kx$ after some finite time (see Proposition 2.3). And the condition (1.4) is verified to hold for $\gamma(t)$ in the new coordinates where '$y = kx$' is taken to be the 'x-axis'. Thus the desired result will follow from Proposition 1.1.

Motivated by previous work, we will also investigate the asymptotic stability of the translating solutions to the curvature flow (1.3). A translating solution to (1.3) is a solution which assumes the form

$$\phi(x, t) = \phi_0(x) + t, \quad \phi_0(x) = \log \sec x, \quad x \in (-\pi/2, \pi/2), \tag{1.9}$$

when it is expressed as a graph over the x-axis. It is frequently called a *hair-pin* in the physics literature and serves as the model for studying tachyon condensation in open string theory. For more physical background one may refer to a recent review paper [3], where the linearized stability of the hair-pin is analyzed. In the mathematics literature the solution (1.9) is well known as the *grim reaper*; it is used to characterize the Type II singularity in the curve shortening problem. Huisken [9] mentioned the convergence of the flow (1.3) to a grim reaper for a special initial curve (see also [14]).

Let $u_0(x)$ be a perturbation of the grim reaper $\phi_0(x)$ on the interval $(-\pi/2, \pi/2)$. See Figure 1. Assume that u_0 satisfies

$$\|u_0(x) - \phi_0(x)\|_{C^{2+\alpha}((-\pi/2,\pi/2))} < C_0, \quad \alpha \in (0, 1), \tag{1.10}$$

for some constant C_0, that

$$(u_0)_{xx} \text{ has finitely many zero points with } (u_0)_{xx} > 0 \text{ as } x \to (\pm\pi/2)^{\mp}, \tag{1.11}$$

and that

$$\int_{-\pi/2}^{\pi/2} (u_0 - \phi_0)\, dx = 0. \tag{1.12}$$

We assert that under the above perturbation the flow will converge to a grim reaper.

THEOREM 1.4. *For initial data $u_0(x)$ satisfying conditions (1.10)–(1.12), problem (1.1)–(1.2) with $I = (-\pi/2, \pi/2)$ has a classical global-in-time solution $u(x, t)$ which satisfies*

$$u(x, t) \to \phi(x, t) \quad \text{as } t \to \infty,$$

on any compact subset of $(-\pi/2, \pi/2)$ in the C^∞ sense, where $\phi(x, t)$ is defined in (1.9).

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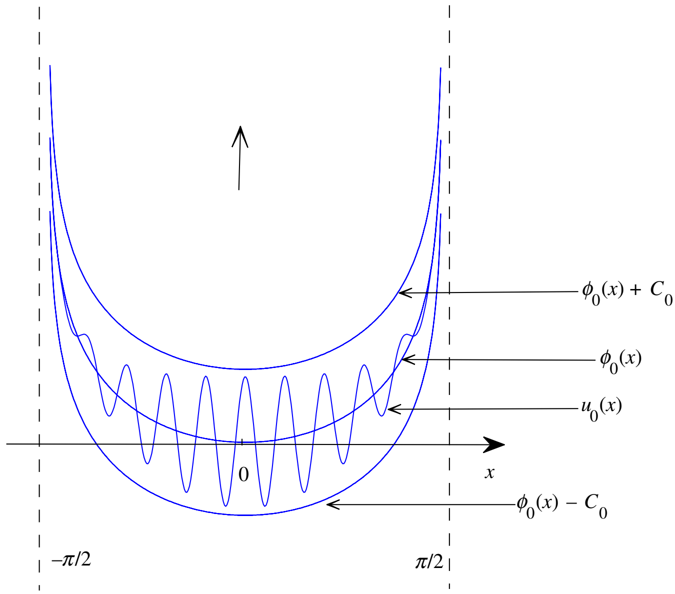

FIGURE 1. The grim reaper $\phi_0(x)$ and its perturbation $u_0(x)$.

To show the long-time solvability of problem (1.1)–(1.2) in Theorem 1.4, an *a priori* estimate is obtained first, which also yields a compactness result. Then a Lyapunov functional is discovered to show the long-time behavior.

This paper is organized as follows. We prove Theorems 1.2 and 1.4 in Sections 2 and 3, respectively. In the last section, we will apply our results to discuss problem (1.1)–(1.2) on the half-plane.

2. The proof of Theorem 1.2

Let us define the function

$$v(x, t) := u(x, t) - kx.$$

Then $v(x, t)$ satisfies the equation

$$v_t = \frac{v_{xx}}{1 + (v_x + k)^2}, \quad x \in \mathbb{R}, t > 0, \tag{2.1}$$

and the initial condition

$$v(x, 0) = u_0(x) - kx. \tag{2.2}$$

In order to prove Theorem 1.2, we first prove some propositions. A well-posedness result about problem (1.1)–(1.2) with $I = \mathbb{R}$ is stated first.

PROPOSITION 2.1. *Assume that* $u_0 \in C^{2+\alpha}_{\text{loc}}(\mathbb{R})$ $(0 < \alpha < 1)$ *satisfies the condition (1.6), that is,* $\|u_0 - kx\|_{C^{2+\alpha}} \le C_0$ *for some constant* $C_0 > 0$ *and* $k \neq 0$. *Then*

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there exists a classical solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\mathbb{R} \times [0,\infty))$ *to problem (1.1)–(1.2). Moreover, we have the inequality*

$$\sup_{x\in R, t>0} |u_x(x,t) - k| \le \sup_{x\in R} |u_{0x} - k| \le C_0. \tag{2.3}$$

PROOF. By considering the equivalent problem (2.1)–(2.2), we can prove this proposition by following the proof of the well-posedness of problem (1.1)–(1.2) for the case $k = 0$ in [13]. □

REMARK 2.2. (i) The global-in-time existence of a complete properly embedded curve under the flow (1.3) has been obtained in [4] by Chou–Zhu. Here, by considering the Cauchy problem (1.1)–(1.2), we can get more information, such as the estimate (2.3).

(ii) We sketch how to show the uniqueness of problem (2.1)–(2.2). By regarding the solution to (1.1)–(1.2) as a graphical solution to problem (1.3), we can use the touch principle (see [2, Theorem 1.3]) to deduce the inequality

$$\sup_{x\in R, t>0} |u(x,t) - kx| \le C_0$$

where C_0 appears in Proposition 2.1. In the class

$$\left\{ v \in C^{2+\alpha,1+\alpha/2}(\mathbb{R} \times [0,\infty)) \;\middle|\; \sup_{x\in R, t>0} |v(x,t)| \le C_0 \right\},$$

only one solution could be found for problem (2.1)–(2.2).

Next we show the decay estimate for u_x by a modification of the method in [13].

PROPOSITION 2.3. *The solution* $u(x,t)$ *of (1.1)–(1.2) satisfies*

$$\sup_{x\in\mathbb{R}} |u_x(x,t) - k| \le C_1(1+t)^{-1/2}, \quad t \ge 0, \tag{2.4}$$

where C_1 *depends only on* u_0.

PROOF. Let $w = \varepsilon(u - kx)$ with ε to be chosen. We consider a function $V(x,t)$ defined by

$$V(x,t) = w^2 + \frac{t}{1+M^2} w_x^2,$$

where $M = |k| + C_0$. Note that $w(x,t)$ satisfies the equation

$$w_t = \frac{w_{xx}}{1+u_x^2}.$$

Then a direct computation yields

$$V_t - \frac{V_{xx}}{1+u_x^2} = -\frac{2w_x^2}{1+u_x^2} + \frac{w_x^2}{1+M^2} - \frac{t}{1+M^2}\left[\frac{4w_{xx}^2 w_x^2}{(1+u_x^2)^2} + \frac{2w_{xx}^2}{1+u_x^2} + \frac{4k w_{xx}^2 w_x}{(1+u_x^2)^2}\right].$$

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From (2.3), we can deduce that

$$-\frac{2w_x^2}{1+u_x^2}+\frac{w_x^2}{1+M^2}\le 0$$

and

$$\left|\frac{4kw_{xx}^2 w_x}{(1+u_x^2)^2}\right|\le 4\varepsilon|k|M\frac{w_{xx}^2}{1+u_x^2}.$$

So, if we choose a small ε such that $4\varepsilon|k|M\le 1$, then

$$V_t-\frac{V_{xx}}{1+u_x^2}\le 0,\quad \forall(x,t)\in\mathbb{R}\times(0,\infty).$$

Then by the maximum principle,

$$V(x,t)\le\|V(x,0)\|_{L^\infty(\mathbb{R})}=\|w(x,0)\|^2_{L^\infty(\mathbb{R})},\quad t\ge 0,$$

which implies that

$$\sup_{x\in\mathbb{R}}|u_x(x,t)-k|\le(1+M^2)^{1/2}t^{-1/2},\quad t>0.$$

That is, (2.4) holds for a positive constant C_1 dependent only on u_0. □

We now introduce a new variable $\xi\in\mathbb{R}$ and define the line $y=kx$ to be the 'ξ-axis'. The origin in the new coordinates is just the one in the old coordinates. Based on Propositions 2.1 and 2.3, we have the following result.

PROPOSITION 2.4. *Let $u(x,t)$ be given in Theorem 1.2. Then there exists a finite time $t_0>0$ such that the curve $(x,u(x,t))$ becomes a graph on the ξ-axis ($y=kx$): $(\xi,w(\xi,t))$ for $t\ge t_0$. Moreover, the function $w(\xi,t)$ solves the equation*

$$w_t=\frac{w_{\xi\xi}}{1+w_\xi^2},\quad \xi\in\mathbb{R},\ t>t_0. \tag{2.5}$$

In addition, $w(\xi,t_0)$ satisfies $\|w(\xi,t_0)\|_{C^{2+\alpha}(\mathbb{R})}\le\infty$ (α is as in Proposition 2.1) and

$$\lim_{R\to+\infty}\sup_{\xi_0\in\mathbb{R}}\frac{1}{2R}\left|\int_{\xi_0-R}^{\xi_0+R}w(\xi,t_0)\,d\xi\right|=0. \tag{2.6}$$

PROOF. The rotational invariance property of the flow (1.3) implies that (2.5) holds. So we only need to prove (2.6). To attain this, we first show that the condition (1.7) is satisfied by $u(x,t)$ for any $t\ge 0$, that is,

$$\lim_{R\to\infty}\sup_{x_0\in\mathbb{R}}\frac{1}{2R}\left|\int_{x_0-R}^{x_0+R}(u(x,t)-kx)\,dx\right|=0,\quad \forall t\ge 0. \tag{2.7}$$

Note that for a graph $(x,u(x,t))$ on $\mathbb{R}$ evolving according to (1.1), we have that, on any finite interval (a,b),

$$\frac{d}{dt}\int_a^b(u(x,t)-kx)\,dx=\int_a^b(\arctan u_x)_x\,dx=\arctan u_x\Big|_a^b. \tag{2.8}$$

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If we fix any $t > 0$, then for any $R > 0$ and $x_0 \in \mathbb{R}$, (2.8) tells us that

$$\left| \int_{x_0-R}^{x_0+R} (u_0(x) - kx)\, dx - \int_{x_0-R}^{x_0+R} (u(x,t) - kx)\, dx \right| \le \pi t,$$

which implies that

$$\sup_{x_0 \in \mathbb{R}} \frac{1}{2R} \left| \int_{x_0-R}^{x_0+R} (u(x,t) - kx)\, dx \right| \le \sup_{x_0 \in \mathbb{R}} \frac{1}{2R} \left| \int_{x_0-R}^{x_0+R} (u_0(x) - kx)\, dx \right| + \frac{\pi t}{2R}.$$

Hence, for any given $\varepsilon > 0$, we will have

$$\sup_{x_0 \in \mathbb{R}} \frac{1}{2R} \left| \int_{x_0-R}^{x_0+R} (u(x,t) - kx)\, dx \right| \le \varepsilon$$

if R is sufficiently large. Thus (2.7) is proved. Note that in the new coordinates it is not difficult to observe that

$$\frac{1}{2R} \left| \int_{\xi_0-R}^{\xi_0+R} w(\xi, t_0)\, d\xi \right| \le \frac{1}{2R} \left| \int_{p(\xi_0-R)}^{p(\xi_0+R)} u(x, t_0)\, dx \right| + \frac{C_0^2 |k|}{R}, \tag{2.9}$$

where the function p is defined as

$$p(z) = \frac{z}{\sqrt{1+k^2}}, \quad \forall z \in \mathbb{R},$$

and C_0 is the constant in (1.6). Then (2.6) follows if we take the limit $R \to +\infty$ in (2.9). □

Proof of Theorem 1.2. Proposition 2.1 gives the existence of the solution. Then the desired convergence result follows from Propositions 1.1 and 2.4. □

3. The proof of Theorem 1.4

In this section, we will prove Theorem 1.4. First, a global existence result is obtained.

Proposition 3.1. *For initial data $u_0(x)$ satisfying condition (1.10), problem (1.1)–(1.2) with $I = (-\pi/2, \pi/2)$ has a classical global-in-time solution $u(x,t)$, which satisfies*

$$\sup_{x \in (-\pi/2, \pi/2)} |u(x,t) - \phi(x,t)| \le C_0, \quad \forall t \in [0, \infty), \tag{3.1}$$

and

$$\|u(x,t) - \phi(x,t)\|_{C^{2+\alpha}([-a,a])} \le C_1(a), \quad \forall t \in [a, \infty), \forall a \in (0, \pi/2), \tag{3.2}$$

where $C_1(a)$ is a constant depending on a.

Proof. Two grim reapers $\phi_0 \pm C_0$ can be adopted as barriers for the solution $u(x,t)$. Indeed, condition (1.10) means that $\phi_0 \pm C_0$ are separated from $u_0(x)$ at

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the beginning; see Figure 1. Then the touch principle in [2] tells us that as long as the solution exists, it will be separated from two evolving grim reapers $\phi_0 \pm C_0 + t$ forever and hence gives the *a priori* estimate (3.1). Write $w = u - \phi$. Then w satisfies the equation

$$w_t = \frac{w_{xx}}{1 + (w_x + \phi_x)^2} + \frac{\phi_{xx}}{1 + (w_x + \phi_x)^2} - 1, \tag{3.3}$$

where $(x, t) \in (-\pi/2, \pi/2) \times (0, T)$. Let $a \in (0, \pi/2)$ and $T > a$. Since (3.1) tells us that the solution $w(x, t)$ has *a priori* estimate $\|w(x, t)\|_{L^\infty(I\times(0,T))} \le C_0$, by employing [11, Theorems 11.18 and 12.2] we can obtain the interior gradient estimate,

$$\|w_x\|_{L^\infty(I'\times(0,T))} \le C_2(a), \tag{3.4}$$

and the interior Hölder estimate, $\|w_x\|_{C^{\delta,\delta/2}(I''\times[a/2,T))} \le C_3(a)$ with $0 < \delta < 1$, where I' and I'' are two open intervals satisfying $[-a, a] \subset\subset I'' \subset\subset I' \subset\subset I$. Then standard regularity theory yields *a priori* estimate (3.2), which implies that the curvature of the curve $(x, u(x, t))$ with $(x, t) \in [-a, a] \times [a, T)$ has a bound only dependent on a. This permits us to use an argument as in [4, Section 1] to show the global existence of the flow. □

In what follows, we will show that the *a priori* estimate (3.2) implies a compactness result. To this end, we introduce a Lyapunov functional,

$$\mathcal{J}(f) = \int_{-\pi/2}^{\pi/2} f^2\, dx,$$

where $f \in L^\infty(I)$.

PROPOSITION 3.2. *Assume that initial data $u_0(x)$ satisfies the conditions (1.10)–(1.12). Then, for any sequence $\{t_i\}_{i=1}^\infty$ tending to ∞, there exists a subsequence $\{t_{i_k}\}$ such that $(u - \phi)(x, t_{i_k}) \to 0$ in $C^2([-a, a])$ for any $a \in (0, \pi/2)$.*

PROOF. Write $w = u - \phi$. The estimate (3.2) ensures that for any sequence $\{t_i\}_{i=1}^\infty$ tending to ∞ we can find a subsequence $\{t_{i_k}\}$ such that $w(x, t_{i_k}) \to v(x)$ in $C^2([-a, a])$ for some function $v(x)$. We first claim that

$$\int_{-a}^{a} v_x^2\, dx = 0. \tag{3.5}$$

To attain this goal, we consider the functional $\mathcal{J}(w)$. A direct computation yields

$$\frac{d}{dt}\mathcal{J}(w) = 2\int_{-\pi/2}^{\pi/2} (u - \phi)(\arctan u_x - \arctan \phi_x)_x\, dx.$$

To go further, we differentiate Equation (1.1) with respect to the variable x to obtain a parabolic equation for the unknown u_{xx} and apply the zero number theory [1] to this equation. It can be seen that u_{xx} will not change its sign near $x = \pm\pi/2$ after a long time. Thus, for large t,

$$u_x(x, t) \to \pm\infty \quad \text{as } x \to \pm\pi/2, \tag{3.6}$$

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due to the estimate (3.1). Then integration by parts yields

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(w) &= -2\int_{-\pi/2}^{\pi/2}(u-\phi)_x(\arctan u_x - \arctan \phi_x)\,dx\\ &= -2\int_{-\pi/2}^{\pi/2}\frac{(u_x-\phi_x)^2}{1+\xi^2(x,t)}\,dx\\ &\leq 0,\end{aligned}$$

where $\xi(x,t)$ is between $u_x(x,t)$ and $\phi_x(x,t)$. So

$$\int_0^T\int_{-\pi/2}^{\pi/2}\frac{(u_x-\phi_x)^2}{1+\xi^2(x,t)}\,dx\,dt \leq \frac{1}{2}\mathcal{J}(u-\phi)(\cdot,0) = \frac{\pi c_0^2}{2}, \quad \forall T>0.$$

Note that the estimate (3.4) implies that

$$\sup_{x\in[-a,a]}|\xi(x,t)| \leq C, \quad \forall t>0, \forall a\in(0,\pi/2).$$

Henceforth, C always denotes a constant dependent on a. Therefore in fact

$$\int_0^T\int_{-a}^{a}(u_x-\phi_x)^2\,dx\,dt \leq C, \quad \forall T>0,$$

which implies that

$$\int_0^\infty\int_{-a}^{a}(u_x-\phi_x)^2\,dx\,dt < \infty. \tag{3.7}$$

To show that (3.5) holds, it is sufficient to show that

$$\int_a^{-a}(u_x-\phi_x)^2\,dx \to 0 \quad \text{as } t\to\infty.$$

If not, there exist a constant $C>0$ and a sequence $\{\tau_j\}_{j=1}^\infty$ tending to ∞ such that

$$\int_a^{-a}(u_x-\phi_x)^2(x,\tau_j)\,dx \geq C.$$

Observing that (3.2) implies that there is an estimate for the norm of $(u-\phi)_t$ in $C^1([-a,a])$ independent of $t\geq a$, we can find a ρ_0 (independent of τ_j) such that

$$\int_a^{-a}(u_x-\phi_x)^2(x,t)\,dx \geq \frac{C}{2} \quad \text{on } [\tau_j, \tau_j+\rho_0].$$

It follows that

$$\int_{\tau_j}^{\tau_j+\rho_0}\int_{-a}^{a}(u_x-\phi_x)^2\,dx\,dt \geq \frac{C\rho_0}{2}. \tag{3.8}$$

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On the other hand, from (3.7) we know that

$$\lim_{j\to\infty}\int_{\tau_j}^{\infty}\int_{-a}^{a}(u_x-\phi_x)^2\,dx\,dt=0,$$

which contradicts (3.8). Thus (3.5) holds.

Next, we show that $v(x)\equiv 0$. In the above argument, take $a=\pi/2-\varepsilon_k$ with $\{\varepsilon_k\}_{k=1}^{\infty}$ tending to 0 and decreasing with respect to k. It is not difficult to produce a sequence $\{t_j\}$ along which $w(x,t_j)$ converges to $v(x)$, pointwise for $x\in(-\pi/2,\pi/2)$; and in $C^2([-a,a])$ for any $a\in(0,\pi/2)$ as $t_j\to\infty$. Another useful observation is that $w(x,t)$ satisfies

$$\int_{-\pi/2}^{\pi/2} w\,dx=0,\qquad \forall t>0. \tag{3.9}$$

This is because

$$\frac{d}{dt}\int_{-\pi/2}^{\pi/2} w\,dx=(\arctan u_x-\arctan\phi_x)\Big|_{-\pi/2}^{\pi/2}=0,$$

where the last equality is due to (3.6). By the dominated convergence theorem, we take the limit in (3.9) along the sequence $\{t_j\}$ and know that $v(x)$ satisfies

$$\int_{-\pi/2}^{\pi/2} v(x)\,dx=0.$$

Thus it must be the case that $v(x)\equiv 0$ in view of (3.5). □

PROOF OF THEOREM 1.4. Again write $w=u-\phi$. Proposition 3.2 shows that for any time sequence approaching infinity, there exists a subsequence $\{t_j\}$ such that

$$w(x,t_j)\to 0\quad \text{in } C^2([-a,a]),\ \forall a\in(0,\pi/2)\text{ as } t_j\to\infty.$$

In fact, standard regularity results on the quasi-linear parabolic equation such as (3.3) imply that all spatial derivatives of $w(\cdot,t)$ have a uniform bound independent of large t on the interval $[-a,a]$ for any $a\in(0,\pi/2)$. So the convergence result in Proposition 3.2 in fact happens in $C^{\infty}([-a,a])$ for any $a\in(0,\pi/2)$. The proof is complete. □

4. Discussion

First, we remark that in Proposition 1.1, under the assumption $\|u_0\|_{C^{2+\alpha}}\le C$, the condition (1.4) is also necessary for the solution u to converge to 0. This is illustrated by an example presented in [13]. In fact, a slight modification of that example will illustrate that condition (1.7) is also necessary for the convergence of the solution in Theorem 1.2. Moreover, we point out that Theorem 1.2 not only extends Nara and Taniguchi's work to a more general case, but also gives a sufficient condition for a line to attract a family of complete, noncompact embedded curves surrounding it.

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Furthermore, Theorem 1.2 can be employed to investigate the curvature flow on the half-plane. Let us consider the problem

$$\begin{cases} u_t = \dfrac{u_{xx}}{1+u_x^2} & \text{if } x > 0,\ t > 0, \\ u(0,t) = b & \text{if } t > 0, \\ u(x,0) = u_0(x) & \text{if } x > 0, \end{cases} \tag{4.1}$$

where $u_0(x)$ satisfies $u_0(0) = b$ with $b \in \mathbb{R}$. We assume that

$$\|u_0 - kx - b\|_{C^{2+\alpha}(\mathbb{R}^+)} \le C_0 \tag{4.2}$$

and

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}^+} \frac{1}{2R}\left|\int_{\max\{x_0-R,0\}}^{x_0+R} (u_0(x) - kx - b)\,dx\right| = 0, \tag{4.3}$$

with $k \in \mathbb{R}$. Under the assumptions (4.2)–(4.3), problem (4.1) has a classical solution converging to $y = kx + b$ $(x \ge 0)$ as time approaches ∞. This can be observed by applying Theorem 1.2 to the function $v(x,t)$ defined as follows:

$$v(x,t) = u(x,t) - b, \quad x \ge 0,\ t \ge 0;$$
$$v(x,t) = -u(-x,t) - b, \quad x < 0,\ t \ge 0.$$

Acknowledgement

The author is grateful to the anonymous referee whose comments helped improve the presentation of this paper.

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